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SLANT SUBMANIFOLDS OF COSYMPLECTIC MANIFOLDS

BY

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Abstract. In this paper we have obtained some results on slant submanifolds of a cosymplectic manifold. We have given a necessary and sufficient condition for a 3-dimensional submanifold of a 5-dimensional cosymplectic manifold to be a minimal proper slant submanifold. In section 4, we have studied an inequality similar to B.Y. Chen's inequality for a submanifold of a cosymplectic manifold.

1. Preliminaries. Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors (φ, ξ, η, g), where φ is a (1,1) tensor field, ξ a vector field, η a 1-form and g is the Riemannian metric on $\overline{M}$. These tensors satisfy [16].

$$(1.1)\qquad \begin{cases} \varphi^2 X = -X + \eta(X)\xi, \;\; \varphi\xi = 0, \eta(\xi) = 1, \quad \eta(\varphi X) = 0 \\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad \eta(X) = g(X,\xi) \end{cases}$$

for any $X, Y \in T\overline{M}$, where $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$. A normal almost contact metric manifold is called a cosymplectic manifold if [1]

$$(1.2)\qquad (\overline{\nabla}_X \varphi)(Y) = 0, \quad \overline{\nabla}_X \xi = 0$$

where $\overline{\nabla}$ denotes the Levi-Civita connection of $(\overline{M}, g)$.

Let M be an m-dimensional Riemannian manifold with induced metric g isometrically immersed in $\overline{M}$. We denote by TM the Lie algebra of vector fields on M and by $T^\perp M$ the set of all vector fields normal to M.

For any $X \in TM$ and $N \in T^\perp M$, we write

$$\varphi X = PX + FX \text{ and } \varphi N = tN + fN, \tag{1.3}$$

where PX (resp. FX) denotes the tangential (resp. normal) component of φX, and tN (resp. fN) denotes the tangential (resp. normal) component of φN.

From now on, we suppose that the structure vector field ξ is tangent to M. Hence, if we denote by D the orthogonal distribution to ξ in TM, we can consider the orthogonal direct decomposition $TM = D \oplus \{\xi\}$.

For each non zero X tangent to M at x, such that it is not proportional to ξ_x, we denote by $\theta(X)$ the Wirtinger angle of X, that is, the angle between φX and T_xM.

The submanifold M is called slant if the Wirtinger angle $\theta(X)$ is a constant, which is independent of the choice $x \in M$ and $X \in T_xM - \{\xi_x\}$[3]. The Wirtinger angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant angle θ equal to 0 and $\pi/2$, respectively. A slant immersion, which is neither invariant nor anti-invariant, is called a proper slant immersion.

Let ∇ be the Riemannian connection on M. Then the Gauss and Weingarten formulae are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1.4}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla^\perp_X N \tag{1.5}$$

for $X, Y \in TM$ and $N \in T^\perp M; h$ and A_N being the second fundamental forms related by

$$g(A_N X, Y) = g(h(X, Y), N) \tag{1.6}$$

and $\nabla^\perp$ being the connection in the normal bundle $T^\perp M$ of M.

The mean curvature vector H is defined by $H = (1/m)$ trace h. Then M is said to be minimal if H vanishes identically.

If P is the endomorphism defined by (1.3), then

$$g(PX,Y)+g(X,PY)=0 \tag{1.7}$$

Thus P^2, which is simply denoted by Q, is self adjoint.

We define the covariant derivatives of Q, P and F by

$$(\nabla_X Q)Y=\nabla_X(QY)-Q(\nabla_X Y) \tag{1.8}$$

$$(\nabla_X P)Y=\nabla_X(PY)-P(\nabla_X Y) \tag{1.9}$$

and

$$(\nabla_X F)Y=\nabla^{\perp}_X FY-F(\nabla_X Y) \tag{1.10}$$

for any, $X, Y \in TM$.

