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HARMONIC MAPS ON FRAMED φ-MANIFOLDS

BY

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Abstract. We study some harmonic maps between metric framed φ -manifolds. Using the almost complex and the almost contact structures induced on the framed manifolds, we study the harmonic maps between a Kähler manifold and a metric framed φ-manifold and the harmonic maps between a contact metric manifold and a metric framed φ-manifold.

1. Introduction. Let M be an m-dimensional smooth manifold endowed with a tensor field φ of type $(1,1)$, satisfying the algebraic condition

$$\varphi^3 + \varphi = 0. \tag{1.1}$$

The geometric structure on M defined by φ is called a φ-structure of rank r if the rank r of φ is constant on M and, in this case, M is called a φ-manifold.It follows easily that r is an even number.

If M is a φ-manifold and if there are $m-r$ vector fields ξ_i and $m-r$ differential 1-forms η_i satisfying

$$\varphi^2 = -I + \sum_{i=1}^{m-r} \eta_i \otimes \xi_i, \tag{1.2}$$

$$\eta_i(\xi_j) = \delta^i_j, \tag{1.3}$$

where $i, j = 1, 2, ..., m-r$, M is said to be globally framed or to have a framed φ-structure. In this case M is called a globally framed φ-manifold or, simply, a framed φ-manifold. From (1.2) and (1.3), one obtains by some algebraic computations

$$\varphi\xi_i = 0, \quad \eta_i \circ \varphi = 0, \quad \varphi^3 + \varphi = 0. \tag{1.4}$$

If $m = 2n+1$ and $rank\ \varphi = 2n$ one obtains an almost contact structure on M.

Let M be an m-dimensional globally framed φ-manifold with structure tensors (φ, ξ_i, η_i) with $rank\ \varphi = r$, and consider the manifold $M \times \mathbb{R}^{m-r}$. We denote a vector field on $M \times \mathbb{R}^{m-r}$ by $(X, \sum_{i=1}^{m-r} f_i \frac{\partial}{\partial t^i})$ where X is tangent to M, $\{t^1, ..., t^{m-r}\}$ are the coordinates on $\mathbb{R}^{m-r}$ and $\{f_1, ..., f_{m-r}\}$ are functions on $M \times \mathbb{R}^{m-r}$. Define an almost complex structure on $M \times \mathbb{R}^{m-r}$ by

$$J(X, \sum_{a=1}^{m-r} f_i \frac{\partial}{\partial t^i}) = (\varphi X - \sum_{i=1}^{m-r} f_i \xi_i, \sum_{i=1}^{m-r} \eta_i(X) \frac{\partial}{\partial t^i}).$$

It is easy to check that $J^2 = -I$. If J is integrable we say that the framed φ-structure is normal. A framed φ-structure is normal if the tensor field S of type (1,2) defined by

$$S = N_\varphi + \sum_{i=1}^{m-r} d\eta_i \otimes \xi_i, \tag{1.5}$$

vanishes, (see [3]), where

(1.6)
$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y], X, Y \in \chi(M),$$

is the Nijenhuis tensor field of φ.

If g is a (semi-)Riemannian metric on M such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^{m-r} \eta_i(X)\eta_i(Y), \tag{1.7}$$

then we say that $(\varphi, \xi_i, \eta_i, g)$ is a metric framed φ-structure and M is called a metric framed φ-manifold.

The metric g is called an associated (semi-)Riemannian metric.

The fundamental 2-form Ω of the considered metric framed φ-manifold M, is defined just like in the case of the almost Hermitian and almost contact metric manifold, by $\Omega = g(X, \varphi Y)$, for any $X, Y \in \chi(M)$.

If on an almost contact manifold (M, φ, ξ, η) it is defined an associated Riemannian metric g then $(M, \varphi, \xi, \eta, g)$ is called an almost contact metric manifold. If on an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ we have $\Omega = d\eta$, where Ω is the fundamental 2-form on M, then we say that $(M, \varphi, \xi, \eta, g)$ is a contact metric manifold.

On an m-dimensional framed φ-manifold $(M, \varphi, \xi_i, \eta_i)$, we consider the tensor field $\widetilde{\varphi}$ of type (1,1) defined by

$$\widetilde{\varphi} = \varphi + \sum_{i=1}^{\left[\frac{m-r}{2}\right]} (\eta_{2i} \otimes \xi_{2i-1} - \eta_{2i-1} \otimes \xi_{2i}). \tag{1.8}$$

One shows easily that $\widetilde{\varphi}$ is an almost complex structure on M if $m = 2n$ and an almost contact structure $(\widetilde{\varphi}, \xi_{2n-2r+1}, \eta_{2n-2r+1})$ on M if $m = 2n + 1$,(see [3]).

We have

Theorem 1.1. ([3]) *Let $(M, \varphi, \xi_i, \eta_i)$ be a $2n$-dimensional globally framed φ-manifold with rank $\varphi = r$, and $i = 1, 2, ..., 2n - r$. Assume that $(M, \varphi, \xi_i, \eta_i)$ is normal. Then the induced almost complex structure on M,*

$$\widetilde{\varphi} = \varphi + \sum_{i=1}^{\left[\frac{2n-r}{2}\right]} (\eta_{2i} \otimes \xi_{2i-1} - \eta_{2i-1} \otimes \xi_{2i}),$$

is integrable.

Concerning the harmonic maps between Riemannian manifolds, we should recall some notions and results as they are presented in [6].

Let (M, g) and (M', g') two Riemannian manifolds with dim $M = m$, and let $f : M \to M'$ be a smooth map. Define the energy density function of f, $e(f) \in C^\infty(M)$, by

$$e(f) = \frac{1}{2} tr_g(f^* g')(x) = \frac{1}{2} \sum_{i=1}^{m} (f^* g')(e_i, e_i) = \frac{1}{2} \sum_{i=1}^{m} g'(f_* e_i, f_* e_i), x \in M$$

where $\{e_1, ..., e_m\}$ is an orthonormal basis for the tangent space $T_x M$ at $x \in M$, and $f_* : T_x M \to T_{f(x)} M'$, is the tangent map of f. If M is compact

we define the energy of f by $E(f)=\int_M e(f)\nu_g$, where ν_g is the volume form of (M,g).
Then f is called a harmonic map if f is a critical point of E in $C^\infty(M,M')$.

