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EBERLEIN-WEAKLY ALMOST PERIODIC SOLUTIONS OF A NON-HOMOGENEOUS LINEAR EQUATION IN A BANACH SPACE

BY

S. FATAJOU

Abstract. We present some results of Eberlein-weakly almost periodic functions with values in a Banach space. Then we apply these results to examine almost periodicity properties of bounded solutions of the non-homogeneous linear equation

$$x'(t) = (A + B)x(t) + f(t), \quad t \in \mathbb{R},$$

using various conditions of the underlying Banach space and on the operators A and B.

1. Introduction. In this paper we study the asymptotic behaviour of solutions to the homogeneous differential equation in a Banach space X:

$$x'(t) = (A + B)x(t), \quad t \in \mathbb{R}, \tag{1}$$

as well as the non-homogeneous case

$$x'(t) = (A + B)x(t) + f(T), \quad t \in \mathbb{R}. \tag{2}$$

To emphasis is on the weak almost periodicity properties (in the sense of Eberlein) of the solution.

Our main results specify conditions on the underlying Banach space X and on the operators A and B, for which all bounded solutions of (1) are weakly almost periodic in the sense of Eberlein and then we deal with bounded solutions of (2) where f is a weak almost periodic function and various supplementary conditions are satisfied.

For existing results on weak almost periodicity properties of the solutions in the context of differential equations connected with evolution systems, linear or nonlinear, see ([1], [2], [12], [14], for instance).

2. Notations and basic properties of weakly almost periodic functions. Throughout the paper, X denotes a (real or complex) Banach space. The dual of X will be denoted by X^*, and B_{X^*} will denote the dual unit ball of X^*. For a subset D of X, the closure and weak closure of D will be denoted by clD and $w-clD$, respectively (w denotes the weak topology on X).

The space of all bounded continuous functions from R^+ into X and from R into X will be denoted by $BC(R^+,X)$ and $BC(R,X)$, respectively, and we shall hereafter assume that each of these spaces is equipped with the supermum norm.

For $J\in\{\mathbb{R},\mathbb{R}^+:=[0,+\infty)\}$, and $\tau\in J$, moreover, we put $f_\tau(f)=f(t+\tau),t\in J$.

Definition 1. A bounded continuous function $f:\mathbb{R}\to X$ is almost periodic, if the orbit of f, the set of translates,

$$\Theta(f):=\{f_\tau:\tau\in\mathbb{R}\}$$

is a relatively compact set in $BC(\mathbb{R},X)$.

We denote these functions by:

$$AP(\mathbb{R},X):=\{f\in BC(\mathbb{R},X):f \text{ is almost periodic}\}.$$

The canonical weakening of the above definition leads to the notion of weakly almost periodic functions, as done by Eberlein in the scalar case [9].

Definition 2. A bounded continuous function $f:J\to X$, $J\in\{\mathbb{R},\mathbb{R}^+:=[0,+\infty)\}$, is weakly almost periodic in the sense of Eberlein (Eberlein-weakly almost periodic) if the orbit of f with respect to J:

$$\Theta_J(f):=\{f_\tau:\tau\in J\}$$

is relatively compact in $BC(\mathbb{R},X)$ with respect to the weak topology.

We denote these functions by

$$W(J,X):=\{f\in BC(J,X):f \text{ is Eberlein-weakly almost periodic}\}.$$

In 1969 Deleeuw and Glicksberg [6], [7] proved that if we consider the space of those Eberlein-weakly almost periodic functions, which contain zero in the weak closure of the orbit (weak topology of $BC(J,X)$), i.e.:

$$W_0(J,X) := \{f \in W(J,X) : \text{for a sequence } (s_n)_{n\in\mathbb{N}} \subseteq J, f_{s_n} \rightharpoonup 0\},$$

the following decomposition

$$W(J,X) = AP(\mathbb{R},X)|_J \oplus W_0(J,X)$$

holds. Moreover, if $f \in W(J,X), g \in AP(\mathbb{R},X)$, and $h \in W_0(J,X)$ with $f = g + h$, then

$$\|\cdot\| - \lim_{T\to+\infty} \frac{1}{T}\int_0^T (f(t+r) - g(t+r))dr = 0,$$

uniformly in $t \in J$.