For proper slant submanifolds of a cosymplectic manifold it can be proved by a direct calculation that

$$(\nabla_X P)Y=0, \tag{1.11}$$

for any $X, Y \in TM$. Moreover, (1.11) shows that

$$(\nabla_X Q)Y=0. \tag{1.12}$$

On the other hand, Gauss and Weingarten formulae together with (1.2) and (1.3) imply

$$(\nabla_X P)Y=A_{FY}X+th(X,Y), \tag{1.13}$$

$$(\nabla_X F)Y=fh(X,Y)-h(X,PY), \tag{1.14}$$

for any $X, Y \in TM$. It is easy to see that (1.11) holds if and only if

$$A_{FY}X=A_{FX}Y \tag{1.15}$$

where we have used (1.13). Using (1.14), a similar calculation shows that

$$(\nabla_X F)Y=0 \text{ if and only if } A_N PY=-A_{fN}Y, \tag{1.16}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$.

2. Slant submanifolds of cosymplectic manifolds. In the present section, we shall prove a characterization theorem for slant submanifolds of a cosymplectic manifold. We mention the following results for latter use.

Theorem A. *[19] Let M be a submanifold of an almost contact metric manifold $\overline{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = -\lambda(I - \eta \otimes \xi). \tag{2.1}$$

Further more, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Corollary A. *[19] Let M be a submanifold of an almost contact metric manifold $\overline{M}$, with slant angle θ. Then, for any $X, Y \in TM$, we have*

$$g(PX, PY) = \cos^2\theta(g(X,Y) - \eta(X)\eta(Y)) \tag{2.2}$$

$$g(FX, FY) = \sin^2\theta(g(X,Y) - \eta(X)\eta(Y)). \tag{2.3}$$

Lemma A. *[3] Let M be a submanifold of an almost contact metric manifold $\overline{M}$, with slant angle θ. Then, at each point x of $M, Q/_D$ has only one eigenvalue $\lambda_1 = -\cos^2\theta$.*

We now prove:

Theorem 2.1. *Let M be a slant submanifold of a cosymplectic manifold $\overline{M}$. Then, $\nabla Q = 0$.*

Proof. Denote by θ the slant angle of M. Then (2.1) implies

$$Q(\nabla_X Y) = -\cos^2\theta(\nabla_X Y) + \cos^2\theta\eta(\nabla_X Y)\xi \tag{2.4}$$

for any $X, Y \in TM$. On the other hand, by taking the covariant derivative of (2.1), we get

$$\begin{aligned} \nabla_X QY &= -\cos^2\theta(\nabla_X Y) + \cos^2\theta\eta(\nabla_X Y)\xi + \\ &\quad + \cos^2\theta g(Y, \nabla_X\xi)\xi + \cos^2\theta\eta(Y)\nabla_X\xi. \end{aligned} \tag{2.5}$$

Now, since M is a submanifold of a cosymplectic manifold, hence from (1.2), we have $\nabla_X \xi = 0$ for any $X \in TM$. Putting the value of $\nabla_X \xi$ in (2.5), we get

$$\nabla_X QY = -\cos^2\theta(\nabla_X Y) + \cos^2\theta\eta(\nabla_X Y)\xi. \tag{2.6}$$

Combining (2.4) and (2.6), we find

$$(\nabla_X Q)Y = 0, \tag{2.7}$$

for any $X, Y \in TM$ where by proving our assertion.

Now, we state the main result of this section.

Theorem 2.2. *Let M be a submanifold of a cosymplectic manifold $\overline{M}$. Then, M is slant if and only if*

1. *The endomorphism $Q/_D$ has only one eigenvalue at each point of M.*
2. *There exists a function $\lambda : M \to [0, 1]$ such that*

$$(\nabla_X Q)Y = 0$$

for any $X, Y \in TM$.

Moreover, in this case, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Proof. Statements 1 and 2 follow directly from lemma A and equation (2.7) respectively. Conversely, let $\lambda_1(x)$ be the eigenvalue of $Q|_D$ at each point x of M and $Y \in D$ be an unit eigenvector associated with λ_1, i.e. $QY = \lambda_1 Y$. Then, by virtue of statement 2, we have

$$X(\lambda_1)Y + \lambda_1\nabla_X Y = \nabla_X(QY) = Q(\nabla_X Y)$$

for any $X \in TM$. Since both $\nabla_X Y$ and $Q(\nabla_X Y)$ are perpendicular to Y, we conclude that λ_1 is constant on M.