Let $f^{-1}(TM')$ be the induced bundle from TM' over M, defined as follows. Denote by $\pi: TM'\to M'$ the projection. Then

$$f^{-1}TM'=\{(x,u)\in M\times TM', \pi(u)=f(x), x\in M\}=\bigcup_{x\in M} T_{f(x)}M'.$$

The set of all $C^\infty-$sections of $f^{-1}TM'$, denoted by $\Gamma(f^{-1}TM')$ is

$\Gamma(f^{-1}TM')=\{V: M\to TM', V$ is a $C^\infty-$map,$V(x)\in T_{f(x)}M', x\in M\}$.

Denote by ∇,∇' the Levi-Civita connections on M and M' respectively and by $\widetilde{\nabla}$ the connection induced by the map f on the bundle $f^{-1}(TM')$. Then the second fundamental form α of f is defined by $\alpha(X,Y)=\widetilde{\nabla}_X f_*Y - f_*(\nabla_X Y)$, for any $X,Y\in\chi(M)$.

The tension field $\tau(f)$ of f is defined by $\tau(f)_x=\sum\limits_{i=1}^{m}\alpha(e_i,e_i)(x)$, where $\{e_1,...,e_m\}$ is an orthonormal basis for the tangent space T_xM at $x\in M$.

One obtains that the map $f: M\to M'$ is a harmonic map if and only if $\tau(f)=0$, (see [6]).

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2. Harmonic maps on even dimensional framed φ-manifold.

Theorem 2.1. *Let $(M,\varphi,\xi_i,\eta_i,g)$ be a normal metric framed φ- manifold of dimension $2m$, where the Riemannian metric g has the fundamental 2-form Ω, and let $(M',\varphi',\xi'_j,\eta'_j,g')$ be a normal metric framed φ'-manifold of dimension $2n$, where the Riemannian metric g' has the fundamental 2-form Ω', such that*

$$d\Omega+\sum_{i=1}^{m-k} d(\eta_{2i-1}\wedge\eta_{2i})=0, \tag{2.1}$$

and

$$d\Omega'+\sum_{j=1}^{n-l} d(\eta'_{2j-1}\wedge\eta'_{2j})=0, \tag{2.2}$$

where rank $\varphi=2k$, and rank $\varphi'=2l$.

Let $f : M \to M'$ be a smooth map. Then f is a harmonic map if and only if

$$\widetilde{\varphi}' f_* = \pm f_* \widetilde{\varphi},$$

where $\widetilde{\varphi} = \varphi + \sum_{i=1}^{m-k} (\eta_{2i} \otimes \xi_{2i-1} - \eta_{2i-1} \otimes \xi_{2i})$, is the induced almost complex structure on M, and $\widetilde{\varphi}' = \varphi' + \sum_{j=1}^{n-l} (\eta'_{2j} \otimes \xi_{2j-1} - \eta'_{2j-1} \otimes \xi_{2j})$, is the induced almost complex structure on M'.

Proof. Let $X, Y \in \chi(M)$. We have

$$g(\widetilde{\varphi}X, \widetilde{\varphi}Y) = g(\varphi X, \varphi Y) + \sum_{i=1}^{2m-2k} \eta_i(X)\eta_i(Y) = g(X, Y),$$

since g is an associated metric on M. Hence g is an almost Hermitian metric on the almost complex manifold $(M, \widetilde{\varphi})$.

Let $\widetilde{\Omega}$ be the fundamental 2-form of the almost Hermitian manifold $(M, g, \widetilde{\varphi})$, defined by $\widetilde{\Omega}(X, Y) = g(X, \widetilde{\varphi}Y)$. We have

$$\widetilde{\Omega}(X, Y) = g(X, \widetilde{\varphi}Y) = g(X, \varphi Y) + \sum_{i=1}^{m-k} [\eta_{2i}(Y)\xi_{2i-1} - \eta_{2i-1}(Y)\xi_{2i}] =$$
$$= \Omega(X, Y) + \sum_{i=1}^{m-k} [\eta_{2i-1}(X)\eta_{2i}(Y) - \eta_{2i}(X)\eta_{2i-1}(Y)] =$$
$$= \Omega(X, Y) + \sum_{i=1}^{m-k} (\eta_{2i-1} \wedge \eta_{2i})(X, Y).$$

Thus $\widetilde{\Omega} = \Omega + \sum_{i=1}^{m-k} (\eta_{2i-1} \wedge \eta_{2i})$, and from (2.1) we have

$$d\widetilde{\Omega} = d\Omega + \sum_{i=1}^{m-k} d(\eta_{2i-1} \wedge \eta_{2i}) = 0,$$

that is $(M, g, \widetilde{\varphi})$ is an almost Kähler manifold. But since $(M, \varphi, \xi_i, \eta_i, g)$ is normal, it follows, from Theorem 1.1, that $N_{\widetilde{\varphi}} = 0$, where $N_{\widetilde{\varphi}}$ is the Nijenhuis tensor of $\widetilde{\varphi}$. Hence $(M, g, \widetilde{\varphi})$ is a Kähler manifold. In the same way one obtains that $(M', g', \widetilde{\varphi}')$ is a Kähler manifold and then $f : M \to M'$

is a map between two Kähler manifolds. This imply that f is a harmonic map if and only if f is a ±holomorphic map, i.e. $\widetilde{\varphi'} f_* = \pm f_* \widetilde{\varphi}$.

Remark. In ([2]) it is proved that if (M, g, J) and (M', g', J') are two Kähler manifolds then a smooth map $f : M \to M'$is a harmonic map if and only if f is a ±holomorphic map. In the same way one obtains that if g and g' are only semi-Riemannian metrics on the complex manifolds (M, J) and (M', J') respectively, which satisfies $g(JX, JY) = g(X, Y), X, Y \in \chi(M), g'(J'X', J'Y') = g'(X', Y'), X', Y' \in \chi(M')$, and the fundamental 2-forms Ω and Ω' on (M, g, J) and (M', g', J') respectively, are closed then any ±holomorphic map f between (M, g, J) and (M', g', J') is a harmonic map. Thus, if in Theorem 2.1 the metrics g and g' are only semi-Riemannian metrics then a smooth map $f : M \to M'$ which satisfies $\widetilde{\varphi'} f_* = \pm f_* \widetilde{\varphi}$ is a harmonic map.

