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CLASSES OF ALMOST ANTI-HERMITIAN STRUCTURES ON THE TANGENT BUNDLE OF A RIEMANNIAN MANIFOLD*

BY

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Abstract. We consider a family of natural almost anti-Hermitian structures (G, J) on the tangent bundle TM of a Riemannian manifold (M, g). The semi-Riemannian metric G is a natural lift to TM of the metric g such that the vertical and horizontal distributions VTM, HTM are maximally isotropic. The almost complex structure J is a natural lift of diagonal type of g (see [9], [20]). We study the conditions under which this almost anti-Hermitian structure belongs to each from the eight classes of anti-Hermitian manifolds obtained in the classification given in [1].

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1. Introduction. Let (M, g) be an n-dimensional Riemannian manifold and denote by $\tau : TM \longrightarrow M$ its tangent bundle. There are several Riemannian and semi-Riemannian metrics induced on TM from the Riemannian metric g on M. Among them, we may quote the Sasaki metric and the complete lift of the metric g. On the other hand, there are the lifts of g of natural type, leading to several new geometric structures with many nice geometric properties (see [5], [4]).

In the present paper we consider a natural almost anti-Hermitian structure (G, J), defined on TM by using some natural lifts of the Riemannian metric g. The vertical distribution VTM and the horizontal distribution HTM are interchanged by the considered almost complex structure J, while

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they are maximally isotropic with respect to the semi-Riemannian metric G. Using the classification of the almost anti-Hermitian structures in eight classes, given in [1] we study the conditions under which the considered almost anti-Hermitian structure (TM, G, J) belongs to each of these eight classes.

The manifolds, tensor fields and other geometric objects we consider in this paper are assumed to be differentiable of class C^∞ (i.e. smooth). We use the computations in local coordinates in a fixed local chart but many results may be expressed in an invariant form by using the vertical and horizontal lifts. The well known summation convention is used throughout this paper, the range of the indices h, i, j, k, l being always $\{1, \ldots, n\}$.

2. An almost anti-Hermitian structure on TM. Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \longrightarrow M$. Recall that TM has a structure of a $2n$-dimensional smooth manifold, induced from the structure of smooth n-dimensional manifold of M. From every local chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ on M it is induced a local chart $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \ldots, x^n, y^1, \ldots, y^n)$, on TM, as follows. For a tangent vector $y \in \tau^{-1}(U) \subset TM$, the first local coordinates $x^1, \ldots, x^n$ are the local coordinates $x^1, \ldots, x^n$ of its base point $x = \tau(y)$ in the local chart (U, φ) (in fact we made an abuse of notation, identifying x^i with $\tau^* x^i = x^i \circ \tau,\ i = 1, \ldots, n$). The last n local coordinates $y^1, \ldots, y^n$ of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis $((\frac{\partial}{\partial x^1})_{\tau(y)}, \ldots, (\frac{\partial}{\partial x^n})_{\tau(y)})$, defined by the local chart (U, φ), i.e. $y = y^i (\frac{\partial}{\partial x^i})_{\tau(y)}$. Due to this special structure of differentiable manifold for TM, it is possible to introduce the concept of M-tensor field on it. The M-tensor fields are defined by their components with respect to the induced local charts on TM (hence they are defined locally), but they can be interpreted as some (partial) usual tensor fields on TM. However, the essential quality of an M-tensor field on TM is that the local coordinate change rule of its components with respect to the change of induced local charts is the same as the local coordinate change rule of the components of a usual tensor field on M with respect to the change of local charts on M. More precisely, an M-tensor field of type (p, q) on TM is defined by sets of n^{p+q} components (functions depending on x^i and y^i), with p upper indices and q lower indices, assigned to induced local charts $(\tau^{-1}(U), \Phi)$ on TM, such that the local coordinate change rule of these components (with respect to induced local charts on TM) is that of the local coordinate components of a tensor field of type (p, q) on the base manifold M (with respect to usual local charts on M), when a change of local charts on M (and hence on TM) is performed (see [7] for further details); e.g., the components y^i, $i = 1, \dots, n$, corresponding to the last n local coordinates of a tangent vector y, assigned to the induced local chart $(\tau^{-1}(U), \Phi)$ define an M-tensor field of type $(1, 0)$ on TM. A usual tensor field of type (p, q) on M may be thought of as an M-tensor field of type (p, q) on TM. If the considered tensor field on M is covariant only, the corresponding M-tensor field on TM may be identified with the induced (pullback by τ) tensor field on TM. Some useful M-tensor fields on TM may be obtained as follows. Let $u : [0, \infty) \longrightarrow \mathbf{R}$ be a smooth function and let $\|y\|^2 = g_{\tau(y)}(y, y)$ be the square of the norm of the tangent vector $y \in \tau^{-1}(U)$. If δ^i_j are the Kronecker symbols (in fact, they are the local coordinate components of the identity tensor field I on M), then the components $u(\|y\|^2)\delta^i_j$ define an M-tensor field of type $(1, 1)$ on TM. Similarly, if $g_{ij}(x)$ are the local coordinate components of the metric tensor field g on M in the local chart (U, φ), then the components $u(\|y\|^2)g_{ij}$ define a symmetric M-tensor field of type $(0, 2)$ on TM. The components $g_{0i} = y^k g_{ki}$ define an M-tensor field of type $(0, 1)$ on TM.