To prove that M is slant in view of (2.1), it is enough to show that there exists a constant μ such that $Q = -\mu(I - \eta \otimes \xi)$. Let X be in TM. Then $X = \widetilde{X} + \eta(X)\xi$, where $\widetilde{X} = X - \eta(X)\xi \in D$. Hence, $QX = Q\widetilde{X}$. Since $Q|_D = \lambda_1 I$, we have, $Q\widetilde{X} = \lambda_1\widetilde{X}$ and so $QX = \lambda_1\widetilde{X} = \lambda_1(X - \eta(X)\xi)$. By taking $\mu = -\lambda_1$, we obtain the result. Moreover, if M is slant, then by (2.7) $\lambda = -\lambda_1 = \mu = \cos^2\theta$, where θ denotes the slant angle of M.

The following corollary directly follows form the above result.

Corollary 2.1. *Let M be a 3-dimensional submanifold of a cosymplectic manifold $\overline{M}$. Then, M is slant if and only if*

$$(\nabla_X Q)Y = 0$$

for any $X, Y \in TM$.

3. Slant submanifolds with dimension three. In this section we shall study three dimensional slant submanifolds of a cosymplectic manifold. To characterize these submanifolds we need the following:

Lemma B. *[4] Let M be a 3-dimensional slant submanifold of an almost contact metric manifold $\overline{M}$ with slant angle θ. Suppose that M is not anti-invariant. If $p \in M$, then in a neighbourhood of p there exist vector fields e_1, e_2 tangent to M such that basis $\{e_1, e_2, \xi\}$ forms a local orthonormal frame satisfying*

$$Pe_1 = \cos\theta e_2, \quad Pe_2 = -\cos\theta e_1$$

Now, we prove the characterization theorem for a three-dimensional slant submanifold of a cosymplectic manifold.

Theorem 3.1. *Let M be a 3-dimensional submanifold of a cosymplectic manifold $\overline{M}$. Then, the following statements are equivalent:*

(i) M is neither an invariant nor an anti-invariant submanifold and

$$(\nabla_X P)Y = 0 \tag{3.1}$$

for any $X, Y \in TM$.

(ii) M is proper slant.

Proof. By virtue of corollary 2.1 and the relations between (1.11) and (1.12), it follows that statements (i) implies (ii). Now we show that statement (ii) implies (i). For this, let $p \in M$ and $\{e_1, e_2\}$ be the orthonormal frame on M defined in a neighbourhood U of p given by lemma B. Put $\xi|_U = e_3$, and let ω_i^j be the structural 1-forms defined by

$$\nabla_X e_i = \sum_{j=1}^{3} \omega_i^j(X) e_j$$

for each vector field X tangent to M.

Notice that by virtue of (1.2), we have

$$(\nabla_X P)e_3 = \nabla_X Pe_3 - P(\nabla_X e_3) = 0.$$

Similarly, we get

$$(\nabla_X P)e_1 = \cos\theta\omega_2^3(X)e_3$$

and

$$(\nabla_X P)e_2 = -\cos\theta\omega_1^3(X)e_3.$$

On the other hand, writing

$$Y = \eta(Y)e_3 + g(Y, e_1)e_1 + g(Y, e_2)e_2,$$

for all $Y \in TM$ and using the above formulae it follows that

$$(\nabla_X P)Y = 0,$$

where we have used $\omega_2^3(X) = \omega_1^3(X) = 0$, for submanifolds of a cosymplectic manifold. Which proves statement (i).

From theorem 3.1 and the equivalence between (1.11) and (1.15), we obtain the following characterization for slant submanifolds of dimension 3 in terms of the Weingarten map.

Corollary 3.1. *Let M be a submanifold of dimension 3 of a cosymplectic manifold $\overline{M}$. Then, M is slant if and only if*

$$A_{FY}X = A_{FX}Y$$

for any $X, Y \in TM$.