If $(M, \varphi, \xi_i, \eta_i)$ is a framed φ-manifold we denote by D the distribution on M which is orthogonal to $span\{\xi_1, ..., \xi_{m-r}\}$, where $\dim M = m$, and $rank\ \varphi = r$, and denote by $\Gamma(D)$ the space of differentiable sections of D.

Proposition 2.2. *Let $(M, \varphi, \xi_i, \eta_i, g)$ and $(M', \varphi', \xi'_j, \eta'_j, g')$ be as in Theorem 2.1 and let $f : M \to M'$ be a smooth map. Then $\widetilde{\varphi'} f_* = \pm f_* \widetilde{\varphi}$ if and only if*

$$\pm f_* \varphi X = \varphi' f_* X + \sum_{j=1}^{n-l} [\eta'_{2j}(f_* X)\xi'_{2j-1} - \eta'_{2j-1}(f_* X)\xi'_{2j}], \tag{2.3}$$

for any $X \in \Gamma(D)$, and

$$f_* \xi_{2i-1} = W_i + \sum_{j=1}^{n-l} (a_i^{2j-1} \xi'_{2j-1} + a_i^{2j} \xi'_{2j}), \tag{2.4}$$

$$f_* \xi_{2i} = \mp \varphi' W_i + \sum_{j=1}^{n-l} (\mp a_i^{2j} \xi'_{2j-1} \pm a_i^{2j-1} \xi'_{2j}), \tag{2.5}$$

for any $i=1, 2, ..., m-k$, where $W_i \in \Gamma(D)$, and $a_i^p : M \to \mathbb{R}, i=1, ..., m-k$; $p = 1, ..., 2n - 2l$.

Proof. It is sufficient to consider the case $\widetilde{\varphi'} f_* = f_* \widetilde{\varphi}$.

First, (2.3) is equivalent to $\widetilde{\varphi}' f_* = f_* \widetilde{\varphi}$, for any $X \in \Gamma(D)$.

From (2.4) and (2.5) we have

$$\widetilde{\varphi}' \ f_* \xi_{2i-1} = \widetilde{\varphi}' W_i + \sum_{j=1}^{n-l} (a_i^{2j-1} \widetilde{\varphi}' \xi'_{2j-1} + a_i^{2j} \widetilde{\varphi}' \xi'_{2j}) =$$

$$= \varphi' W_i + \sum_{j=1}^{n-l} (-a_i^{2j-1} \xi'_{2j} + a_i^{2j} \xi'_{2j-1}) = -f_* \xi_{2i} = f_* \widetilde{\varphi} \xi_{2i-1},$$

since $\widetilde{\varphi}\xi_{2i-1} = -\xi_{2i}$ and $\widetilde{\varphi}\xi_{2i} = \xi_{2i-1}, i = 1, 2, ..., m-k$. Similarly, we have $\widetilde{\varphi}' f_* \xi_{2i} = f_* \widetilde{\varphi} \xi_{2i}$. Thus we have $\widetilde{\varphi}' f_* = f_* \widetilde{\varphi}$.

Conversely, assume that $\widetilde{\varphi}' f_* = f_* \widetilde{\varphi}$, and

$$f_* \xi_{2i-1} = W_i + \sum_{j=1}^{n-l} (a_i^{2j-1} \xi'_{2j-1} + a_i^{2j} \xi'_{2j}),$$

for any $i{=}1, 2, ..., m{-}k$, where $W_i{\in}\Gamma(D)$, and $a_i^p : M \to \mathbb{R}, i{=}1, ..., m{-}k$; $p = 1, ..., 2n - 2l$.

Then we have

$$f_* \xi_{2i} = \mp \varphi' W_i + \sum_{j=1}^{n-l} (\mp a_i^{2j} \xi'_{2j-1} \pm a_i^{2j-1} \xi'_{2j}),$$

since $\widetilde{\varphi}\xi_{2i-1} = -\xi_{2i}$, $\widetilde{\varphi}\xi_{2i} = \xi_{2i-1}, i = 1, 2, ..., m-k$, and $\widetilde{\varphi}' \xi'_{2i-1} = -\xi'_{2i}$, $\widetilde{\varphi}' \xi'_{2j} = \xi'_{2j-1}, j = 1, 2, ..., n-l$.

Remark. If dim M = dim $M' = 2m$ and $rank \ \varphi = rank \ \varphi' = 2k$, then the conditions (2.3), (2.4) and (2.5) are verified if $f : M \to M'$ is an isomorphism, i.e. $f_* \varphi = \varphi' f_*$ and $f_* \xi_i = \xi'_i, i = 1, 2, ..., 2m - 2k$. Thus if $f : M \to M'$ is an isomorphism, then the map f is harmonic.

Proposition 2.3. *Let (M, g, J) be a Kähler manifold and let $(N, \varphi, \xi_i, \eta_i, h)$ be a $2n$-dimensional normal framed φ-manifold, where g is a Riemannian metric, with the fundamental 2-form Ω satisfying*

$$d\Omega + \sum_{j=1}^{n-l} d(\eta_{2j-1} \wedge \eta_{2j}) = 0, \tag{2.6}$$

where rank $\varphi = 2l$.

Let $f : M \to N$ *be a smooth map. If*

$$f_* J = \pm \varphi f_*, \tag{2.7}$$

then f *is a harmonic map.*

Proof. Let $X \in \chi(M)$. Then, from (2.7), we have

$$-f_* X = \pm \varphi f_* J X.$$

That is $f_* X \in \Gamma(D)$, for any $X \in \chi(M)$, where D is the distribution on N which is orthogonal to $span\{\xi_1, ..., \xi_{2n-2l}\}$. Then $\varphi f_* X = \widetilde{\varphi} f_* X$, for any $X \in \chi(M)$, where $\widetilde{\varphi}$ is the induced almost complex structure on N.

Thus, from (2.7) we have $f_* J = \pm \widetilde{\varphi} f_*$, and since by definition and (2.6) $(N, g, \widetilde{\varphi})$ is a Kähler manifold it follows that f is a harmonic map.