For more detailed informations about the decomposition and the ergodic result we refer to [11]. Thus the question of Eberlein-weakly almost periodic solutions to a differential equation is part of the asymptotics.

In order to prove the weak compactness of translates, Ruess and Summers in [13] extended the double limits criterion of Grothendieck [10] to the following:

Proposition 1. *1. A subset $A \subseteq BC(J,X)$ is relatively weakly compact if and only if,*

(i) A is bounded in $BC(J,X)$, and

(ii) for all $(h_m)_{m\in\mathbb{N}} \subseteq A, (t_n)_{n\in\mathbb{N}} \subseteq J$ and $(x_n^*)_{n\in\mathbb{N}} \subseteq B_{X^*}$ the following double limits condition holds:

$$\lim_{n\to+\infty}\lim_{m\to+\infty}, < h_m(t_n), x_m^* >= \lim_{m\to+\infty}\lim_{n\to\infty} < h_m(t_n), x_m^* >,$$

whenever the iterated limits exist.

This result will be the main tool in verifying weak almost periodicity. For the other task, we will use:

Proposition 2. *Eberlein-weakly almost periodic functions are uniformly continuous with relatively weak compact ranges.*

Proposition 3. *Eberlein-weakly almost periodic functions have uniformly convergent means, i.e.: if $f \in W(J,X)$, then*

$$\|\cdot\| - \lim_{T\to+\infty} \frac{1}{T}\int_t^{T+t} f(r)dr = x \in X, \text{ exists uniformly over } t \in J.$$

For more details on the theory of almost and weak almost periodic functions, including examples, c.f., for instance, [3], [4], [9] and [13].

3. Existence of Eberlein-weakly almost periodic solutions. We first prove a number of preliminary results, which will be fundamental for our subsection results on Eberlein-weakly almost periodic solution to homogeneous differential equation (1) and the corresponding non-homogeneous case, equation (2).

3.1. Preliminary results.

Lemma 1. *Let $f : \mathbb{R} \to X$ be a Eberlein-weak almost periodic function and $T = (T(t))_{t\in\mathbb{R}}$ be a bounded C_0-group of continuous linear operators such that*

(i) $T(.)x : \mathbb{R} \to X$ is Eberlein-weakly almost periodic, for each $x \in X$, and

(ii) for every weakly compact subset K of X such that $(K, w|_K)$ is metrizable, the map

$$\begin{aligned} \Upsilon : (K, w|_K) &\to W(\mathbb{R}, X) \\ x &\to (T(.)x)(t) := T(t)x \end{aligned}$$

is continuous.

Then $\{t \mapsto T(t)f(t)\}$ is Eberlein-weakly almost periodic.

Proof. To apply the characterization of vector valued weakly almost periodic functions from Proposition 1, let $(t_m)_{m\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and $(x_m^*)_{m\in\mathbb{N}} \subseteq B_{X^*}$ be given. We have to verify the following identity :

$$\begin{aligned} &\lim_{n\to+\infty}\lim_{m\to+\infty} < T(t_m+w_n)f(t_m+w_n), x_m^* >= \\ &= \lim_{m\to+\infty}\lim_{n\to+\infty} < T(t_m+w_n)f(t_m+w_n), x_m^* > \end{aligned}$$

whenever the iterated limits exist.