We shall use the horizontal distribution HTM, defined by the Levi Civita connection $\dot{\nabla}$ of g, in order to define some natural lifts to TM of the Riemannian metric g on M. Denote by $VTM = \text{Ker } \tau_* \subset TTM$ the vertical distribution on TM. Then we have the direct sum decomposition

$$TTM = VTM \oplus HTM. \tag{1}$$

If $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$ is a local chart on TM, induced from the local chart $(U, \varphi) = (U, x^1, \dots, x^n)$, the local vector fields $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}$ on $\tau^{-1}(U)$ define a local frame for VTM over $\tau^{-1}(U)$ and the local vector fields $\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}$ define a local frame for HTM over $\tau^{-1}(U)$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i}\frac{\partial}{\partial y^h}, \quad \Gamma^h_{0i} = y^k \Gamma^h_{ki}$$

and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of g.

The set $(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n})$ defines a local frame on TM, adapted to the direct sum decomposition (1). Remark that

$$\frac{\partial}{\partial y^i} = (\frac{\partial}{\partial x^i})^V, \quad \frac{\delta}{\delta x^i} = (\frac{\partial}{\partial x^i})^H,$$

where X^V and X^H denote the vertical and horizontal lifts of the vector field X on M.

Lemma 1. *If $n > 1$ and u, v are smooth functions on TM such that*

$$ug_{ij} + vg_{0i}g_{0j} = 0,\ g_{0i} = y^h g_{hi},\ y \in \tau^{-1}(U)$$

on the domain of any induced local chart on TM, then $u = 0,\ v = 0$.

The proof is obtained easily by transvecting the given relation with g^{ij} and y^j (Recall that the functions $g^{ij}(x)$ are the entries of the inverse of the matrix $(g_{ij}(x))$, associated to g in the local chart (U, φ) on M; moreover, the components $g^{ij}(x)$ define a tensor field of type $(2, 0)$ on M).

Remark. From the relation

$$u\delta^i_j + vg_{0j}y^i = 0,\ y \in \tau^{-1}(U),$$

it is obtained, in a similar way, $u = v = 0$.

Let $C = y^i \frac{\partial}{\partial y^i}$ be the Liouville vector field on TM and consider the horizontal vector field $\widetilde{C} = y^i \frac{\delta}{\delta x^i}$ on TM, defined in a similar way.

Since we work in a fixed local chart (U, φ) on M and in the corresponding induced local chart $(\tau^{-1}(U), \Phi)$ on TM, we shall use the following simpler notations

$$\frac{\partial}{\partial y^i} = \partial_i,\quad \frac{\delta}{\delta x^i} = \delta_i.$$