Proposition 2.4. *Let* (M, g, J) *and* $(N, \varphi, \xi_i, \eta_i, h)$ *as in Proposition 2.3 and let* $f : N \to M$ *be a smooth map. Then* f *is a harmonic map if and only if*

$$J f_* X = \pm f_* \varphi X, \tag{2.8}$$

for any $X \in \Gamma(D)$*, where* D *is the distribution on* N *which is orthogonal to* $span\{\xi_1, ..., \xi_{2n-2l}\}$*, and*

$$f_* \xi_{2i} = \mp J f_* \xi_{2i-1}, \tag{2.9}$$

for any $i = 1, 2, ..., 2n - 2l$.

Proof. From (2.8) and (2.9) we have $J f_* = \pm f_* \widetilde{\varphi}$, where $\widetilde{\varphi}$ is the induced almost complex structure on N and since $(N, g, \widetilde{\varphi})$ is a Kähler manifold, f is a harmonic map.

Conversely, if f is a harmonic map, and since $(N, g, \widetilde{\varphi})$ is a Kähler manifold, we have

$$J f_* = \pm f_* \widetilde{\varphi}. \tag{2.10}$$

If $X \in \Gamma(D)$, then $\widetilde{\varphi} X = \varphi X$, and from (2.10) it follows (2.7). Finally, we have $\widetilde{\varphi} \xi_{2i-1} = -\xi_{2i}$, and from (2.10) one obtains

$$f_* \widetilde{\varphi} \xi_{2i-1} = -f_* \xi_{2i} = \pm J f_* \xi_{2i-1}.$$

3. Harmonic maps on odd dimensional framed φ-manifolds. In ([4]) the authors prove the following two results

Theorem 3.1. *Let M and M' be contact metric manifolds defined by (φ, ξ, η, g) and $(\varphi', \xi', \eta', g')$ respectively and let $f : M \to M'$ be a non constant map. Then we have*

1. If $f_*\varphi = \varphi' f_*$ then there exists $a \in \mathbb{R}, a \gneq 0$ such that $f_{*x}\xi_x = a\xi'_{f(x)}$ for any point $x \in M$.

2. If $f_*\varphi = -\varphi' f_*$ then there exists $a \in \mathbb{R}, a \lneq 0$ such that $f_{*x}\xi_x = a\xi'_{f(x)}$ for any point $x \in M$.

Theorem 3.2. *Let M and M' be contact metric manifolds and let $f : M \to M'$ be a smooth map such that $f_*\varphi = \pm\varphi' f_*$. Then f is a harmonic map.*

For odd dimensional framed φ-manifolds one obtains

Theorem 3.3. *Let $(M, \varphi, \xi_i, \eta_i, g)$ and $(M', \varphi', \xi'_i, \eta'_i, g')$ be two metric framed φ (and φ' respectively)-manifolds, with* dim $M = 2m + 1$ *and* dim $M' = 2n + 1$, *where g and g' are Riemannian metrics. Let Ω and Ω' be the fundamental 2-forms on M and M' respectively, such that*

$$\Omega + \sum_{i=1}^{m-k} (\eta_{2i-1} \wedge \eta_{2i}) = d\eta_{2m-2k+1}, \tag{3.1}$$

and

$$\Omega' + \sum_{j=1}^{n-l} (\eta'_{2j-1} \wedge \eta'_{2j}) = d\eta'_{2n-2l+1}, \tag{3.2}$$

where rank $\varphi = 2k$ and rank $\varphi' = 2l$.

Let $f : M \to M'$ be a smooth map such that

$$f_*\widetilde{\varphi} = \pm\widetilde{\varphi'} f_*, \tag{3.3}$$

where $(\widetilde{\varphi}, \xi_{2m-2k+1}, \eta_{2m-2k+1})$ and $(\widetilde{\varphi'}, \xi_{2n-2l+1}, \eta_{2n-2l+1})$ are the induced almost contact structures on M and M' respectively. Then f is a harmonic map.

Proof. Since g is an associated metric on M we have

$$g(\widetilde{\varphi}X,\widetilde{\varphi}Y)=g(\varphi X,\varphi Y)+\sum_{i=1}^{2m-2k}\eta_i(X)\eta_i(Y)=$$
$$=g(X,Y)-\eta_{2m-2k+1}(X)\eta_{2m-2k+1}(Y).$$

Thus g is an associated metric on the almost contact manifold $(M,\widetilde{\varphi},\xi_{2m-2k+1},\ \eta_{2m-2k+1})$.

If $\widetilde{\Omega}$ is the fundamental 2-form of the almost contact metric manifold $(M,\widetilde{\varphi},\xi_{2m-2k+1},\eta_{2m-2k+1},g)$, that is $\widetilde{\Omega}\ (X,Y)=g(X,\widetilde{\varphi}Y)$, then we have $\widetilde{\Omega}(X,Y)=g(X,\widetilde{\varphi}Y)=g(X,\varphi Y)+\sum\limits_{i=1}^{m-k}(\eta_{2i-1}\wedge\eta_{2i})(X,Y)$.

It follows that

$$\widetilde{\Omega}=\ \Omega+\sum_{i=1}^{m-k}(\eta_{2i-1}\wedge\eta_{2i}),$$

and from (3.1) we have $\widetilde{\Omega}=d\eta_{2m-2k+1}$.

That means $(M,\widetilde{\varphi},\xi_{2m-2k+1},\eta_{2m-2k+1},g)$ is a contact metric manifold.

Similarly one obtains that $(M',\widetilde{\varphi'},\xi'_{2n-2l+1},\eta'_{2n-2l+1},g')$ is a contact metric manifold.

From (3.3), using Theorem 3.2 it follows that f is a harmonic map.

