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STABILITY CRITERIA FOR DIFFERENCE EQUATIONS

BY

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Abstract. We shall give sufficient conditions for the zero solution of system

$$\Delta y(n) = f(n, y(n))$$

to be asymptotically and uniformly stable with respect to the initial conditions. We also point out the impact of summary excitations of the Lyapunov's stability of sets of difference equations.

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1. Introduction. The aim of this paper is to extend a part of the results concerning the asymptotic properties of an differential system to the case of corresponding discrete system. Our paper is divided into two sections.

In the first Section (Section 2) we consider the system of difference equation

$$\Delta y(n) = f(n, y(n)) \tag{1}$$

where $y = [y^1, \ldots, y^k]$, $f(n, y) = [f^1(n, y), \ldots, f^k(n, y)]$, and we look for the sufficient conditions for the solution $y = 0$ of the system (1) to be asymptotically and uniformly stable with respect to the initial conditions.

The Section 3 examines the impact of summary term on Lyapunov's stability of solution to difference equation

$$x(n+1) = A(n)x(n) + \sum_{m=0}^{n-1} B(n-m-1)x(m) \tag{1'}$$

where $A(n)$, $B(n)$ are $k \times k$ matrices.
The method presented here makes it possible to determine the conditions under which unstable (stable, respectively) linear difference equation become stable (unstable, respectively) with summary terms.

The purpose of this paper is to extend some results given by P.Talpalaru [4].

Throughout, we shall use concepts of stability, uniform stability given in [1].
Let $N(n_0) = \{n_0, n_0 + 1, \dots\}$, where $n_0 \geq 0$ is a natural number or zero.
Let R^k denote the space (R denotes the real line) of sequence $u=[u^1, \dots, u^k]$ of real numbers, with the usual scalar product

$$(u, v) = \sum_{i=1}^{k} u^i v^i$$

and the norm $|u| = (u, u)^{\frac{1}{2}}$.
Let D denote the set

$$D = N(n_0) \times D_a,$$

where $D_a = \left\{ y = [y^1, \dots, y^k] : |y| < a \right\}, \ D_a \subset R^k, \ a > 0$ and in particular $a = +\infty$ is possible. Let

$$Q(n_1, \beta) = \{(n, y) : n = n_1, |y| < \beta, 0 < \beta < a\}$$

for fixed n_1, β.
For the convenience of the reader we shall give a definition for the solution $y = 0$ of the system (1) to be asymptotically and uniformly stable with respect to the initial condition.

Definition. The solution $y = 0$ of system (1) is said to be asymptotically stable in $N(n_0)$ uniformly with respect to the initial conditions given on the set $Q(n_1, \beta)$ if

1° $y = 0$ is stable in $N(n_0)$ and that

2° for any $\varepsilon > 0$ and for any $n_1, \ n_1 > n_0$ there exists an $A > 0$ dependent on ε and on n_1 such that for an arbitrary solution $y = \phi(n)$ of (1) issuing from an arbitrary point of the set $Q(n_1, \beta)$, the inequality $|\phi(n)| < \varepsilon$ hold if only $n > n_1 + A$.

2. Stability and asymptotic stability. We start with two Lemmas.

Lemma 1. *Assume that*

1° *the function* $f : D \longrightarrow D_a$ *in the system (1) is continuous with respect to the second argument,*

2° $2y \cdot f(n, y) + f(n, y) \cdot f(n, y) =$
$= \sum_{i=1}^{k} \left[2y^i f^i(n, y) + f^i(n, y) \cdot f^i(n, y)\right] \leq 0$ *for each* $(n, y) \in D$,

3° $f(n, 0) = 0$ *for* $n \in N(n_0)$.

Then the zero solution of system (1) is Lyapunov stable.