Denote by

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y, y) = \frac{1}{2}g_{ik}(x)y^i y^k,\quad y \in \tau^{-1}(U) \tag{2}$$

the energy density defined by g in the tangent vector y. We have $t \in [0, \infty)$ for all $y \in TM$. Consider the real valued smooth functions a_1, a_2, b_1, b_2 defined on $[0, \infty) \subset \mathbf{R}$ and consider a 1-st order natural almost complex structure on TM, by using these coefficients and the Riemannian metric g, just like the 1-st order natural lifts of g to TM are obtained in [5]. The expression of J is given by (see [9], [20])

$$\begin{cases} JX^H_y = a_1(t)X^V_y + b_1(t)g_{\tau(y)}(y, X)C_y, \\ \\ JX^V_y = -a_2(t)X^H_y - b_2(t)g_{\tau(y)}(y, X)\widetilde{C}_y. \end{cases} \tag{3}$$

The expression of J in adapted local frames is given by

$$J\delta_i = a_1(t)\partial_i + b_1(t)g_{0i}C,$$
$$J\partial_i = -a_2(t)\delta_i - b_2(t)g_{0i}\widetilde{C}.$$

Proposition 2. *The operator J defines an almost complex structure on TM if and only if*

$$a_1a_2 = 1\ , \qquad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1. \tag{4}$$

Proof. The relations are obtained easily from the property $J^2 = -I$ of J and Lemma 1.

Remark. From the conditions (4) we have that the coefficients a_1, a_2, $a_1 + 2tb_1, a_2 + 2tb_2$ cannot vanish and have the same sign. We assume that $a_1 > 0,\ a_2 > 0,\ a_1 + 2tb_1 > 0,\ a_2 + 2tb_2 > 0$ for all $t \geq 0$.

The integrability problem for the almost complex structure J has been studied in [8], [9], [10], [11] by finding the conditions under which the Nijenhuis tensor field N_J of J vanishes.

Theorem 3. *Let (M, g) be an $n(> 2)$-dimensional connected Riemannian manifold. The almost complex structure J defined by (3) on TM is integrable if and only if (M, g) has constant sectional curvature c and the function b_1 is given by*

$$b_1 = \frac{a_1a_1' - c}{a_1 - 2ta_1'}. \tag{5}$$

Remark. The relations (4) allow us to express two of the coefficients $a_1,\ a_2, b_1,\ b_2$ as functions of the other two; e.g. we have

$$a_2 = \frac{1}{a_1}, \quad b_2 = \frac{-a_2b_1}{a_1 + 2tb_1} = \frac{-b_1}{a_1(a_1 + 2tb_1)}. \tag{6}$$

Remark. In the case where the almost complex structure J is integrable, we have:

$$b_2 = \frac{c - a_1a_1'}{a_1(a_1^2 - 2ct)}$$

(compare with the corresponding expressions from [9] and [20]).

Now, we consider a particular 1-st order natural lift G of g to TM, defining a semi-Riemannian metric of signature (n, n) on TM . This lift is defined by two real valued smooth functions $u, v : [0, \infty) \to \mathbf{R}$ and is given by

$$(7)\quad \begin{cases} G_y(X^H, Y^H) = 0,\ G_y(X^V, Y^V) = 0, \\ G_y(X^H, Y^V) = G_y(Y^V, X^H) = G_y(X^V, Y^H) = G_y(Y^H, X^V) = \\ = u(t)g_{\tau(y)}(X, Y) + v(t)g_{\tau(y)}(y, X)g_{\tau(y)}(y, Y). \end{cases}$$

The expression of G in local adapted frames is defined by the conditions

$$G(\delta_i, \delta_j) = 0, \quad G(\partial_i, \partial_j) = 0,$$

$$G(\partial_i, \delta_j) = G(\delta_i, \partial_j) = ug_{ij} + vg_{0i}g_{0j}.$$

Remark that G is defined, essentially, by the symmetric M-tensor field $G_{ij} = ug_{ij} + vg_{0i}g_{0j}$ of type $(0, 2)$. The condition for G to be nondegenerate is assured if

$$(8)\quad u \neq 0,\ u + 2tv \neq 0.$$

The condition for G to be anti-Hermitian with respect to the almost complex structure J, considered above, is given by

$$(9)\quad G(JX, JY) = -G(X, Y),$$

for all vector fields X, Y on TM (see [1], [2], [14] for details). Some authors call such a metric a Norden metric.

The property (9) of the semi-Riemannian metric G and the almost complex structure J can be checked easily by a straightforward computation in the adapted local frames. Hence we may state

Proposition 4. *The semi-Riemannian metric G is anti-Hermitian with respect to the almost complex structure J, i.e. (TM, G, J) is an almost anti-Hermitian manifold.*

Next we can obtain local coordinate expression of the Levi Civita connection of G.