As from DUNFORD/SCHWARTZ [8], we recall, that the weak topology on weak compact subsets in separable Banach spaces is a metric topology. Nothing that continuous images of separable spaces are separable, we obtain $Y = span\{f(\mathbb{R})\}$, for $f \in W(J, X)$ is separable, hence the weak topology on $K := w - cl(f(\mathbb{R}))$ is metric on K. Thus by a use of classical diagonal process for the double sequence

$$z_{n,m} := f(t_m + w_n)$$

we may pass to subsequences $(t_{m_l})_{l\in\mathbb{N}}$ and $(w_{n_k})_{k\in\mathbb{N}}$, such that the iterated limits of $\{z_{n_k,m_l}\}_{k,l\in\mathbb{N}}$ exist in the weak topology of K. Using that we only have to prove the equality, thus, without loss of generality, we assume that

$$\lim_{n\to+\infty}\lim_{m\to+\infty} d(z_{n,m}, z_0) = \lim_{m\to+\infty}\lim_{n\to+\infty} d(z_{n,m}, z_0) = 0, \text{ for some } z_0 \in K.$$

But our hypothesis, we have $T(.)z_0 : \mathbb{R} \to X$ is Eberlein-weakly almost periodic, thus, in order to avoid subindices, we assume that the iterated limits exist for the sequence

$$\{< T(t_m + w_n)z_0, x_m^* >\}_{n,m\in\mathbb{N}}.$$

Let h be the double limit.

Claim:

$$\lim_{n\to+\infty}\lim_{m\to+\infty} < T(t_m + w_n)f(t_m + w_n), x_m^* >= h.$$

Now,

$$\begin{aligned}
&|\lim_{n\to+\infty}\lim_{m\to+\infty} < T(t_m + w_n)f(t_m + w_n), x_m^* > -h| \leq \\
&\leq |\lim_{n\to+\infty}\lim_{m\to+\infty} < T(t_m + w_n)f(t_m + w_n), x_m^* > - \\
&- < T(t_m + w_n)f(t_m + w_n), x_n^* > |+ \\
&+|\lim_{n\to+\infty}\lim_{m\to+\infty} < T(t_m + w_n)(f(t_m + w_n) - z_0), x_m^* > |+ \\
&+|\lim_{n\to+\infty}\lim_{m\to+\infty} < T(t_m + w_n)z_0, x_m^* > -h|.
\end{aligned}$$

From the convergent of

$\{< T(t_m + w_n)f(t_m + w_n), x_n^* >\}_{n,m\in\mathbb{N}}$ and

$\{< T(t_m + w_n)z_0, x^*_m >\}_{n,m\in\mathbb{N}}$,
we obtain that for every $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$, such that for $n \geq n_0$, there exists an m_n, such that

$$\lim_{n\to+\infty}\lim_{m\to+\infty} < T(t_m + w_n)f(t_m + w_n), x^*_m > -$$
$$- < T(t_m + w_n)f(t_m + w_n), x^*_m > | < \tfrac{\epsilon}{3},$$

and

$$\lim_{n\to+\infty}\lim_{m\to+\infty} < T(t_m + w_n)z_0, x^*_m > -h| < \frac{\epsilon}{3}, \text{ for all } m \geq m_n.$$

To estimate the second term. Applying the continuity of the map Υ on z_0, f or $\epsilon > 0$, there is an $\delta > 0$, such that for all $x \in K$ with $d(x, z_0) < \delta$, we have $\|T(t)x - T(t)z_0\| \leq \frac{\epsilon}{3}, \forall t \in \mathbb{R}$. Using the double limits conditions of the sequence $\{z_{n,m}\}_{n,m\in\mathbb{N}}$, for $\delta > 0$, there exists $n_1 \in \mathbb{N}$, such that for $n \geq n_1$, there exists and m^1_n, such that

$$d(z_{n,m}, z_0) < \delta, \text{ for all } m \geq m^1_n,$$

hence

$$\|T(t)z_{n,m} - T(t)z_0\| < \frac{\epsilon}{3}, \forall t \in \mathbb{R} \text{ and } m \geq m^1_n.$$

The yields, by a standard estimate, that

$$\lim_{n\to+\infty}\lim_{m\to+\infty} < T(t_m + w_n)f(t_m + w_n), x^*_m >= h.$$

Nothing that the same routine works for interchanged limit of

$$\{< T(t_m + w_n)f(t_m + w_n), x^*_m >\}_{n,m\in\mathbb{N}},$$

and the proof is complete.