Proof. Let $(\bar{n_0}, \bar{y})$ be an arbitrary point of D. It follows from assumption of Lemma that each solution of system (1) issuing from the point $(\bar{n}_0, \bar{y})$ is defined in $N(\bar{n}_0)$. We shall demonstrate that for each $\varepsilon > 0$ and $n_1 \in N(n_0)$ there exists $\delta > 0$ such that each solution $y = \varphi(n)$ of the system (1) satisfying the condition $|\varphi(n_1)| < \delta$ satisfies also the inequality $|\varphi(n)| < \varepsilon$ for $n \in N(n_1)$.
From Assumption 2° we have that the function $\rho(n) = |\varphi(n)|$ is non-increasing. To prove it we define the function

$$u(n) = |\varphi(n)|^2 = \sum_{i=1}^{k} (\varphi^i(n))^2.$$

Hence using Assumption 2°, we may write

$$\Delta u(n) = \sum_{i=1}^{k} \left[(\varphi^i(n+1))^2 - (\varphi^i(n))^2\right] =$$
$$= \sum_{i=1}^{k} (\varphi^i(n+1) - \varphi^i(n))(\varphi^i(n+1) + \varphi^i(n)) =$$
$$= \sum_{i=1}^{k} f^i(n, \varphi(n)) \left[2\varphi^i(n) + f^i(n, \varphi(n))\right] \leq 0$$

For a fixed $\varepsilon > 0$ and $n_1 \in N(n_0)$ we take $0 < \delta \leq \varepsilon$ and from the initial inequality $|\varphi(n_1)| < \delta$ we obtain the inequality $|\varphi(n)| < \delta$ for $n \in N(n_1)$

and hence also $|\varphi(n)| < \varepsilon$ for $n \in N(n_1)$. It completes the proof of Lemma 1.

Lemma 2. *If*

1° *the function u is defined in $N(n_0)$,*

2° $\lim\limits_{n\to\infty} sup \Delta u(n) = \delta < 0$

then $\lim\limits_{n\to\infty} u(n) = -\infty$

Proof. If follows from Assumption 2° that there exists $m \in N(n_0)$ and $\delta_1, \delta < \delta_1 < 0$ such that $\Delta u(n) \leq \delta_1$ for $n \geq m$. Summing the above inequalities on both sides over $j = m, m+1, \ldots, n$ we obtain the inequality

$$u(n+1) \leq u(m) - (n-m)\delta_1.$$

Taking the limit in the above inequality we get

$$\lim_{n\to\infty} u(n) = -\infty.$$

Now we can formulate:

Theorem 3. *Let be satisfied the assumption of Lemma 1. Moreover, assume that*

4° *exactly one solution of (1) passes trough every point $(n, y) \in D$,*

5° $\lim\limits_{\substack{n\to\infty \\ y\to\bar{y}}} sup\, [2y \cdot f(n, y) + f(n, y) \cdot f(n, y)] = \delta < 0$
for any $\bar{y} \in D_a$, $|\bar{y}| > 0$.

Then the zero solution $y = 0$ of system (1) is asymptotically stable in $N(n_0)$ uniformly with respect to the initial conditions given in the set $Q(n_1, \beta)$.

Proof. It follows from Lemma 1 and assumption 4° that the solution $y = 0$ of (1) is Lyapunow stable. Now we shall show that for any $n_1 \in N(n_0)$ the solution $y = \varphi(n)$ of system (1) satisfying condition $|\varphi(n_1)| < \beta$ satisfies also the condition

$$\lim_{n\to\infty} |\varphi(n)| = 0.$$

As in the proof of Lemma 1 we get that the function

$$u(n) = |\varphi(n)|^2 = \sum_{i=1}^{k} (\varphi^i(n))^2 \tag{2}$$

is non-increasing, because

$$\Delta u(n) = \sum_{i=1}^{k} \left[(\varphi^i(n+1))^2 - (\varphi^i(n))^2 \right] =$$

$$= \sum_{i=1}^{k} \Delta\varphi^i(n) \left[\varphi^i(n+1) + \varphi^i(n) \right] \leq 0. \tag{3}$$