Proposition 5. *The Levi Civita connection* ∇ *of the pseudo - Riemannian metric* G *on* TM *has the following expression in the local adapted frame* $(\partial_i, ..., \partial_n, \delta_i, ..., \delta_n)$

$$\nabla_{\partial_i}\partial_j = Q^h_{ij}\partial_h,\ \nabla_{\delta_i}\partial_j = \Gamma^h_{ij}\partial_h + P^h_{ji}\delta_h,$$

$$\nabla_{\partial_i}\delta_j = P^h_{ij}\delta_h,\ \nabla_{\delta_i}\delta_j = \Gamma^h_{ij}\delta_h + S^h_{ij}\partial_h,$$

where the $M-$*tensor fields* P^h_{ij}, Q^h_{ij}, S^h_{ij} *are given by*

$$P^h_{ij} = \frac{u'-v}{2u}(g_{0i}\delta^h_j - \frac{u}{u+2tv}g_{ij}y^h - \frac{v}{u+2tv}g_{0i}g_{0j}y^h),$$

$$Q^h_{ij} = \frac{u'+v}{2u}(g_{0i}\delta^h_j + g_{0j}\delta^h_i) + \frac{v}{u+2tv}g_{ij}y^h + \frac{v'u-u'v-v^2}{u(u+2tv)}g_{0i}g_{0j}y^h,$$

$$S^h_{ij} = R^h_{j0i} + \frac{v}{u+2tv}R_{0ij0}y^h,$$

R_{likj} *denoting the local coordinate components of the Riemann-Christoffel tensor of* $\dot{\nabla}$ *on* M *and* $R_{0ikj} = R_{likj}y^l$, $R_{0ij0} = R_{lijk}y^ly^k$.

3. Classes of almost anti-Hermitian structures on TM**.** In this section we shall study the existence of some classes of almost anti-Hermitian structures (G, J) on TM of the type defined in the previous section. These classes belong to the classification given in [1]. To this end we consider the usual tensor field F of type $(0, 3)$ defined on TM by using the metric G and the covariant differential ∇J

$$F(X, Y, Z) = G((\nabla_X J)Y, Z),\ X, Y, Z \in \Gamma(TM).$$

The tensor field F has the following invariance and symmetry properties

$$F(X, Y, Z) = F(X, JY, JZ) = F(X, Z, Y). \tag{10}$$

The space $\mathcal{F}$ of tensor fields F of type (0,3) with the properties (10) has been decomposed (see [1]) into a direct sum of three irreducible components $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ which are invariant under the natural representation of the semi-unitary (pseudo-unitary) group on $\mathcal{F}$, induced from its standard representation on the model space of TM. From this decomposition one obtains eight (=2^3) classes of almost anti-Hermitian manifolds. We shall describe these classes and their characterizations. Introduce the following 1-form ϕ, associated with F (see also [1])

$$\phi(X) = G^{ij}F(E_i, E_j, X),\ X \in \Gamma(TM),\ i, j = 1, ..., 2n,$$

where $(E_1, ..., E_{2n})$ is a local frame in TTM and G^{ij} are the entries of the inverse of the matrix of (G_{ij}) associated to G in the local frame $(E_1, ..., E_{2n})$.

Then the considered almost anti-Hermitian structure (G, J) on TM can belong to one of the following classes of manifolds

1. the anti-Kählerian manifolds, when

$$F(X, Y, Z) = 0, \tag{11}$$

or, equivalently, $\nabla J = 0$,

2. the conformally anti-Kählerian manifolds, or ω_1-*manifolds, when*

$$2nF(X, Y, Z) = G(X, Y)\phi(Z) + G(X, Z)\phi(Y)+ \\ +G(X, JY)\phi(JZ) + G(X, JZ)\phi(JY), \tag{12}$$