In the same way as in Proposition 2.2 and using Theorem 3.1 one obtains

Proposition 3.4. *Let* $(M,\varphi,\xi_i,\eta_i,g)$ *and* $(M',\varphi',\xi'_j,\eta'_j,g')$ *be as in Theorem 3.3 and let* $f:M\to M'$ *be a smooth map. Then* $f_*\widetilde{\varphi}=\pm\widetilde{\varphi'}f_*$ *if and only if*

$$\pm f_*\varphi X=\varphi' f_*X+\sum_{j=1}^{n-l}[\eta'_{2j}(f_*X)\xi'_{2j-1}-\eta'_{2j-1}(f_*X)\xi'_{2j}], \tag{3.4}$$

where $X\in\Gamma(D)$,

$$f_*\xi_{2i-1}=W_i+\sum_{j=1}^{n-l}(a_i^{2j-1}\xi'_{2j-1}+a_i^{2j}\xi'_{2j}), \tag{3.5}$$

$$f_*\xi_{2i}=\pm\varphi' W_i+\sum_{j=1}^{n-l}(\mp a_i^{2j}\xi'_{2j-1}+a_i^{2j-1}\xi'_{2j}), \tag{3.6}$$

$$f_*\xi_{2m-2k+1} = a\xi'_{2n-2l+1}, \tag{3.7}$$

for any $i = 1, 2, ..., m-k$, *where* $W_i \in \Gamma(D')$, D' *is the distribution on* M' *orthogonal to* $span\{\xi'_1, .., \xi_{2n-2l+1}\}$, $a_i^p : M \to \mathbb{R}$, $p = 1, 2, ..., 2n-2l$ *and* $a \in \mathbb{R}$, $a \gneq 0$ *if* $f_*\widetilde{\varphi} = \widetilde{\varphi}' f_*, a \lneq 0$ *if* $f_*\widetilde{\varphi} = -\widetilde{\varphi}' f_*$.

Proposition 3.5. *Let* $(M, \varphi, \xi, \eta, g)$ *be a contact metric manifold and let* $(N, \varphi', \xi_i, \eta_i, h)$ *be a* $2n+1$*-dimensional metric framed* φ'*-manifold, where rank* $\varphi' = 2l$, *and the Riemannian metric* h *has the fundamental* 2*-form* Ω *such that*

$$\Omega + \sum_{j=1}^{n-l} (\eta_{2j-1} \wedge \eta_{2j}) = d\eta_{2n-2l+1}. \tag{3.8}$$

Let $f : M \to N$ *be a smooth map. If*

$$f_*\varphi X = \pm \varphi' f_* X, \tag{3.9}$$

for any $X \in \Gamma(D)$, *and*

$$f_*\xi = a\xi_{2n-2l+1}, \tag{3.10}$$

where $a : M \to \mathbb{R}$, *and* D *denote the distribution on* M *orthogonal to* ξ, *then* f *is a harmonic map.*

Proof. Let $(\widetilde{\varphi}', \xi_{2n-2l+1}, \eta_{2n-2l+1})$ be the almost contact structure induced on N. Then, in the same way as in Theorem 3.3 one obtains that $(N, \widetilde{\varphi}', \xi_{2n-2l+1}, \eta_{2n-2l+1}, h)$ is a contact metric manifold.

If $X \in \Gamma(D)$, from (3.9) we have

$$f_*\varphi^2 X = -f_* X = \pm \varphi' f_* \varphi X.$$

Then $f_* X \in \Gamma(D')$. That means $\varphi' f_* X = \widetilde{\varphi}' f_* X$, and then, using (3.9), we have

$$f_*\varphi X = \pm \widetilde{\varphi}' f_* X, \tag{3.11}$$

for any $X \in \Gamma(D)$.

Next, from (3.10), we have $\widetilde{\varphi}' f_* \xi = a\ \widetilde{\varphi}' \xi_{2n-2l+1} = 0$, and then

$$\widetilde{\varphi}' f_* \xi = \ f_* \varphi \xi = 0. \tag{3.12}$$

Thus, from (3.11) and (3.12) we have $f_* \varphi = \ \widetilde{\varphi}' f_*$. Hence f is a harmonic map.

Proposition 3.6. *Let M and N be as in Proposition 3.5 and let $f : N \to M$ be a smooth map. If*

$$\pm f_* \varphi' X = \varphi f_* X, \tag{3.13}$$

for any $X \in \Gamma(D')$,

$$f_* \xi_{2i} = \mp \varphi f_* \xi_{2i-1}, \tag{3.14}$$

$$f_* \xi_{2i-1} = \mp \varphi f_* \xi_{2i}, \tag{3.15}$$

for any $i = 1, 2, ..., 2n - 2l$, and

$$f_* \xi_{2n-2l+1} = a\xi, \tag{3.16}$$

where $a : N \to \mathbb{R}$, then f is a harmonic map.

Proof. From the conditions (3.13), (3.14), (3.15) and (3.16) one obtains by a straightforward computation that $\varphi f_* = f_* \widetilde{\varphi}'$, and since M and N are contact metric manifolds it follows that f is a harmonic map.

4. Harmonic maps on tangent bundle of a cosymplectic manifold. In this section we shall consider an example related to the Theorem 2.1.

Let M be a differentiable manifold of dimension $2n + 1$, and let $\pi : TM \to M$ be its tangent bundle. Then TM can be organized as a $(4n+2)$-dimensional manifold as follows. A local coordinate neighborhood $(U; x^i)$, $i=1, ..., 2n+1$, in M induces a local coordinate neighborhood $(\pi^{-1}(U); x^i, y^j)$, $i, j = 1, ..., 2n + 1$, on TM, where we denote $x^i \circ \pi$ by x^i and y^j are the coordinates of the vectors on $\pi^{-1}(U)$ in natural basis $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^{2n+1}$.

If ω is a differentiable 1-form on M then it can be regarded as a function on TM which we denote by $\iota\omega$.

If f is a function on M, we define the vertical lift f^V of f by $f^V = f \circ \pi$, and the complete lift f^C of f by $f^C = \iota(df)$. We have $f^C = y^i \frac{\partial f}{\partial x^i} = y^i \partial_i f = \partial f$ with respect to the induced coordinates in TM, where ∂_i denote $\frac{\partial}{\partial x^i}$ and ∂ denote $y^i \partial_i$. The vertical lift $(df)^V$ of the 1-form df is defined by $(df)^V = d(f)^V$. For two function f and g on M we have $(gdf)^V = g^V(df)^V = g^V(df^V)$.