We will show that $\lim_{n\to\infty} u(n) = 0$. For this purpose we assume that the last condition does not hold.
Then $u(n) > 0$ and $\Delta u(n) \leq 0$ for any $n \in N(n_0)$ implies that

$$\lim_{n\to\infty} u(n) = \gamma, \ \gamma \geq 0. \tag{4}$$

From (2),(3) and (4) we conclude that there exists the limit

$$\lim_{n\to\infty} sup\varphi(n) = \bar{\bar{y}} \neq 0$$

and from assumption 5° we infer that

$$\lim_{n\to\infty} sup\Delta u(n) = \lim_{n\to\infty} sup \sum_{i=1}^{k} \Delta\varphi^i(n) \left[\varphi^i(n+1) + \varphi^i(n) \right] = \delta < 0.$$

Now from Lemma 2 we have

$$\lim_{n\to\infty} u(n) = -\infty,$$

what contradicts (4). Therefore the stability is indeed asymptotic.
To complete the proof of the Theorem we must show that this asymptotic stability is uniform with respect to the initial conditions given on a fixed set $Q(n_1, \beta)$.
Assume on the contrary that for some $\varepsilon_0 > 0$ no such $A > 0$ can be chosen that if

$$|\varphi(n_1)| \leq \beta$$

then

$$|\varphi(n)| < \varepsilon_0 \ for \ n > n_1 + A$$

for all solutions $y = \varphi(n)$ of (1) issuing from $Q(n_1, \beta)$ with fixed n_1 and β. This would means that for each positive integer m there exists a point $(n_1, \Psi(m)) \in Q(n_1, \beta)$ such that the solution

$$y = \varphi^m(n)$$

issuing from this point satisfies the condition

$$|\varphi^m(n_1 + m)| > \varepsilon_0 \tag{5}$$

As $\bar{Q}(n_1, \beta)$ is a compact set, therefore the sequence $\{(n_1, \Psi(m))\}$ contains the subsequences $\{(n_1, \Psi(m_l))\}$ convergent to some point $(n_1, \bar{\Psi}) \in \bar{Q}(n_1, \beta)$.
Denote by $y = \bar{\Psi}(n)$ the solution of (1) satisfying the initial condition

$$\bar{\Psi}(n_1) = \bar{\Psi}.$$

Because the solution $y = 0$ of (1) is asymptoticaly stable,

$$\lim_{n \to \infty} \left|\bar{\Psi}(n)\right| = 0$$

and then there exists an $N_0 \in N(n_0)$ such that

$$\left|\bar{\Psi}(n_1 + N_0)\right| < \varepsilon_0$$

The solution of (1) depends continuously on the initial conditions and therefore, there exists a neighbourhood $S(n_1, \bar{\Psi})$ such that all solutions of (1) issuing from $S(n_1, \bar{\Psi})$ satisfy the inequality

$$|\varphi(n_1 + N_0)| < \varepsilon_0$$

For a sufficiently large m_l the points $(n_1, \Psi(m_l))$ belong to $S(n_1, \bar{\Psi})$ therefore

$$|\varphi^{m_l}(n_l + N_0)| < \varepsilon_0$$

for the sufficiently large m_l, what contradicts inequality (5).
The proof is complete.

Remark 1. If in assumption $5°$, $\delta = 0$ for $|\bar{y}| > 0$, the zero solution of (1) may be asymptotically stable.

Example 1. Consider the system

$$\Delta y_1(n) = -\frac{y_1(n)}{n},\ \Delta y_2 = -\frac{y_2(n)}{n}$$

in the set D.

$$\lim_{\substack{n\to\infty \\ y\to\bar{y},|\bar{y}|>0}} sup\left\{-\frac{(y_1(n))^2}{n}\left(2-\frac{1}{n}\right)-\frac{(y_2(n))^2}{n}\left(2-\frac{1}{n}\right)\right\}=0$$

The system has the solution

$$y_1(n+1) = y_1(n_0)\frac{n_0-1}{n-1},\ y_2(n+1) = y_2(n_0)\frac{n_0-1}{n-1},\ n_0>1,$$

which is asymptotically stable.