If M is an invariant submanifold of a cosymplectic manifold $\overline{M}$, then (3.1) also holds and $\nabla F = 0$ is automatically satisfied. On the other hand, if M is an anti-invariant submanifold, it is obvious that $\nabla P = 0$, i.e. (3.1) holds. We also know that

$$(\nabla_X F)Y = 0, \quad \text{for any } X, Y \in TM.$$

Now, we shall calculate the value of ∇F for a three-dimensional slant submanifold M of a cosymplectic manifold $\overline{M}$ with $dim\overline{M} = 5$.

Suppose that M is proper slant with slant angle θ. Then, for a unit tangent vector field e_1 of M perpendicular to ξ, we put

$$e_2 = (sec\,\theta)Pe_1, \;\; e_3 = \xi, \;\; e_4 = (csc\,\theta)Fe_1, \;\; e_5 = (csc\,\theta)Fe_2.$$

Then $e_1 = -(sec\;\theta)Pe_2$ and by virtue of (2.2) and (2.3), e_1, e_2, e_3, e_4, e_5 form an orthonormal frame such that e_1, e_2, e_3 are tangent to M and e_4, e_5 are normal to M. We call such an orthonormal frame an adapted slant frame. We also have:

$$te_4 = -\sin\theta e_1, \;\; te_5 = -\sin\theta e_2, \;\; fe_4 = -\cos\theta e_5, \;\; fe_5 = \cos\theta e_4.$$

If we put $h^r_{ij} = g(h(e_i, e_j), e_r), i, j = 1, 2, 3, r = 4, 5$, then we have the following:

Lemma 3.1. *Under the above conditions, we have*

$$h^4_{12} = h^5_{11}, \quad h^4_{22} = h^5_{12} \tag{3.2}$$

$$h^4_{13} = h^4_{32} = h^4_{33} = h^5_{13} = h^5_{23} = h^5_{33} = 0. \tag{3.3}$$

Proof. We obtain (3.2) by virtue of corollary 3.1, while (3.3) holds because $\overline{M}$ is a cosymplectic manifold.

Theorem 3.2. *Let M be a submanifold of dimension 3 of a cosymplectic manifold $\overline{M}$ of dimension 5. Then M is a minimal proper slant submanifold of $\overline{M}$ if and only if*

$$(\nabla_X F)Y = 0 \tag{3.4}$$

for any $X, Y \in TM$.

Proof. To prove (3.4), we choose e_1, e_2, e_3, e_4, e_5 as an adapted slant frame. Then (3.4) follows from (1.15), (3.2) and (3.3). To see the converse, we choose an unit local vector field e_1 perpendicular to ξ, such that

$$P^2 e_1 = -\cos^2\theta_1 e_1,$$

where $\theta_1 = \theta(e_1) \in (0, \pi/2)$ denotes the Wirtinger angle of e_1. Define a local orthonormal frame formed by e_1, e_2, e_3, e_4, e_5 such that

$$e_2 = (sec\,\theta_1)Pe_1, \;\; e_3 = \xi, \;\; e_4 = (csc\,\theta_1)Fe_1, \;\; e_5 = (csc\,\theta_1)Fe_2$$

and

$$te_4 = -\sin\theta_1 e_1, \;\; te_5 = -\sin\theta_1 e_2, \;\; fe_4 = -\cos\theta_1 e_5, fe_5 = \cos\theta_1 e_4.$$

Since $(\nabla_X F)Y = -\eta(Y)FX$, from (1.16), we know that

$$A_N PY = -A_{fN}Y$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$. Therefore, we can show that

$$A_{Fe_1}e_2 = A_{Fe_2}e_1 \text{ and } A_{Fe_1}e_3 = A_{Fe_2}e_3 = 0.$$

It can be obtained by a simple calculation that

$$A_{FX}Y = A_{FY}X,$$

for any, $X, Y \in TM$. From corollary 3.1, we can say that M is a proper slant submanifold. It is easy to show that M is also minimal.