Remark. The vector field $C = y^i \frac{\partial}{\partial x^i}$ is the Liouville vector field.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a vector field on M. We define the vertical lift X^V of X by $X^V(\iota\omega) = (\omega(X))^V$, ω being an arbitrary 1-form on M, and the complete lift X^C of X by $X^C f^C = (Xf)^C$, f being an arbitrary function M. One obtains with respect to the induced coordinates in TM

$$X^V = X^i \frac{\partial}{\partial y^i}, \quad X^C = X^i \frac{\partial}{\partial x^i} + \partial X^i \frac{\partial}{\partial y^i}.$$

Let $\eta = \eta_i dx^i$ be a differentiable 1-form on M. We define the vertical lift η^V of η by $\eta^V = (\eta_i)^V (dx_i)^V$, and the complete lift η^C of η by $\eta^C(X^C) = (\omega(X))^C$, X being an arbitrary vector field on M. Then, we have with respect to the induced coordinates in TM

$$\eta^V = \eta_i dx^i, \quad \eta^C = \partial \eta_i dx^i + \eta_i dy^i.$$

The vertical and the complete lifts of a tensor field on M can be defined, using the conditions

$$(P+Q)^V = P^V + Q^V, \quad (P \otimes Q)^V = P^V \otimes Q^V,$$

$$(P+Q)^C = P^C + Q^C, \quad (P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C,$$

where P, Q are tensor fields on M. Let $\varphi = \varphi_i^h \frac{\partial}{\partial x^h} \otimes dx^i$ be a tensor field of type (1,1) on M. Then one obtains, for the complete lift φ^C of φ,

$$\varphi^C = \varphi_i^h \frac{\partial}{\partial x^h} \otimes dx^i + \varphi_i^h \frac{\partial}{\partial y^h} \otimes dy^i + \partial \varphi_i^h \frac{\partial}{\partial y^h} \otimes dx^i,$$

with respect to the induced coordinates in TM.

Let $g = g_{ij} dx^i \otimes dx^j$ be a tensor field of type (0,2) on M. Then one obtains, for the complete lift g^C of g,

$$g^C = \partial g_{ij} dx^i \otimes dx^j + g_{ij} dx^i \otimes dy^j + g_{ij} dy^i \otimes dx^j$$

with respect to the induced coordinates in TM.

Let (M,φ,ξ,η) be a $(2n+1)$-dimensional almost contact manifold and let TM be its tangent bundle.

It is easy to check that

$$(\varphi^C)^2 = -I + \eta^C \otimes \xi^V + \eta^V \otimes \xi^C, \eta^C \circ \varphi^C = 0, \eta^V \circ \varphi^C = 0,$$

$$\varphi^C \xi^V = 0, \varphi^C \xi^C = 0, \eta^V(\xi^V) = 0, \eta^C(\xi^C) = 0, \eta^V(\xi^C) = 1, \eta^C(\xi^V) = 1.$$

Thus $(TM, \varphi^C, \xi^V, \xi^C, \eta^C, \eta^V)$ is a $(4n+2)$-dimensional framed φ^C- manifold.

If N_φ denote the Nijenhuis tensor of φ and N_{φ^C} the Nijenhuis tensor of φ^C we have, $N_{\varphi^C} = (N_\varphi)^C$, (see [7]).

Then we have

Proposition 4.1. *If the almost contact manifold (M,φ,ξ,η) is normal, then $(TM, \varphi^C, \xi^V, \xi^C, \eta^C, \eta^V)$ is a normal framed φ^C-manifold.*

Proof. Since (M,φ,ξ,η) is normal we have $N_\varphi + d\eta \otimes \xi = 0$. Then if we consider the complete lift one obtains $(N_\varphi + d\eta \otimes \xi)^C = 0$. That is

$$N_{\varphi^C} + d\eta^C \otimes \xi^V + d\eta^V \otimes \xi^C = 0.$$

Let $\widetilde{\varphi} = \varphi^C + \eta^V \otimes \xi^V - \eta^C \otimes \xi^C$ the almost complex structure induced on $(TM, \varphi^C, \xi^V, \xi^C, \eta^C, \eta^V)$. Then from Proposition 4.1 and Theorem 1.1 we have

Proposition 4.2. *If the almost contact manifold (M,φ,ξ,η) is normal, then $\widetilde{\varphi}$ is a complex structure on TM.*

Let (M,φ,ξ,η,g) be a normal almost contact metric manifold. We say that (M,φ,ξ,η,g) is a cosymplectic manifold if $d\eta = 0$ and $d\omega = 0$, where ω is the fundamental 2-form on M. We have

Theorem 4.3.([1]) *An almost contact metric manifold (M,φ,ξ,η,g) is cosymplectic if and only if φ is parallel.*

Let (M,φ,ξ,η,g) be a cosymplectic manifold and let TM be its tangent bundle. Then $(TM, \varphi^C, \xi^V, \xi^C, \eta^C, \eta^V)$ is a normal framed φ^C-manifold.

Next we consider $G = g^C + (\eta^V - \eta^C) \otimes (\eta^V - \eta^C)$. We can prove that G is a semi-Riemannian metric on TM.

Then by a straightforward computation, one obtains

$$G(\varphi^C X^C, \varphi^C Y^C) = G(X^C, Y^C) - \eta^V(X^C)\eta^V(Y^C) - \eta^C(X^C)\eta^C(Y^C),$$

$$G(\varphi^C X^V, \varphi^C Y^V) = G(X^V, Y^V) - \eta^V(X^V)\eta^V(Y^V) - \eta^C(X^V)\eta^C(Y^V),$$

$$G(\varphi^C X^C, \varphi^C Y^V) = G(X^C, Y^V) - \eta^V(X^C)\eta^V(Y^V) - \eta^C(X^C)\eta^C(Y^V),$$

for any $X, Y \in \chi(M)$.

It follows that $G(\varphi^C \widetilde{X}, \varphi^C \widetilde{Y}) = G(\widetilde{X}, \widetilde{Y}) - \eta^V(\widetilde{X})\eta^V(\widetilde{Y}) - \eta^C(\widetilde{X})\eta^C(\widetilde{Y})$, for any $\widetilde{X}, \widetilde{Y} \in \chi(TM)$. Then G is an associated metric on TM.

Denote by Ω the fundamental 2-form of $(TM, \varphi^C, \xi^V, \xi^C, \eta^C, \eta^V, G)$, which is defined by $\Omega(\widetilde{X}, \widetilde{Y}) = G(\widetilde{X}, \varphi^C \widetilde{Y})$, for any $\widetilde{X}, \widetilde{Y} \in \chi(TM)$, and denote by $\widetilde{\Omega}$ the fundamental 2-form of the manifold $(TM, G, \widetilde{\varphi})$, which is defined by $\widetilde{\Omega}(\widetilde{X}, \widetilde{Y}) = G(\widetilde{X}, \widetilde{\varphi}\widetilde{Y})$, for any $\widetilde{X}, \widetilde{Y} \in \chi(TM)$. Then we have $\widetilde{\Omega} = \Omega + \eta^C \wedge \eta^V$.