3. the special complex anti-Hermitian manifolds, or ω_2 *-manifolds, when*

$$\phi = 0,\ F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0, \tag{13}$$

4. the quasi-anti-Kählerian manifolds, or ω_3-*manifolds, when*

$$F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) = 0, \tag{14}$$

5. the complex anti-Hermitian manifolds, or $\omega_1 \oplus \omega_2-$*manifolds, when*

$$F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0, \tag{15}$$

6. the semi-anti-Kählerian manifolds, or $\omega_2 \oplus \omega_3-$ *manifolds, when*

$$\phi = 0, \tag{16}$$

7. the $\omega_1 \oplus \omega_3-$*manifolds, when*

$$n\{F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y)\} = G(X, Y)\phi(Z)+ \\ +G(Z, X)\phi(Y) + G(Y, Z)\phi(X) + G(X, JY)\phi(JZ)+ \\ +G(Y, JZ)\phi(JX) + G(Z, JX)\phi(JY), \tag{17}$$

8. the general class of the almost anti-Hermitian manifolds, or $\omega_1 \oplus \omega_2 \oplus \omega_3-$ *manifolds, when no special condition is imposed.*

The almost anti-Hermitian structure (G, J) on TM from the last class has been obtained in Proposition 4. From this class we shall obtain almost anti-Hermitian structures belonging to some other classes.

First we have the following technical result, obtained by a straightforward computation

Proposition 6. *The expression of* F *and* ϕ *in the case of* (TM, G, J) *are given in the adapted local frame* (∂_i, δ_i) *by*

$$F(\partial_i, \partial_j, \partial_k) = \frac{a_1 v + a_1' u}{a_1^2} g_{0i} g_{jk} + (\frac{b_1 u}{a_1(a_1 + 2tb_1)} + \frac{v + u'}{2a_1})(g_{0j} g_{ik} + g_{0k} g_{ij}) +$$

$$+[\frac{(a_1^2 b_1' - 2a_1 b_1 a_1' - 2a_1 b_1^2 - 2ta_1' b_1^2)(u + 2tv)}{a_1^2 (a_1 + 2tb_1)^2} - \frac{b_1 u'}{a_1(a_1 + 2tb_1)} +$$

$$+\frac{v'}{a_1 + 2tb_1} + \frac{a_1' v}{a_1^2}] g_{0i} g_{0j} g_{0k},\ F(\partial_i, \partial_j, \delta_k) = F(\partial_i, \delta_k, \partial_j) = 0,$$

$$F(\partial_i, \delta_j, \delta_k) = (a_1' u + a_1 v) g_{0i} g_{jk} + \frac{a_1 u' + 2tb_1 u' + a_1 v + 2tb_1 v + 2b_1 u}{2} (g_{0j} g_{ik} +$$

$$g_{0k} g_{ij}) + (a_1' v + a_1 v' + 2tb_1 v' + b_1' u + 2tb_1' v + 3b_1 v) g_{0i} g_{0j} g_{0k},$$

$$F(\delta_i, \partial_j, \delta_k) = F(\delta_i, \delta_k, \partial_j) = \frac{u}{a_1} R_{0ijk} + \frac{b_1 u}{a_1(a_1 + 2tb_1)} g_{0j} R_{0ik0} -$$

$$\frac{u' - v}{2} \{a_1 g_{0j} g_{ik} - (a_1 + 2tb_1) g_{0k} g_{ij} + b_1 g_{0i} g_{0j} g_{0k}\},$$

$$F(\delta_i, \partial_j, \partial_k) = F(\delta_i, \delta_j, \delta_k) = 0,$$

$$\phi(\partial_k) = 0,\ \phi(\delta_k) = \frac{1}{a_1} g^{ij} R_{0ijk} + \frac{n(a_1 u' + 2tb_1 u' + b_1 u)}{u} g_{0k} +$$

$$[\frac{2v(a_1 + 2tb_1)}{u + 2tv} + a_1' + b_1 + \frac{2tv(a_1 + 2tb_1)}{u + 2tv} + 2tb_1' - \frac{2tu' v(a_1 + 2tb_1)}{u(u + 2tv)}] g_{0k}.$$

The condition under which the almost anti-Hermitian manifold (TM, G, J) is anti-Kählerian, i.e. condition $F = 0$ is fulfilled, can be obtained by studying the vanishing the components of the tensor field F. By an easy integration (see [12]), it follows that this condition is equivalent to the property of (M, g) to have constant sectional curvature c and the functions b_1, u and v must be expressed as follows

$$b_1 = \frac{a_1 a_1' - c}{a_1 - 2ta_1'}, \quad u = \frac{Aa_1}{a_1^2 - 2ct}, \quad v = -\frac{Aa_1'}{a_1^2 - 2ct}, \tag{18}$$

where A is a nonzero real constant and a_1 is an arbitrary positive function such that $a_1^2 - 2ct \neq 0$ and $a_1 - 2ta_1' \neq 0$. Remark that the first condition (18) is just the integrability condition (5) of J, presented in Theorem 3.