Example 2. Consider the system

$$\Delta y_1(n) = -\frac{y_1(n)}{n},\ \Delta y_2(n) = -y_2(n)$$

in the set D.
The solution $y_1(n)=0$, $y_2(n)=0$ is asymptotically stable, although

$$\lim_{\substack{n\to\infty \\ y\to\bar{y},|\bar{y}|>0}} sup\left\{-2\frac{(y_1(n))^2}{n}-2(y_2(n))^2+\frac{(y_2(n))^2}{n}+(y_2(n))^2\right\}=-(\bar{y}_2)^2.$$

This limit is equal zero for $\bar{y}_2 = 0$.

Example 3. Consider the following system of difference equation

$$\Delta y_1(n) = -\frac{y_1(n)}{n},\ \Delta y_2(n) = -\frac{-(y_1(n))^2 y_2(n)}{n} \tag{6}$$

in the set

$$D=\{(n,y_1,y_2);\ n\in N(n_0)\ n_0>1,\ (y_1)^2+(y_2)^2<\alpha=2\}.$$

Functions occuring on the right hand side of(6) satisfy assumptions $1^\circ - 4^\circ$ of Theorem 3, but

$$\lim_{\substack{n\to\infty \\ y\to\bar{y},\bar{\bar{y}}>0}} sup\{[y_1,y_2]colon[f_1(n,y_1,y_2),f_2(n,y_1,y_2)]\}=0.$$

The general solution of (6) has the form

$$y_1(n) = y_1(n_0)\frac{n_0 - 1}{n - 1}, y_2(n) = y_2(n_0)\prod_{s=n_0}^{n-1}\left[1 - \frac{(y_1(n_0))^2(n_0 - 1)^2}{(s-1)^2 \cdot s}\right].$$

Hence the solution $y_1(n) = 0$, $y_2(n) = 0$ of system (6) is stable, but not asymptotically.

3. The impact of summary excitations on the Lyapunov's stability. Talpalaru [4] has considered discrete system

$$(*) \qquad x(n+1) = A(n)x(n) + \sum_{m=0}^{n-1} B(n, m, x(m)),$$

which can be thought of as a perturbation of the system

$$(**) \qquad y(n+1) = A(n)y(n).$$

He obtained some results on the asymptotic behaviour (uniformly stable, asymptotically stable, uniformly asymptotically stable) of solutions of (*) using Lemma 2.1 [4] where the stability properties of the system (**) are related to the fundamental matrix $Y(n)$.
In this section we consider two problems:

1° we give sufficient conditions for the stability and asymptotic stability solutions of (1') assuming that the trivial solution of (**) is unstable,

2° we give sufficient conditions for the unstable solutions of (1') assuming that the trivial solution of (**) is stable.

We adopt the following notation:
N is the set of all non-negative integers, $|x| = \sum_{i=1}^{k} |x_i|$, $x = (x_1, \ldots, x_k) \in R^k$ is the norm in R^k, M^k is the space of all $k \times k$ matrices $A = \lfloor a_{ij} \rfloor$ with norm $|\cdot|$.
We will consider the initial value problem

$$(7) \qquad x(n+1) = Ax(n) + \sum_{m=0}^{n-1} B(n - m - 1)x(m), \; x(0) = x_0$$

and the initial value problem corresponding to the linear system

$$y(n+1) = Ay(n),\ y(0) = y_0 = x_0 \tag{8}$$

where $A, B(n)$ are from M^k.
In the following we assume that

$$\sum_{i=s}^{k} u(i) = 0,\ \prod_{i=s}^{k} u(i) = 1 \text{ where } s > k \text{ and } detA \neq 0.$$

In paper [3] the authors have considered the difference equation

$$x(n) = f(n) + \sum_{m=0}^{n-1} a(n-m-1)x(m) \tag{9}$$

where $a(n) \in M^k,\ n \in N$.
In our analysis we will use the resolvent [3] of equation (9).
The resolvent matrix $r(n)$ associated with the equation (9) satisfies the resolvent matrix equation

$$r(n) = -a(n) + \sum_{m=0}^{n-1} a(n-m-1)r(m),\ n \in N. \tag{10}$$

If $r(n)$ is a solution of (10) then for $x(n)$ given by (9) we can write

$$x(n) = f(n) - \sum_{m=0}^{n-1} r(n-m-1)f(m). \tag{11}$$

To investigate the stability behaviour of linear difference equation (7) we introduce the following relation