The following example shows that in the general case the weak-to-norm continuity of Υ is essential, and that weak-to-weak continuity is not sufficient even if additional algebraic structure is given

Example 1. There is a bounded C_0-group $T = (T(t))_{t\in\mathbb{R}}$ on l^2 such that $T(.)x : \mathbb{R} \to X$ is 2π-periodic, for each $x \in l^2$, and for which

$$\begin{aligned} \Upsilon : X &\to W(\mathbb{R}, l^2) \\ x &\mapsto (T(.)x)(t) := T(t)x \end{aligned}$$

is linear and continuous, (i.e. weak-weak and norm-norm) but we find $f \in W(\mathbb{R}, l^2)$ such that $\{t \mapsto T(t)f(t)\}$ is not Eberlein-weakly almost periodic.

We define $T = (T(t))_{t\in\mathbb{R}}$ by

$$\begin{aligned} T(t) : l^2 &\to l^2 \\ \{x_n\}_{n\in\mathbb{N}} &\mapsto \{e^{int}x_n\}_{n\in\mathbb{N}}, \end{aligned}$$

and if 1_A denotes the indicator function of the set A, we let

$$\begin{aligned} f : \mathbb{R} &\to l^2 \\ t &\mapsto \{\sin^2(t)1_{[n\pi,(n+1)\pi)}(t)\}_{n\in\mathbb{N}}. \end{aligned}$$

Using [5, Example 1.3.15], we obtain that f is Eberlein-weakly almost periodic easily. Moreover, an easy computation gives that for fixed $x \in l^2, T(t+2\pi)x = T(t)x$. Now, for the sequences $s_n = (n+\frac{1}{2})\pi + \frac{1}{n}, t_n = (n+\frac{1}{2})\pi - \frac{1}{n}$, some calculations lead to the identity:

$$\lim_{n\to\infty} \|T(s_n)f(s_n) - T(t_n)f(t_n)\|^2 = 2\sin(1)\cos^2(\frac{1}{2}),$$

hence $\{t \mapsto T(t)f(t)\}$ is not uniformly continuous, hence not Eberlein-weakly almost periodic Proposition 2.

Corollary 1. *Let $f : \mathbb{R} \to X$ be a Eberlein-weak almost periodic function with a relatively compact range and $T = (T(t))_{t\in\mathbb{R}}$ be a bounded C_0-group of linear operators such that $T(.)x : \mathbb{R} \to X$ is Eberlein-weakly almost periodic, for each $x \in X$. Then $\{t \mapsto T(t)f(t)\}$ is Eberlein-weakly almost periodic.*

Proof. This following from the previous Lemma, since $(K, w|_K) = (K, \|\cdot\|)$ and the map

$$\begin{aligned} \Upsilon : X &\to W(\mathbb{R}, l^2) \\ x &\mapsto (T(.)x)(t) := T(t)x, \end{aligned}$$

is norm-norm continuous.

3.2. Application to the ordinary differential equation $x'(t) = Ax(t) + f(t)$. In this subsection, we show how the preceding analysis comes into play in the discution of Eberlein-weak almost periodicity properties of the solutions of the following type of differential equation:

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbb{R}, \tag{3-1}$$

where A denotes a closed linear operator.

3.2.1 The case $A = \lambda \in C$.

Theorem 1. *If A is a scalar $\lambda \in C$ and $f : \mathbb{R} \to X$ a Eberlein-weak almost periodic function, then the solution of (3-1) is given by the Eberlein-weak almost periodic function*

$$x_1(t) = -\int_t^{+\infty} e^{\lambda(t-r)} f(r)dr, \quad Re\,\lambda > 0,$$

and

$$x_2(t) = \int_t^{-\infty} e^{\lambda(t-r)} f(r)dr, \quad Re\,\lambda < 0.$$

Proof. It is obvious that both functions $x_1(.)$ and $x_2(.)$ are solution of (3-1). It remains to prove that they are Eberlein-weakly almost periodic.