Hence we may state

Theorem 7. *The almost anti-Hermitian manifold* (TM, G, J) *is an anti-Kählerian manifold if and only if the base manifold* (M, g) *has constant sectional curvature* c *and the functions* b_1, u, v *satisfy relations* (18).

Now we shall study the Case 4 (the quasi anti-Kähler manifolds) since the obtained conditions are quite similar to that obtained in the Case 1. In the Case 4 we have to check the following essential relations obtained from (14)

$$F(\partial_i, \partial_j, \partial_k) + F(\partial_j, \partial_k, \partial_i) + F(\partial_k, \partial_i, \partial_j) = 0,$$

$$F(\delta_i, \partial_j, \delta_k) + F(\partial_j, \delta_k, \delta_i) + F(\delta_k, \delta_i, \partial_j) = 0.$$

Using the first relation, one can express v as a function of a_1, a_1', b_1, u, u'. Then the second relation can be expressed in the following form

$$(R_{hkij} - R_{hijk})y^h = \alpha(g_{jk}g_{0i} - 2g_{ik}g_{0j} + g_{ij}g_{0k}) + \frac{2b_1}{a_1 + 2b_1 t} g_{0j} R_{hikl} y^h y^l,$$

where α is a function expressed with the help of $u, u', a_1, a_1', a_1'', b_1$. After a cyclic permutation of the indices i, j, k, a substraction, then using the first Bianchi identity, one gets

$$y^h R_{hijk} = \beta(g_{ik}g_{0j} - g_{ij}g_{0k}) + *,$$

where β is another function expressed with the help of $u, u', a_1, a_1', a_1'', b_1$ and $*$ denotes a homogeneous polynomial of 3-rd degree in the components y^h.

Differentiating with respect to y^h, then taking $y = 0$, it follows that the manifold (M, g) must have constant sectional curvature c. Then, after some quite long but standard computations we obtain the following relations that must be fulfilled.

$$
\begin{aligned}
&v = -\tfrac{a_1^2 a_1' u + 2ca_1 u + 2cta_1' u - a_1^3 u' + 2cta_1 u'}{2a_1^3 + 4cta_1}, \quad b_1 = \tfrac{cu - a_1^2 u'}{a_1 u + 2ta_1 u'},\\
&u''(a_1^4 u^2 - 2a_1^3 a_1' tu^2 + 2a_1^2 ctu^2 - 4a_1 a' ct^2 u^2) +\\
&+ a_1''(a_1^3 u^3 + 2a_1 ctu^3 + 2a_1^3 tu^2 u' + 4a_1 ct^2 u^2 u') +\\
&+ 4a_1 a_1' cu^3 - 2a_1^3 a_1' u^2 u' + 4a_1^2 cu^2 u' + 4a_1 a_1' ctu^2 u' -\\
&- 4a_1'^2 ctu^3 - 8a_1'^2 ct^2 u^2 u' - 4a_1^4 uu'^2 + 2a_1^3 a_1' tuu'^2 +\\
&+ 4a_1^2 ctuu'^2 + 4a_1 a_1' ct^2 uu'^2 - 2a_1^4 tu'^3 + 4a_1^2 ct^2 u'^3 = 0.
\end{aligned} \tag{19}
$$

The last relation could be thought of as a differential equation of second order in u, where the function a_1 is considered as a parameter. There are no essential restrictions in choosing the function a_1; however, recall that we must have $a_1 > 0, a_1 + 2tb_1 > 0$. Remark that the function u given in (18) is a solution of this differential equation, depending on one integration constant. In this case, one obtains from (19) the same expressions (18) for b_1 and v. However the general solution u of the last differential equation (19) should depend on two arbitrary constants, hence it is more general than that obtained in (18). Hence we may state

Theorem 8. *The almost anti-Hermitian manifold* (TM, G, J) *is quasi-anti-Kählerian if and only if the base manifold* (M, g) *has constant sectional curvature* c *and the functions* b_1, u, v *satisfy relations* (19).