4. B.Y. Chen's inequality and its application to slant immersions into cosymplectic manifolds. In the study of submanifold theory, one of the important problems is to find a relationship between the main extrinsic invariants and the main intrinsic invariants of a submanifold. Recently, B.Y. Chen ([6], [9]) has given a best possible inequality between the sectional curvature function K and the scalar curvature function τ (intrinsic invariants) and the mean curvature function $|H|$ (extrinsic invariants) of a submanifold M of a real space form of constant curvature c. For that purpose, he somewhat non-standardly defined the scalar curvature τ and Riemannian invariant δ_M as follows:

Let M be an m-dimensional Riemannian manifold and $\{e_1, \cdots, e_m\}$ be an orthonormal basis of the tangent space T_xM. The scalar curvature τ at x is defined by

$$\tau = \sum_{1 \le i < j \le m} K(e_i \wedge e_j). \tag{4.1}$$

For each point $x \in M$, we put

$$(\inf K)(x) = \inf\{K(\Pi) : \Pi \subset T_x M, \dim \Pi = 2\},$$

where $K(\Pi)$ denotes the sectional curvature of M associated with Π. Obviously, $\inf K$ is a well defined function on M. Let δ_M denote the difference between the scalar curvature and $\inf K$ i.e.

$$\delta_M(x) = \tau(x) - \inf K(x). \tag{4.2}$$

Then δ_M is a well defined Riemannian invariant which is trivial when $m = 2$. For an m-dimensional submanifold M to be a real form $\overline{M}(c)$, Chen gave the following basic inequality involving the intrinsic invariant δ_M and the squared mean curvature of the immersion:

$$\delta_M \leq \frac{m^2(m-2)}{2(m-1)}|H|^2 + \frac{1}{2}(m+1)(m-2)c. \tag{4.3}$$

It was remarked in [10] that the above inequality is also true for anti-invariant submanifolds in complex space form $\overline{M}(4c)$. Later, Chen generalized the above situation by establishing an inequality for an arbitrary submanifold of dimension greater than 2 in a complex space form [5]. By applying this inequality, he showed that (4.3) holds for arbitrary submanifolds in the complex hyperbolic space $H^m(4c)(c < 0)$ as well.

In contact geometry, Defever, Miahi and Verstraelen obtained an inequality similar to (4.3) for C-totally real submanifolds of a Sasakian space form with constant φ-sectional cuvature c as [17]:

$$\delta_M \leq \frac{m^2(m-2)}{2(m-1)}|H|^2 - \frac{1}{2}(m+1)(m-2)\frac{c+3}{4}. \tag{4.4}$$

To obtain an inequality similar to (4.3) which can be used to characterized slant submanifolds into a cosymplectic manifold, we need the following definitions and results. If $\overline{M}$ is a cosymplectic manifold, then

$$h(X, \xi) = 0. \tag{4.5}$$

Given a local orthonormal frame $\{e_1, \cdots, e_m\}$ of D, we can define the squared norms of P and F by

$$|P|^2 = \sum_{i,j=1}^{m} g^2(e_i, Pe_j), \quad |F|^2 = \sum_{i=1}^{m} |Fe_i|^2, \textit{ respectively.} \tag{4.6}$$

Both $|P|^2$ and $|F|^2$ are independent of the choice of the above orthonormal frame. From lemma 2.3.8 of [2] we have

$$\sum_{i,j=1}^{m} g^2(e_i, \varphi e_j) = \cos^2\theta \tag{4.7}$$

for any $i = 1, \cdots, m$ where $\{e_1, \cdots, e_m, \xi\}$ is a local orthonormal frame of TM.

A φ-section of a cosymplectic manifold $\overline{M}$ is a plane section Π in the tangent space T_xM at x which is spaned by a vector X orthogonal to ξ and φX. The sectional curvature $K(\Pi)$ with respect to a φ-section Π is defined by

$$K(\Pi) = K(X, \varphi X) = g(\overline{R}(X, \varphi X)\varphi X, X)$$

The sectional curvature $K(\Pi)$ is called a φ-holomorphic sectional curvature. Similar to the case of a Kaehler manifold, it is verified that if a Sasakian manifold has a φ-holomorphic sectional curvature c which does not depend on the φ-section at each point, then c is a constant. Such a cosymplectic manifold with constant φ-holomorphic sectional curvature c is called a cosymplectic space form $\overline{M}(c)$. The expression for the curvature tensor $\overline{R}$ is given by