Nothing that for $\mathrm{Re}\,\lambda > 0$, the mapping

$$\begin{aligned} J_\lambda : W(\mathbb{R}, X) &\to BC(\mathbb{R}, X) \\ f &\mapsto x(t) := \int_0^{+\infty} e^{\lambda(t-r)} f(r)dr \end{aligned}$$

is a bounded linear operator, commuting with the translation C_0-group, the proof is straightforward.

The proof of Eberlein-weak almost periodicity of $x_2(.)$ is analogous.

Corollary 2. Let A be a linear operator $\mathbb{R}^n \to \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}^n$ a Eberlein-weak almost periodic function. Then, every bounded solution of (3-1) is Eberlein-weakly almost periodic.

Proof. Let B be an invertible linear operator $\mathbb{R}^n \to \mathbb{R}^n$ such that $B^{-1}AB$ is triangular with the representation

$$B^{-1}AB = \begin{pmatrix} \lambda_1 & c_1 & . & . & . & c_{1n} \\ 0 & \lambda_2 & . & . & . & c_{2n} \\ 0 & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & . & . & . & . & \lambda_n \end{pmatrix}$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A.

Let $x(.)$ be a bounded solution of (3-1) and put $y(.) = B^{-1}x(.)$.

Then $y(.)$ is also bounded and it satisfies the equation:

$$y'(t) = B^{-1}ABy(t) + B^{-1}f(t), \ \ t \in \mathbb{R}$$

It is obvious that $B^{-1}f : \mathbb{R} \to \mathbb{R}^n$ is Eberlein-weakly almost periodic since B^{-1} is a bounded linear operator.

Let $y(t) = (y_i(t))_{i=1}^{i=n} \in \mathbb{R}_n$ and $B^{-1}f(t) = (h_i(t))_{i=1}^{i=n} \in \mathbb{R}^n$. Now, $y_n(.)$ is a Eberlein-weakly almost periodic solution to the equation $y_n'(t) = \lambda_n y_n(t) + h_n(t), t \in \mathbb{R}$, (Theorem 1). We can see $y_{n-1}(.)$ is also Eberlein-weakly almost periodic and proceed until $y_1(.)$. Which prove that $y(.)$ is Eberlein-weakly almost periodic and consequently $x(.) = By(.)$ is Eberlein-weakly almost periodic too.

3.2.2. The case A is the generator of a bounded C_0-group of continuous operators. We assume that $f \in C(\mathbb{R}, X)$ and A generates a C_0-group of continuous linear operators $T = (T(t))_{t\in\mathbb{R}}$ on X. Let us first recall the following definition:

Definition 3. A function $x(.) \in C(\mathbb{R}, X)$ with the integral representation

$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(s)ds$$

is called a mild solution of the differential equation (3-1).

Remark 1. Every strong solution of (3-1) is a mild solution of (3-1). Conversely, any mild solution of (3-1) which is also in $C^1(\mathbb{R}, X)$ is a strong solution of (3-1).

Before we present the next theorem, let us observe that under similar assumptions to those in Lemma 1 if equation (3-1) has a Eberlein-weak almost periodic solution $x(.)$. Then,

$$\|.\| - \lim_{T\to+\infty} \frac{1}{T} \int_r^{T+r} \int_0^t T(-s)f(s)ds = x \in X,$$

exists uniformly over $r \in \mathbb{R}$.

Indeed, we note that $T(.)x(0) : \mathbb{R} \to X$ is Eberlein-weakly almost periodic. Now, let

$$v(t) := \int_0^t T(t-s)f(s)ds, t \in \mathbb{R}.$$

Since $v(t) = x(t) - T(t)x(0)$ is Eberlein-weakly almost periodic. By Lemma 1, $t \to T(-t)v(t)$ is Eberlein-weakly almost periodic.