In the following we study the situation when condition (15) is fulfilled (the case 5, where (TM, G, J) is a complex anti-Hermitian manifold). The following two essential relations are obtained

$$
\begin{gathered}
F(\partial_i, \partial_j, \partial_h)(a_1\delta_k^h + b_1 y^h g_{0k}) - F(\partial_j, \delta_k, \delta_h)(a_2\delta_i^h + b_2 y^h g_{0i}) -\\
- F(\delta_k, \partial_i, \delta_h)(a_2\delta_j^h + b_2 y^h g_{0j}) = 0,\\
F(\delta_i, \delta_j, \partial_h)(a_1\delta_k^h + b_1 y^h g_{0k}) + F(\delta_j, \delta_k, \partial_h)(a_1\delta_i^h + b_1 y^h g_{0i}) +\\
+ F(\delta_k, \delta_i, \partial_h)(a_1\delta_j^h + b_1 y^h g_{0j}) = 0.
\end{gathered}
$$

After a straightforward computation one obtains that the second relation is identically fulfilled. The first one is fulfilled if and only if the base manifold (M, g) has constant sectional curvature c and the function a_1, b_1 are related by the relation

$$
b_1(a_1 - 2ta_1') = a_1 a_1' - c. \tag{20}
$$

Hence we may state

Theorem 9. *The almost anti-Hermitian manifold* (TM, G, J) *is a complex anti-Hermitian manifold if and only if the base manifold* (M, g) *has constant sectional curvature* c *and the functions* a_1, b_1 *are related by* (20).

Remark. The obtained result is just the result of the Theorem 3. This fact is quite natural since in the integrability condition for the almost complex structure J there are not involved the functions u, v which are used in order to obtain the semi-Riemannian metric G.

Now, we study condition (16) (the case 6) fulfilled by the anti-Hermitian manifold (TM, G, J) when it is a semi-anti-Käherian manifold. From the local expression of ϕ it follows that that the condition $\phi = 0$ is equivalent to the property of the base manifold M to be an Einstein manifold, i.e. the Ricci tensor satifies $R_{hk} = \kappa g_{hk}$, where κ is a constant. Moreover, the functions a_1, b_1, u, v must satisfy the relation

$$
\begin{aligned}
(21)\qquad & a_1a_1'u^2 + a_1b_1u^2 - \kappa u^2 + 2ta_1b_1'u^2 + 2a_1^2uv + 2ta_1a_1'uv+ \\
& +6ta_1b_1uv - 2\kappa tuv + 4t^2a_1b_1'uv - 2ta_1^2u'v - 4t^2a_1b_1u'v+ \\
& +2ta_1^2uv' + 4t^2a_1b_1uv' + a_1n(b_1u + a_1u' + 2tb_1u')(u + 2tv) = 0.
\end{aligned}
$$

Hence we have

Theorem 10. *The almost anti-Hermitian manifold* (TM, G, J) *is a semi-anti-Kählerian manifold if and only if the base manifold* (M, g) *is an Einstein manifold and the functions* a_1, b_1, u, v *satisfy relation* (21).

Taking into account that condition (16) is, in fact, the first condition in (13) and condition (15) is, in fact, the second condition in (13), combining the above results in the cases 5 and 6, we obtain the following characterization of special complex anti-Hermitian manifolds (the case 3).

Theorem 11. *The almost anti-Hermitian manifold* (TM, G, J) *is a special complex anti-Hermitian manifold if and only if the base manifold* (M, g) *has constant sectional curvature* c *and the functions* a_1, b_1, u, v *satisfy the following relations*

$$b_1 = \frac{a_1a_1' - c}{a_1 - 2ta_1'},$$

$$n(a_1 - 2ta_1')(a_1^2a_1'u - 2ca_1u + 2cta_1'u + a_1^3u' - 2cta_1u')(u + 2tv)+$$

$$2(a_1^2-2ct)(a_1a_1'u^2-t(a_1')^2u^2+ta_1a_1''u^2+a_1^2uv-2t^2(a_1')^2uv+2t^2a_1a_1''uv-$$
$$ta_1^2u'v+2t^2a_1a_1'u'v+ta_1^2uv'-2t^2a_1a_1'uv')=0.$$