For Ω, we have

$$\Omega(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \varphi_j^h \partial g_{ih} + g_{ih} \partial \varphi_j^h = \partial(\varphi_j^h g_{ih}), \tag{4.1}$$

$$\Omega(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}) = \varphi_j^h g_{ih}, \tag{4.2}$$

$$\Omega(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = 0, \tag{4.3}$$

with respect to the induced coordinates in TM, where φ_j^h are the components of φ, and g_{ih} are the components of g.

From (4.1), (4.2) and (4.3) one obtains $\Omega = \omega^C$, where ω^C is the complete lift of ω.

Thus, if ω_{ij} are the components of ω, we have

$$\begin{aligned} \widetilde{\Omega} = \Omega + \eta^C \wedge \eta^V = & \sum_{1 \le i \lneq j \le 2n+1} (\partial\omega_{ij} + \eta_j \partial\eta_i - \eta_i \partial\eta_j) dx_i \wedge dx_j + \\ & + \sum_{1 \le i,j \le 2n+1} (\omega_{ij} - \eta_i \eta_j) dx_i \wedge dy_j. \end{aligned} \tag{4.4}$$

After a straightforward computation one obtains

$$
\begin{aligned}
d\widetilde{\Omega} = & \sum_{1\leq i\lneq j\lneq k\leq 2n+1} [\partial(\partial_i\omega_{jk} - \partial_j\omega_{ik} + \partial_k\omega_{ij})+ \\
& +\partial_i(\eta_k\partial\eta_j - \eta_j\partial\eta_k) - \partial_j(\eta_k\partial\eta_i - \eta_i\partial\eta_k)+ \\
& +\partial_k(\eta_j\partial\eta_i - \eta_i\partial\eta_j)]dx_i \wedge dx_j \wedge dx_k+ \\
& + \sum_{1\leq i\lneq j\leq 2n+1} \sum_{k=1}^{2n+1} [\partial_k\omega_{ij} + \partial_i(\omega_{jk} - \eta_k\eta_j)- \\
& -\partial_j(\omega_{ik} - \eta_k\eta_i) + \eta_j\partial_k\eta_i - \eta_i\partial_k\eta_j]dx_i \wedge dx_j \wedge dy_k.
\end{aligned} \tag{4.5}
$$

Then, since $(M, \varphi, \xi, \eta, g)$ is a cosymplectic manifold, we have $d\widetilde{\Omega} = 0$. We have

Proposition 4.4. *If $(M, \varphi, \xi, \eta, g)$ is a cosymplectic manifold, then $(TM, \varphi^C, \xi^V, \xi^C, \eta^C, \eta^V, G)$ is a $(4n+2)$-dimensional normal metric framed φ^C-manifold with the fundamental 2-form Ω satisfying*

$$d\Omega + d(\eta^C \wedge \eta^V) = 0.$$

Remark. From (4.5) one obtains that $d\widetilde{\Omega}=0$ if and only if $(M, \varphi, \xi, \eta, g)$ is a cosymplectic manifold.

Let $f : M \to N$ be a smooth map between two cosymplectic manifolds $(M, \varphi, \xi, \eta, g)$ and $(N, \varphi', \xi', \eta', g')$. Denote by $\widetilde{\varphi}, \widetilde{\varphi'}$ the induced almost complex structures on TM and TN respectively. Let D and D' be the distributions on M and N, respectively, orthogonal to ξ and ξ', respectively.

Denote by $F = f_* : TM \to TN$ the tangent map induced by f.

Then, from Remark at Theorem 2.1 and from Proposition 4.4, it follows that if $F_*\widetilde{\varphi} = \pm\widetilde{\varphi'}F_*$ then F is a harmonic map.

Next suppose that $F_*\widetilde{\varphi} = \widetilde{\varphi'}F_*$ or $F_*\widetilde{\varphi} = -\widetilde{\varphi'}F_*$. It is sufficient to consider the case $F_*\widetilde{\varphi} = \widetilde{\varphi'}F_*$. For a vector field $X\in\Gamma(D)$, we have $F_*\widetilde{\varphi}X^V = \widetilde{\varphi'}F_*X^V$.

But $F_*\widetilde{\varphi}X^V = F_*(\varphi X)^V = (f_*\varphi X)^V$, (see [7]), and $\widetilde{\varphi'}F_*X^V = \widetilde{\varphi'}(f_*X)^V = \varphi'^C(f_*X)^V - \eta'^C((f_*X)^V)\xi'^C$.

Then $(f_*\varphi X)^V = (\varphi' f_*X)^V - \eta'^C((f_*X)^V)\xi'^C$. Computing with η'^V one obtains $\eta'^C((f_*X)^V) = 0$. It follows that $(f_*\varphi X)^V = (\varphi' f_*X)^V$, and then

$$f_*\varphi X = \varphi' f_*X, \tag{4.6}$$

for any $X \in \Gamma(D)$.

Next, since $F_*\widetilde{\varphi}\xi^C = \widetilde{\varphi}' F_*\xi^C, F_*\widetilde{\varphi}\xi^C = F_*\xi^V = (f_*\xi)^V$ and

$$\widetilde{\varphi}' F_*\xi^C = (\varphi' f_*\xi)^C + (\eta'(f_*\xi))^V \xi'^V - (\eta'(f_*\xi))^C \xi'^C$$

we have

$$(f_*\xi)^V = (\varphi' f_*\xi)^C + (\eta'(f_*\xi))^V \xi'^V - (\eta'(f_*\xi))^C \xi'^C. \tag{4.7}$$

Computing with η'^V one obtains $(\eta'(f_*\xi))^C = 0$, that is $\eta'(f_*\xi) = a$, where $a \in \mathbb{R}$, is a constant.

Next suppose that $\varphi' f_*\xi$ does not vanishes identically. Then there exists $W \in \Gamma(D')$ such that $f_*\xi = W + a\xi'$. From (4.7) we have $W^V = \varphi'^C W^C$, and using the local coordinates one obtains that W vanishes identically. Thus

$$f_*\xi = a\xi', \tag{4.8}$$

where $a \in \mathbb{R}$.