$$x(n) = z(n) + \sum_{m=0}^{n-1} Q(n-m-1)x(m), \tag{12}$$

where $Q(n-m-1) = -A^{-1}B(n-m-1)$.
Then from (11) we have

$$x(n) = z(n) - \sum_{m=0}^{n-1} r(n-m-1)z(m) \tag{13}$$

where

$$r(n) = -Q(n) + \sum_{m=0}^{n-1} Q(n-m-1)r(m). \tag{14}$$

Setting (13) to the equation (7) we obtain

$$z(n+1) = Cz(n) + \sum_{m=0}^{n-1} r(n-m)z(m),\ z(0) = x_0 \tag{15}$$

where $C = A - A^{-1}B(0)$.
Let X(n) be the fundamental matrix solution of the equation

$$z(n+1) = Cz(n),$$

then (15) can be written as

$$z(n) = X(n)X^{-1}(0)z_0 + \sum_{s=0}^{n-1} X(n)X^{-1}(s+1)\sum_{m=0}^{s-1} r(s-m)z(m). \tag{16}$$

Theorem 4. *Let the following conditions hold:*

1° *the trivial solution of (8) is unstable,*

2° *there exist constants $\alpha > 0,\ c_0 \geq 1$ such that* $|X(n)X^{-1}(s+1)| \leq c_0 e^{-\alpha(n-s)}$ *for $n \geq s$,*

3° *there exists an $\omega > 0$ such that* $\sum\limits_{s=0}^{\infty}\left(\sum\limits_{m=0}^{s-1} e^{\alpha(s-m)}|r(s-m)|\right) \leq \omega$,

4° *there exists an $r_0 > 0$ such that $|r(m)| \leq r_0$ for all $m \in N$.*

Then the trivial solution of (7) is

(i) asymptotically stable if the conditions $1^\circ - 3^\circ$ hold,

(ii) stable if the conditions 1°, 2°, 4° hold.

Proof. From (16) and condition 2° we obtain

$$|z(n)| \leq c_0 e^{-\alpha n}|z_0| + c_0 \sum_{s=0}^{n-1} e^{-\alpha(n-s)} \sum_{m=0}^{s-1} |r(s-m)||z(m)|.$$

By introducing the new variable $v(n) = z(n)e^{\alpha n}$, we see that

$$v(n) \leq c_0|z_0| + c_0 \sum_{s=0}^{n-1} e^{\alpha(s+1)} \sum_{m=0}^{s-1} e^{-\alpha m}|r(s-m)|v(m).$$

Setting

$$H(n) = c_0|z_0| + c_0 \sum_{s=0}^{n-1} e^{\alpha(s+1)} \sum_{m=0}^{s-1} e^{-\alpha m}|r(s-m)|v(m),$$

we get

$$H(n) \leq H(0) + c_0 \sum_{s=0}^{n-1} \left[e^{\alpha(s+1)} \sum_{m=0}^{s-1} e^{-\alpha m}|r(s-m)| \right] H(s).$$

Now, using the discrete version of Gronwall inequality [2], we have

$$H(n) \leq H(0) exp \left(c_0 \sum_{s=0}^{n-1} e^{\alpha(s+1)} \sum_{m=0}^{s-1} e^{-\alpha m}|r(s-m)| \right).$$

Hence, assumption 3° implies

$$v(n) \leq c_0|z_0|e^{c_0 \omega e^{\alpha}} \text{ forall } n \in N.$$

This in turn yields

$$|z(n)| \leq c_0|z(0)|e^{c_0 \omega e^{\alpha}} e^{-\alpha n} \text{ for } n \in N.$$

From this and from (13) we obtain the relation

$$\begin{aligned} |x(n)| &\leq |z(n)| + \sum_{m=0}^{n-1} |r(n-m-1)||z(m)| \leq \\ &\leq c_0|z_0|e^{c_0 \omega e^{\alpha}} e^{-\alpha(n-1)} \left[1 + \sum_{l=0}^{n-1} |r(l)|e^{\alpha l} \right]. \end{aligned} \tag{17}$$

Assumption 3° implies the inequality

$$\sum_{i=0}^{n-1} |r(l)|e^{\alpha l} \leq \beta, \text{ for } n \in N \text{ and } \beta > 0.$$

This and (17) give that the zero solution of (7) is asymptotically stable. If $|r(n)| \leq r_0 = const$ for all $n \in N$, then from (17) the stability follows.