(4.8)

$$\begin{aligned}\overline{R}(X,Y)Z \; &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \\ &+\eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi - g(\varphi X, Z)\varphi Y + \\ &+g(\varphi Y, Z)\varphi X + 2g(X, \varphi Y)\varphi Z\}.\end{aligned}$$

For an orthonormal basis $\{e_1 \cdots e_m, e_{m+1}\}$ of the tangent space T_xM at $x \in M$, if we put $e_{m+1} = \xi_x$. The scalar curvature τ at x of M, takes the following form

$$2\tau = \sum_{i\neq j}^{m} K(e_i \wedge e_j) + 2\sum_{i=1}^{m} K(e_i \wedge \xi). \tag{4.9}$$

From (4.6), (4.8) and (4.9), we obtain the following relation between the scalar curvature and the mean curvature vector of M.

$$2\tau = (m+1)^2|H|^2 - |h|^2 + \frac{c}{4}m(m-1) + \frac{3c}{4}|P|^2. \tag{4.10}$$

The following lemma is very useful in established Chen's inequality.

Lemma C. *[9] Let $a_1, \cdots, a_k, c$ be $k+1 (k \geq 2)$ real numbers such that*

$$(\sum_{i=1}^{k} a_i)^2 = (k-1)(\sum_{i=1}^{k} a_i^2 + C)$$

Then $2a_1a_2 \geq c$, and the equality holds if and only if $a_1+a_2 = a_3 = \cdots = a_k$.

Now, we have the following:

Theorem 4.1. *Let $\psi : M^{m+1} \to \overline{M}^{2m+1}$ be an isometric immersion from a Riemannian $(m+1)$-dimensional manifold M into a cosymplectic space form $\overline{M}^{2m+1}(c)$ of constant curvature such that $\xi \in TM$. Then, for any point $x \in M$ and any plane section $\Pi \subset D_x$, we have*

$$\tau - K(\Pi) = \frac{(m+1)^2(m-1)}{2m}|H|^2 + \frac{1}{2}(m+1)(m-2)\frac{c}{4} + \frac{3}{2}|P|^2\frac{c}{4} - 3g^2(e_1, \varphi e_2)\frac{c}{4}. \tag{4.13}$$

Equality in (4.13) holds at $x \in M$ iff there exists an orthonormal basis $\{e_1, \cdots, e_m, e_{m+1}\}$ of T_xM and an orthonormal basis $\{e_{m+2}, \cdots, e_{2m+1}\}$ of $T_x^{\perp}$ such that (a)$e_{m+1} = \xi_x$, (b) Π is spanned by e_1, e_2 and (c) the shape operators take the following forms:

$$A_{m+2} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0_{m-1} \end{pmatrix} \tag{4.14}$$

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 \\ h_{12}^r & -h_{11}^r & 0 \\ 0 & 0 & 0_{m-1} \end{pmatrix}, \quad r = m+3, \cdots, 2m+1. \tag{4.15}$$

Proof. Let M^{m+1} be a submanifold of $\overline{M}^{2m+1}(c)$. Put

$$\varepsilon = 2\tau - \frac{(m+1)^2(m-1)}{m}|H|^2 - (m+1)(m-2)\frac{c-3}{4} - 3\frac{c+1}{4}|P|^2. \tag{4.16}$$

Then, from (4.12) and (4.16), we get

$$\varepsilon = \frac{(m+1)^2}{m}|H|^2 - |h|^2 + 2\frac{c}{4}$$

or,

$$(m+1)^2|H|^2 = m|h|^2 + m(\varepsilon - 2\frac{c}{4}). \tag{4.17}$$

Let $\Pi \subset D_x$ be a plane section. We choose an orthonormal basis $\{e_1, \cdots, e_m, e_{m+1}\}$ of T_xM and $\{e_{m+2}, \cdots, e_{2m+1}\}$ of $T_x^{\perp}M$ such that $e_{m+1}=\xi_x, \Pi$ is spanned by e_1, e_2 and e_{m+2} is in the direction of the mean curvature vector H. Hence, (4.17) gives