Now,

$$T(-t)v(t) := \int_0^t T(-s)f(s)ds, t \in \mathbb{R},$$

and since $t \to T(-t)v(t)$ is Eberlein-weakly almost periodic, by Proposition 3,

$$\|.\| - \lim_{T \to +\infty} \frac{1}{T} \int_r^{T+r} \int_0^t T(-s)f(s)ds = x \in X,$$

exists uniformly over $r \in \mathbb{R}$.

Conversely, let us proof the following theorem:

Theorem 2. *Under the assumption of Lemma 1, if we suppose*

$$\|.\| - \lim_{T \to +\infty} \frac{1}{T} \int_r^{T+r} \int_0^t T(-s)f(s)ds = x \in X, \qquad (C-1)$$

exists uniformly over $r \in \mathbb{R}$, then, every bounded mild solution of (3-1) is Eberlein-weakly almost periodic.

Proof. Let $x(t) = T(t)x(0) + \int_0^t T(t-s)f(s)ds$ be a bounded mild solution od (3-1). We note that $u(t) = T(t)x(0)$ is Eberlein-weakly almost periodic.

Let $v : \mathbb{R} \to X$ be defined by

$$v(t) := \int_0^t T(t-s)f(s)ds, \quad t \in \mathbb{R}.$$

Claim: $v(.)$ is Eberlein-weakly almost periodic.

We have $v(t) = T(t) \int_0^t T(-s)f(s)ds$, for all $t \in \mathbb{R}$. Thus by Lemma 1, the same claim holds if we prove that $\{t \to \int_0^t T(-s)f(s)ds\}$ is Eberlein-weakly almost periodic.

In order to prove that $\{t \to \int_0^t T(-s)f(s)ds\}$ is Eberlein-weakly almost periodic, by double limits criterion [13], we have to verify that for given sequences $(t_m)_{m\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and $(x_m^*)_{m\in\mathbb{N}} \subseteq B_{X^*}$.

$$\lim_{n\to+\infty} \lim_{m\to+\infty} \int_0^{t_m+w_n} < T(-s)f(s), x_m^* > ds =$$

$$= \lim_{m\to+\infty} \lim_{n\to+\infty} \int_0^{t_m+w_n} < T(-s)f(s), x_m^* > ds$$

whenever the iterated limits exist.

Assuming that the iterated limits exist and by the fact that we only have to prove the equality of them, we may pass to subsequences for the verifications. By Proposition 2, we have that every Eberlein-weak almost periodic function is uniformly continuous, hence for every $x^*_m \in B_{X^*}, x^*_m T(.)f(.)$ is uniformly continuous and the C_0-group of translations on $BC(\mathbb{R}), S = (S(t))_{t\in\mathbb{R}}$, is strongly continuous.

Since for all $m \in \mathbb{N}$,

$$\Theta(x^*_m T(.)f(.)) = S(\mathbb{R})(x^*_m T(.)f(.)),$$

$w - cl\Theta(x^*_m T(.)f(.))$ is a weakly compact subset of a closed and separable subspace of $BC(\mathbb{R})$, hence $w - cl\Theta(x^*_m T(.)f(.))$ is compact metrizable in the weak topology.

As a consequence of the metric weak compactness of $w - cl\Theta(x^*_m T(-.)f(.))$, for all $m \in \mathbb{N}$, we may pass to subsequences of $(t_m)_{m\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}}$ and $(x^*_m)_{m\in\mathbb{N}}$, such that the iterated limits of $\{x^*_m(T(-.)f(.))_{t_m+w_n}\}_{m,n\in\mathbb{N}}$ exist in the weak topology of $BC(\mathbb{R})$, and without loss of generality the sequences are chosen in this way. From the interchangeable double limits condition we obtain

$$\begin{aligned} &w - \lim_{n\to\infty} w - \lim_{m\to\infty} x^*_m(T(-.)f(.))_{t_m+w_n} = \\ &= w - \lim_{m\to\infty} w - \lim_{n\to\infty} x^*_m(T(-.)f(.))_{t_m+w_n}. \end{aligned}$$