Now, we do not assume that $A(n)$ in (7) is constant.

We consider the system

$$x(n+1) = A(n)x(n) + \sum_{m=0}^{n-1} B(n-m-1)x(m), \; x(0) = b \tag{18}$$

as a perturbed system of

$$y(n+1) = A(n)y(n) \tag{8'}$$

where $A(n) \in M^k$ and det $A(n) \neq 0$ for all $n \in N$.

The substitution

$$x(n) = z(n) + \sum_{m=0}^{n-1} Q(n-m-1)x(m), \tag{19}$$

where $Q(n-m-1) = -A^{-1}(n)B(n-m-1)$, and formula (11) reduce the system (18) to the system

$$z(n+1) = C(n)x(n) + \sum_{m=0}^{n-1} r(n-m)z(m) \tag{20}$$

where

$$C(n) = A(n) - A^{-1}(n)B(0).$$

Let $V_1(n)$ be a fundamental solution matrix of the system

$$z(n+1) = C(n)z(n) \tag{21}$$

where $detV_1(n) \neq 0$ for all $n \in N$.

Then, any solution of (20) satisfies the relation

$$z(n) = V_1(n)z(0) + \sum_{l=0}^{n-1} V_1(n)V_1^{-1}(l+1) \sum_{m=0}^{l-1} r(l-m)z(m). \tag{22}$$

Theorem 5. *If the following conditions are satisfied:*

1° *the trivial solution of (8') is unstable,*

2° *the trivial solution of (21) is stable (asymptotically stable),*

3° $\sum_{l=0}^{\infty}\left(\sum_{m=0}^{l-1}|V_1^{-1}(l+1)r(l-m)V_1(m)|\right) \le \omega_1 < \infty,$

4° $r_1(n) = \sum_{m=0}^{n-1}|r(n-m-1)V_1(m)| < r_1$*=const. for all* $n \in N$

($r_1(n) \longrightarrow 0$ *as* $n \longrightarrow \infty$ *if trivial solution of (21) is asymptotically stable). Then the trivial solution of (18) is stable (or asymptotically stable, respectively).*

Proof. Multiplying (22) by $V_1^{-1}(n)$ we get

$$V_1^{-1}(n)z(n) = x_0 + \sum_{l=0}^{n-1} V_1^{-1}(l+1)\sum_{m=0}^{l-1} r(l-m)z(m).$$

Let $w(n) = V_1^{-1}(n)z(n), then$

$$|w(n)| \le |x_0| + \sum_{l=0}^{n-1}\sum_{m=0}^{l-1}|V_1^{-1}(l+1)r(l-m)V_1(m)||w(m)|.$$

Similarly, we can show that (see Theorem 4)

$$|z(n)| \le |x_0||V_1(n)|e^{\omega_1}.$$

This inequality combined with relation

$$x(n) = z(n) - \sum_{m=0}^{n-1} r(n-m-1)z(m)$$

gives the inequality

$$|x(n)| \le |x_0|e^{\omega_1}\left(|V_1(n)| + \sum_{m=0}^{n-1}|r(n-m-1)z(m)|\right).$$

Hence, by 4°, it follows that

$$|x(n)| \le (|V_1(n)| + r_1)|x_0|e^{\omega_1}.$$

Now, assumption 2° implies the zero solution of (18) is stable.
Moreover, if

$$|V_1(n)| \longrightarrow 0 \text{ and } r_1(n) \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

then the zero solution of (18) is asymptoticaly stable.
Up to now we have considered the stability problem for the systems (7) and (18) assuming that the zero solution of (8) and (8') is unstable, respectively.