$$(\sum_{i=1}^{m+1} h_{ii}^{m+2})^2 = m\{\sum_{i=1}^{m+1}(h_{ii}^{m+2})^2 + \sum_{i\neq j}(h_{ij}^{m+2})^2 + \sum_{r=m+3}^{2m+1}\sum_{i,j}(h_{ij}^r)^2 + \varepsilon - \frac{2c}{4}\}$$

and so lemma C, we obtain

$$2h_{11}^{m+2}h_{22}^{m+2} \geq \sum_{i\neq j}(h_{ij}^{m+2})^2 + \sum_{r=m+3}^{2m+1}\sum_{i,j}(h_{ij}^r)^2 + \varepsilon - \frac{2c}{4}. \tag{4.18}$$

On the other hand from (4.8), we find

$$\begin{aligned} K(\Pi) &= R(e_1, e_2, e_2, e_1) = g(h(e_1, e_1), h(e_2, e_2)) - \\ &-g(h(e_1, e_2), h(e_1, e_2)) + \frac{c}{4} + 3\frac{c}{4}g^2(e_1, \varphi e_2) \end{aligned}$$

or,
(4.19)

$$\begin{aligned} K(\Pi) &= \sum_{r=m+2}^{2m+1} g(h(e_1, e_1), e_r)g(h(e_2, e_2), e_r) - \\ &- \sum_{r=m+2}^{2m+1} g(h(e_1, e_2), e_r)g(h(e_1, e_2), e_r) + \frac{c}{4} + 3\frac{c}{4}g^2(e_1, \varphi e_2). \end{aligned}$$

Then from (4.18) and (4.19), we get

$$
\begin{aligned}
K(\Pi) &\geq \sum_{m+2}^{2m+1} \sum_{j>2} \{(h^r_{1j})^2 + (h^r_{2j})^2\} + \frac{1}{2} \sum_{i \neq j > 2} (h^{m+2}_{ij})^2 + \\
&+ \frac{1}{2} \sum_{r=m+3}^{2m+1} \sum_{i,j>2} (h^r_{ij})^2 + \frac{1}{2} \sum_{r=m+3}^{2m+1} (h^r_{11} + h^r_{22})^2 + \\
&+ \frac{\varepsilon}{2} + \frac{3c}{4} g^2(e_1, \varphi e_2) \geq \frac{\varepsilon}{2} + \frac{3c}{4} g^2(e_1, \varphi e_2).
\end{aligned} \tag{4.20}
$$

Finally combining (4.16) and (4.20) we obtain (4.13).

Now, if the equality in (4.13) holds, then the inequality in (4.18) and (4.20) become equalities. Thus, we have:

$$h^{m+2}_{1j} = h^{m+2}_{2j} = h^{m+2}_{ij} = 0, \quad i \neq j > 2;$$

$$h^r_{1j} = h^r_{2j} = h^r_{ij} = 0, r = m+3, \ldots, 2m+1, \quad i, j = 3, \ldots, m+1$$

$$h^{m+3}_{11} + h^{m+3}_{22} = \ldots = h^{2m+1}_{11} + h^{2m+1}_{22} = 0,$$

Now, we may choose e_1, e_2 such that $h^{m+2}_{12} = 0$. Moreover, by applying lemma C and the equation (4.5), we get

$$h^{m+2}_{11} + h^{m+2}_{22} = h^{m+2}_{33} = 0 = h^{m+2}_{44} \ldots = h^{m+2}_{m+1m+1}$$

Therefore, the shape operator will take the form (4.14) and (4.15) with respect to the basis $\{e_1 \ldots e_{m+1}, e_{m+2} \ldots, e_{2m+1}\}$.

The converse follows from a direct calculation.