Let $(g_n)_{n\in\mathbb{N}}, (h_m)_{m\in\mathbb{N}} \subseteq BC(\mathbb{R})$ and $g, h \in BC(\mathbb{R})$ such that

$$\begin{aligned} &w - \lim_{m\to\infty} x^*_m(T(-.)f(.))_{t_m+w_n} = g_n, \quad w - \lim_{n\to\infty} g_n = g, \\ &w - \lim_{n\to\infty} x^*_m(T(-.)f(.))_{t_m+w_n} = h_m, \quad w - \lim_{m\to\infty} h_m = h, \end{aligned}$$

from the interchangeable double limits condition above, we have $g = h$. Let $m, n \in \mathbb{N}$; then we can write

$$\begin{aligned} &\int_0^{t_m+w_n} < T(-s)f(s), x^*_m > ds = \\ &= \frac{1}{T}\int_0^T \int_0^{t+t_m+w_n} < T(-s)f(s), x^*_m > dsdt- \end{aligned}$$

$$-\frac{1}{T}\int_0^T\int_{t_m+w_n}^{t+t_m+w_n} < T(-s)f(s), x_m^* > dsdt =$$
$$= \frac{1}{T}\int_0^T\int_0^{t+t_m+w_n} < T(-s)f(s), x_m^* > dsdt-$$
$$-\frac{1}{T}\int_0^T\int_0^t x_m^*(T(-.)f(.))_{t_m+v_n} dsdt.$$

Given $\epsilon > 0$, we use condition (C-1), to choose $T_0 > 0$ such that

$$|\int_0^{t_m+w_n} < T(-s)f(s), x_m^* > ds+$$
$$+\frac{1}{T}\int_0^T\int_0^t x_m^*(T(-.)f(.))_{t_m+w_n} dsdt- < x, x_m^* > | < \epsilon$$

for all $m, n \in \mathbb{N}$. Further, since $(x_m^*)_{m\in\mathbb{N}}$ is a bounded sequence in X^*, there exist x^*, there exist $x^* \in X^*$ and a subsequence $(x_{m_l}^*)_{l\in\mathbb{N}} \subseteq (x_m^*)_{m\in\mathbb{N}}$, which is w^*-convergent to x^*. Without loss of generality we can suppose $w^* - \lim\limits_{m\to\infty} x_m^* = x^*$.

Starting with $\lim\limits_{m\to\infty}$ then $\lim\limits_{n\to\infty}$, and in the reverse order, by the last inequality, since $g = h$ we obtain that

$$\lim_{n\to+\infty}\lim_{m\to+\infty}\int_0^{t_m+w_n} < T(-s)f(s), x_m^* > ds =$$
$$= \lim_{m\to+\infty}\lim_{n\to+\infty}\int_0^{t_m+w_n} < T(-s)f(s), x_m^* > ds.$$

The proof of theorem is complete.

3.3. The equation $x'(t) = (A+B)x(t) + f(t)$. In this section, we consider the following homogeneous differential equation in Banach space X :

(1) $$x'(t) = (A+B)x(t), t \in \mathbb{R},$$

and the corresponding non-homogeneous case

(2) $$x'(t) = (A+B)x(t) + f(t), t \in \mathbb{R}.$$

We make the following assumptions:

$(H1)$ A is the infinitesimal generator of a C_0-group of bounded operators $T = (T(t))_{t\in\mathbb{R}}$ such that:

(i) $T(.)x : \mathbb{R} \to X$ is Eberlein-weakly almost periodic, for each $x \in X$, and

(ii) for every weakly compact subset K of X such that $(K, w|_K)$ is metrizable, the map

$$\begin{aligned} \Upsilon : (K, w|_K) \ &\to W(\mathbb{R}, X) \\ x \ &\mapsto (T(.)x)(t) := T(t)x \end{aligned}$$

is continuous.

$(H2)$ There exists a finite-dimensional space X_1 of X such that $D(A) \cap X_1$ is dense in X.