Now, we consider condition (12) which must be fulfilled by (TM,G,J) in order to be a conformally anti-Kählerian manifold (the case 2). The essential relations, obtained from (12), are

$$2nF(\partial_i,\delta_j,\delta_k)-(G_{ij}\phi(\delta_k)+G_{ik}\phi(\delta_j))=0,$$
$$2nF(\delta_i,\partial_j,\delta_k)-(G_{ij}\phi(\delta_k)-G_{ih}(a_1\delta_k^h+b_1y^hg_{0k})\phi(\delta_l)(a_2\delta_j^l+b_2y^lg_{0j}))=0,$$
$$2nF(\partial_i,\partial_j,\partial_k)-(G_{ih}(a_2\delta_j^h+b_2y^hg_{0j})\phi(\delta_l)(a_2\delta_k^l+b_2y^lg_{0k})+$$
$$+G_{ih}(a_2\delta_k^h+b_2y^hg_{0k})\phi(\delta_l)(a_2\delta_j^l+b_2y^lg_{0j}))=0.$$

It follows by a straightforward computation that these relations are satisfied if and only if the base manifold (M,g) has constant sectional curvature c and the functions a_1,b_1,u,v satisfy the following relations

$$b_1=\frac{a_1a_1'-c}{a_1-2ta_1'},\ v=-\frac{a_1'u}{a_1}. \tag{22}$$

Hence we state

Theorem 12. *The almost anti-Hermitian manifold (TM,G,J) is a conformally anti-Kählerian manifold if and only if the base manifold (M,g) has constant sectional curvature c and the functions a_1,b_1,u,v satisfy relations* (22).

Remark. The functions u,a_1 are quite arbitrary. However, they must satisfy the usual conditions $a_1>0,a_1+2tb_1>0,u\neq 0,u+2tv\neq 0$.

Finally, we study condition (17) (the case 7) which must be fulfilled by the almost anti-Hermitian manifold (TM,G,J) in order to be an $\omega_1\oplus\omega_3-$manifold. We get by a straightforward computation that this condition is fulfilled if and only if the base manifold (M,g) has also constant sectional curvature c and the functions a_1,b_1,u,v satisfy relations

$$v=-\frac{a_1a_1'u+a_1b_1u+cu+2ta_1'b_1u}{2a_1(a_1+2tb_1)}, \tag{23}$$

$$b_1'=\frac{a_1^3a_1''+3a_1^2a_1'b_1-a_1^2b_1^2+c^2-2ta_1(a_1')^2b_1+4ta_1^2a_1''b_1}{a_1(a_1-2ta_1')(a_1+2tb_1)}+$$

$$\frac{6ta_1a_1'b_1^2 - 4t^2(a_1')^2b_1^2 + 4t^2a_1a_1''b_1^2}{a_1(a_1 - 2ta_1')(a_1 + 2tb_1)}.$$

So, we have

Theorem 13. *The almost anti-Hermitian manifold* (TM, G, J) *is an* $\omega_1 \oplus \omega_3-$*manifold if and only if the base manifold* (M, g) *has constant sectional curvature* c *and the functions* a_1, b_1, u, v *satisfy relations* (23).

4. Some final remarks.

(i) The family of general natural almost anti-Hermitian structures on the tangent bundle of a Riemannian manifold and the integrability conditions of these structures have been studied by the first author in [12]

(ii) The case 1, i.e. the condition under which the considered almost anti-Hermitian manifold (TM, G, J) is anti-Kählerian (Theorem 7), has been also obtained by the first author in [12].

(iii) The particular case from this paper where the functions a_1, b_1, u, v satisfy the conditions $a_1 = u$ and $b_1 = v$ has been studied by the second author in [17]. He has obtained specific examples only for five from the seven essential (the general case is not considered) classes of almost anti-Hermitian manifolds (almost complex manifolds with Norden metric).

(iv) In this paper we obtained specific examples (distinguished characterizations) for all seven classes of almost anti-Hermitian manifolds obtained in the classification from [1].

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