Now, for each point $x \in M$, we define

$$(\inf_D K)(x) = \inf\{K(x);\ \text{plane section}\ \Pi \subset D_x\}.$$

We denote by δ^D_M the difference between the scalar curvature and $\inf_D K$, i.e.

$$\delta^D_M(x) = \tau(x) - (\inf_D) K(x). \tag{4.21}$$

Then from (4.2) and (4.21), it is clear that

$$\delta_M^D \leq \delta_M. \tag{4.22}$$

Corollary 4.1 *Let M be an $(m+1)$-dimensional anti-invariant submanifold of cosymplectic space form $\overline{M}(c)$ such that $\xi \in TM$. Then, we have*

$$\delta_M^D \leq \frac{(m+1)^2(m-1)}{2m} + \frac{1}{2}(m+2)(m-1)\frac{c}{4}. \tag{4.23}$$

Proof. From (4.13), by putting $|P|^2 = 0, \varphi e_2 = Fe_2$ i.e. $g(e_1, \varphi e_2) = 0$ for anti-invariant submanifolds, we obtain (4.23).

Now, we are going to study inequality (4.13) when M is a slant submanifold.

Theorem 4.2 *Let $\psi : M^{m+1} \to \overline{M}^{2m+1}(c)$ be a θ-slant immersion of a Riemannian $(m+1)$-dimensional manifold into a cosymplectic space form $\overline{M}(c)$. Then for any point $x \in M$ and any plane section $\Pi \subset D_x$, we have*

$$\begin{aligned} \tau - K(c) \quad &\leq \frac{(m+1)^2(m-1)}{2m}|H|^2 + \frac{1}{2}(m+2)(m-1)\frac{c}{4} + \\ &+ \frac{3}{4}c(\frac{m}{2}\cos^2\theta - g^2(e_1, \varphi e_2)). \end{aligned} \tag{4.24}$$

Proof. For a slant submanifold of a cosymplectic manifold,

$$|P|^2 = \sum g^2(e_i, Pe_j) = m\cos^2\theta.$$

Putting the above value of $|P|^2$ in (4.13), we get (4.24).

In particular we can state the following result for 3-dimensional slant submanifolds.

Corollary 4.2. *In the above condition if $m = 2$, then*

$$\delta_M^D \leq \frac{9}{4}|H|^2, \tag{4.25}$$

with equality holding if and only if M is minimal.

Proof. If $m = 2$, it is clear that $\delta_M^D = \tau - K(D)$ and $g^2(e_1, \varphi e_2) = \cos^2\theta$. Thus (4.25) follows from (4.24).

On the other hand, we have

$$\tau = K(e_1 \wedge e_2) + K(e_1 \wedge \xi) + K(e_2 \wedge \xi)$$

and $K(D) = K(e_1 \wedge e_2)$.

Therefore,

$$\tau - K(D) = K(e_1 \wedge \xi) + K(e_2 \wedge \xi) = 0.$$

Consequently, equality in (4.25) holds if and only if $|H| = 0$ i.e. M is minimal.

Now, we study some special plane sections, which are orthogonal to ξ. Let M^{m+1} be a submanifold of a cosymplectic space from $\overline{M}(c)$ such that $\xi \in TM$. We define that a plane section $\Pi \subset T_xM$ is a P-section if there exists a tangent vector $X \in D_x$ such that Π is spanned by X and PX.

For each point $x \in M$, we can define

$$(\inf_P K)(x) = \inf\{K(\Pi) : P-\text{ sections } \Pi\}$$

and,

$$\delta_M^P(x) = \tau(x) - \inf_P K(x).$$

Since every P-section is orthogonal to ξ, it is clear that $\delta_M^P(x) \leq \delta_M^D$. In the case of slant submanifolds we have the following inequality for δ_M^P:

Theorem 4.3. *Let $\psi : M^{m+1} \to \overline{M}^{2m+1}(c)$ be a non-anti-invariant θ-slant immersion of a Riemannian $(m+1)$-dimensional manifold into a cosymplectic-space-form $\overline{M}(c)$. Then*

$$\begin{aligned}\delta_M^P \leq & \frac{(m+1)^2(m-1)}{2m}|H|^2 + \frac{1}{2}(m+1)(m-2)\frac{c}{4} + \\ & + \frac{1}{2}(m-2)\frac{3c}{4}\cos^2\theta.\end{aligned} \tag{4.26}$$

Proof. We can choose two tangent vectors e_1, e_2 such that Π is spanned by e_1 and $e_2 = \sec\theta P e_1$ for a P-section Π. Then $g^2(e_1, \varphi e_2) = \cos^2\theta$, and from (4.24), we get (4.26).

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