$(H3)$ The projection operator $P : X \to X_1$, commute with A.

$(H4)$ B is a continuous linear operator such that $B(X) = X_1$.

We will prove Eberlein-weak almost periodicity of bounded solutions of equation (1) and we will deal with bounded solution of equation (2).

Theorem 3. *Under assumptions* $(H1) - (H4)$*, every bounded solution of equation (1) is Eberlein-weakly almost periodic.*

Proof. It is well known that P is a bounded linear operator and has the property

$$X = X_1 \oplus \operatorname{Ker} P$$

where $\operatorname{Ker} P$ is the kernel of P. We set $Q = I - P$.

Now if (.) is a bounded solution of (1), then

$$x(t) = x_1(t) + x_2(t), \quad t \in \mathbb{R},$$

where $x_1(t) = Px(t) \in X_1$ and $x_2(t) = Qx(t) \in \operatorname{Ker} P$, are also bounded functions. Let us prove that $x_2(.)$ is Eberlein-weakly almost periodic.

We have,

$$x_2'(t) = \frac{d}{dt}Qx(t) = QAx(t) + QBx(t), \quad t \in \mathbb{R}.$$

Since $QBx(t) = 0$ and P and A commute, we obtain

$$x_2'(t) = Ax_2(t).$$

Thus we can write $x_2(t) = T(t)x_2(0), t \in \mathbb{R}$, which proves that $x_2(.)$ is Eberlein-weakly almost periodic.

Now if we apply P to equation (1) and use the commutativity of A and P, we get

$$x_1'(t) = (A + PB)x_1(t) + PBx_2(t), \quad t \in \mathbb{R}.$$

It is clear that $A+PB = A+B$ is a linear operator restricted to $D(A)\cap X_1$, which appears to be X_1 by assumption (H2).

Since $x_1(.)$ is bounded, then it is Eberlein-weakly almost periodic (Corollary 2.) Finally, $x(.)$ is Ebelrein-weakly almost periodic as the sum of two Eberlein-weak almost periodic functions.

The proof is complete.

We now consider the non-homogeneous case, where $f : \mathbb{R} \to X$ is Eberlein-weakly almost periodic.

Theorem 4. *Assuming that (H1)-(H4) are satisfied and the function $f : \mathbb{R} \to X$ is Eberlein-weakly almost periodic such that condition (C-1) holds. Then, every bounded solution of (2) is Eberlien-weakly almost periodic.*

Proof. Let $x(.)$ be a bounded solution of (2). Then we have the decomposition

$$x(t) = x_1(t) + x_2(t), \quad t \in \mathbb{R}$$

Observe that $x_2(.)$ is a bounded function and

$$x_2'(t) = \frac{d}{dt}Qx(t) = QAx(t) + Bx(t) + Qf(t), \quad \text{for all } t \in \mathbb{R}.$$

Since $QBx(t) = 0$ and P and A commute, we obtain

$$x_2'(t) = Ax_2(t) + Qf(t), \quad \text{for all } t \in \mathbb{R}.$$

The function $Qf(.)$ is Eberlein-weakly almost periodic since Q is a bounded linear operator. Now, the equation $x_2'(t) = Ax_2(t) + Qf(t)$ holds true in $\operatorname{Ker} P$. Hence $x_2(.)$ is Eberlein-weakly almost periodic (Theorem 2).

Let us apply P to equation (1) and use the commutativity of A and P, we get

$$x_1'(t) = (A + PB)x_1(t) + h(t), \quad t \in \mathbb{R},$$

where $h(t) = PBx_2(t)+Pf(t)$ is a Eberlein-weakly almost periodic function. The last equation holds in the finite dimensional space X_1. Its solution $x_1(.)$ is bounded X_1.

We deduce that is Eberlein-weakly almost periodic (Corollary 2.)
The proof is complete.

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University Cadi Ayyad,
Department of Mathematics (Semlalia),
B.P.S. 15, Marrakech,
Morocoo,
fatajou@yahoo.fr