Conversely, if we suppose that f verify (4.6) and (4.8) it is easy to see that $F_*\widetilde{\varphi} = \widetilde{\varphi}' F_*$.

Remark. In the case $F_*\widetilde{\varphi} = -\widetilde{\varphi}' F_*$ one obtains

$$f_*\varphi X = -\varphi' f_* X, \tag{4.9}$$

for any $X \in \Gamma(D)$ and $f_*\xi = a\xi'$, where $a \in \mathbb{R}$.

We have

Proposition 4.5. *Let $f : M \to N$ and $F : TM \to TN$ be as above. If $f_*\varphi = \pm\varphi' f_*$ and $f_*\xi = a\xi'$, $a \in \mathbb{R}$, then F is a harmonic map. Moreover $F_*\widetilde{\varphi} = \pm\widetilde{\varphi}' F_*$ if and only if $f_*\varphi = \pm\varphi' f_*$ and $f_*\xi = a\xi'$, $a \in \mathbb{R}$.*

Proposition 4.6. *Let $f : M \to N$ and $F : TM \to TN$ be as above. If $F_*\widetilde{\varphi} = \pm\widetilde{\varphi}' F_*$ then f is a harmonic map.*

Proof. Denote by ∇, ∇' the Levi-Civita connections on M and N respectively and by $\widetilde{\nabla}$ the connection induced by the map f on the bundle $f^{-1}(TN)$. Then since $(M, \varphi, \xi, \eta, g)$ is cosymplectic, we have $\nabla\varphi = 0$. From this, for any $X \in \Gamma(D)$ we have $(\nabla_X\varphi)\varphi X = (\nabla_{\varphi X}\varphi)X = 0$. Thus

$$\nabla_X X + \nabla_{\varphi X}\varphi X = \varphi\left[\varphi X, X\right],$$

for any $X \in \Gamma(D)$.

Similarly, on N we have $\nabla'_{X'}X' + \nabla'_{\varphi'X'}\varphi'X' = \varphi'[\varphi'X', X']$, for any $X' \in \Gamma(D')$.

Let α be the second fundamental form of f, defined by

$$\alpha(X, Y) = \widetilde{\nabla}_X f_* Y - f_*(\nabla_X Y),$$

for any $X, Y \in \chi(M)$.

Then, if $\{e_1, ..., e_n, \varphi e_1, ..., \varphi e_n, \xi\}$ is a local orthonormal φ-basis on M,where $\dim M = 2n + 1$, we have $\alpha(e_j, e_j) + \alpha(\varphi e_j, \varphi e_j) = 0$, since $f_*\varphi = \pm\varphi' f_*$.

On the other hand, from $f_*\xi = a\xi'$, $a \in \mathbb{R}$, and $\nabla_\xi \xi = 0$, $\nabla_{\xi'}\xi' = 0$, we have $\alpha(\xi, \xi) = 0$.

Then

$$\tau(f) = \alpha(e_j, e_j) + \alpha(\varphi e_j, \varphi e_j) + \alpha(\xi, \xi) = 0,$$

where $\tau(f)$ is the tension field of f. So, f is a harmonic map.

Let ∇^G be the Levi-Civita connection on $(TM, \varphi^C, \xi^V, \xi^C, \eta^C, \eta^V, G)$. Let ∇^C the Levi-Civita connection of g^C. Then, we have $\nabla^C \eta^V = (\nabla\eta)^V = 0$, $\nabla^C \eta^C = (\nabla\eta)^C = 0$, (see [7]), since $(M, \varphi, \xi, \eta, g)$ is cosymplectic. Thus, by definition of ∇^C one obtains $\nabla^C G = 0$. That is $\nabla^G = \nabla^C$.

Using $\nabla^C \varphi^C = (\nabla\varphi)^C$, (see [7]), and since $(M, \varphi, \xi, \eta, g)$ is cosymplectic we have $\nabla^C \varphi^C = 0$.

From all this one obtains, for a local φ^C-basis $\{e_1, ..., e_{2n}, \varphi^C e_1, ..., \varphi^C e_{2n}, \xi^V, \xi^C\}$ in TM

$$\nabla^G_{e_i} e_i + \nabla^G_{\varphi^C e_i} \varphi^C e_i = \varphi^C[\varphi^C e_i, e_i], \tag{4.10}$$

and

$$\nabla^G_{\xi^V} \xi^V = 0, \nabla^G_{\xi^C} \xi^C = 0, \tag{4.11}$$

since $\nabla^C_{\xi^V} \xi^V = 0$ and $\nabla^C_{\xi^C} \xi^C = 0$, (see [7]).

We have

Proposition 4.7. *Let $\pi : TM \to M$ be the projection map. Then π is a harmonic map.*

Proof. Let $f : M \to \mathbb{R}$ a smooth function on M. Then, for $X \in \chi(M)$ we have $(\pi_* X^C)f = X^C(f \circ \pi) = X^C f^V = (Xf)^V = Xf$, and $(\pi_* X^V)f = X^V f^V = 0$, (see [7]).

That is $\pi_* X^C = X$, and $\pi_* X^V = 0$. It follows that $\pi_* \varphi^C = \varphi\ \pi_*$, and $\pi_* \xi^V = 0, \pi_* \xi^C = \xi$.

Then, for a local orthonormal φ^C-basis $\{e_1, ..., e_{2n}, \varphi^C e_1, ..., \varphi^C e_{2n}, \xi^V, \xi^C\}$ in TM, from (4.10) and (4.11), we have $\alpha(e_j, e_j) + \alpha(\varphi^C e_j, \varphi^C e_j) = 0$, and $\alpha(\xi^V, \xi^V) = 0$, $\alpha(\xi^C, \xi^C) = 0$, where α is the second fundamental form of π. Thus $\tau(\pi) = 0$.

Remark. From (4.10) and (4.11), one obtains that any smooth map $f : TM \to M$, which satisfies $f_* \varphi^C = \varphi\ f_*$, and $f_* \xi^V = a\xi$, $f_* \xi^C = b\xi$, where $a, b \in \mathbb{R}$, is a harmonic map.

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