Now, we consider another initial value problem, i.e. we assume that the zero solution of (8) ((8')) is stable and we prove that zero solution of (7) ((18)) is unstable.

Definition. The zero solution of (20) is said to be unstable if given fixed $T > 0$ there exists $\delta(T) > 0$ such that for $|z_0| < \delta(T)$, $|z(n)| \geq T$ for $n \longrightarrow \infty$.
Denote by $V[V(0) = I]$ and $W[W(0) = I]$ the fundamental matrices of the equations (18) and (20) respectively.

Theorem 6. *Assume that*

1° *the zero solution of (8') is stable,*

2° *the zero solution of (21) is unstable,*

3° *the following conditions hold*

$$\sum_{m=0}^{\infty}\sum_{l=0}^{m-1}\left|V_1^{-1}(m+1)r(m-l-1)V_1(l)\right| = q \leq \frac{1}{2},$$
$$\sum_{m=0}^{n-1}|r(n-m-1)V_1(m)| \leq p < \infty \text{ for all } n \in N.$$

Then the zero solution of the equation (18) is unstable.

Proof. The solutions of equations (18) and (20) can be expressed by

$$x(n) = V(n)b \text{ and } z(n) = W(n)b. \tag{23}$$

Since

$$x(n) = z(n) - \sum_{m=0}^{n-1} r(n-m-1)z(m),$$

it follows from (23) that

$$\left[V(n) - W(n) + \sum_{m=0}^{n-1} r(n-m-1)W(m)\right] b = 0.$$

Thus for arbitrary $b, b \neq 0$ we get

$$W(n) = V(n) + \sum_{m=0}^{n-1} r(n-m-1)W(m).$$

Using fundamental matrix $V_1(n)$, we get

$$W(n) = V_1(n) + \sum_{m=0}^{n-1} V_1(n)V_1^{-1}(m+1) \sum_{l=0}^{m-1} r(m-l-1)W(l). \tag{24}$$

We set $h(n) = V_1^{-1}(n)W(n)$, so that we have

$$h(n) = I + \sum_{m=0}^{n-1} V_1^{-1}(m+1) \sum_{l=0}^{m-1} r(m-l-1)V_1(l)h(l).$$

Proceeding as in Theorem 4, we obtain

$$|h(n)| \leq exp\left[\sum_{m=0}^{n-1}\sum_{l=0}^{m-1} |V_1^{-1}(m+1)r(m-l-s)V_1(s)|\right].$$

Applying the assumption 3° to the above inequality, we get

$$|V_1^{-1}(n)W(n)| \leq e^q \text{ for all } n \in N. \tag{25}$$

Now by (24) and(25) we have

$$|W(n)| \geq |V_1(n)| \left[1 - \sum_{m=0}^{n-1}\sum_{l=0}^{m-1} |V_1^{-1}(m+1)r(m-l-1)V_1(l)| \cdot |V_1^{-1}(n)W(n)\right] \geq$$

$$\geq \lfloor |V_1(n)|(1-qe^q)\rfloor .$$

Then, the unstability of the trivial solution of (21) and $q \leq \frac{1}{2}$ implies that the $|W(n)|$ is unbounded.
Now from the formula

$$V(n) = W(n) - \sum_{m=0}^{n-1} r(n-m-1)W(m),$$

and assumption 3°, we get

$$|V(n)| \geq |W(n)| - \sum_{m=0}^{n-1} |r(n-m-1)W(m)| \geq$$

$$\geq |W(n)| - \sum_{m=0}^{n-1} |r(n-m-1)V_1(m)||V_1^{-1}(m)W(m)| \geq |W(n)| - pe^q.$$

It implies that the zero solution of the equation (18) is unstable.

Remark 2. Results obtained in this part can be extended to the difference equation of the form

$$x(n+1) = A(n)x(n) + \sum_{m=0}^{n-1} B(n,m)x(m) + F\left(n, x(n), \sum_{m=0}^{n-1} G(n,m,x(m))\right).$$

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