ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul L, s.I a, Matematică, 2004, f.1.

SEMI-DYNAMICAL SYSTEMS ON CONES OF MEASURES

BY

EUGEN POPA

1. Abstract. The purpose of this short communication is to present a few results about semi–groups acting on measures. As such, this is a particular case of the notion of semi–dynamical system on algebraic structure (see:

E. Popa, Semi–dynamical systems on algebraic structures, An. şt. Univ. Iaşi t. XLIV, sIa, Matematica, (1998), Supliment, pg. 585–596)

namely an ordered convex cone X and a map $\Phi : (0, \infty) \times X \to X$ satisfying the semi–group property $\Phi(s+t, x) = \Phi(s, \Phi(t, x))$ and some other compatibility conditions. Each map $x \to \Phi(t, x)$ is a "kernel" on X: this notion of kernel, together with the corresponding notion of a resolvent, was first considered by:

A. Cornea, G. Licea: **Order and Potential. Resolvent Families of Kernels**, Lecture Notes in Math. vol. 494, Springer-Verlag 1975.

We are concerned now only with case $X=$ the/an ordered convex cone of positive measures on a measurable space. Let us recall that each true kernel (i.e. on functions) defines a kernel on measures, but the converse is not true. Characterization for this situation is provided, as well as for the important case of absolutely continuous kernels.

The main result presents a sufficient condition for a resolvent to have an associated semi–group of kernels on measures (a Hiile–Yosida type result). Such a result was already proved in

E. Popa: Resolvents associated with semi-dynamical systems in duality, to appear in An. şt. Univ. Iaşi

but in a duality setting. Hence, instead of the unsatisfactory duality between functions and measures, we rather use the fact that the cone of (positive) measures is a dual. The Ray type condition is considered here in its classical form: as a separation (of the compact base space) by some excessive like functions, which are specific to the case of kernels on measures.

The results from this paper were announced in [6].

2. Subcones and quotient cones of measures. Let $(A, \mathcal{A}, \mu)$ be a space with positive measure. We present some subcones and quotient cones of measures which will be used in what follows. $\mathcal{F}$ resp. $\mathcal{M}$ stand for the ordered convex cones of positive, measurable functions; resp. positive measures on $(A, \mathcal{A})$.

We denote:

$$\mathcal{M}_\mu := \{\nu \in \mathcal{M} | \nu = \bigvee_n (\nu \wedge n\mu)\}$$

$\mathcal{M}_\mu$ is a solid convex subcone in $\mathcal{M}$, since:

$$(\nu + \nu') \wedge 2n\mu \geq \nu \wedge n\mu + \nu' \wedge n\mu$$

and:

$$\bigvee_n \nu \wedge \mu_n = \nu \wedge \bigvee_n \mu_n$$

(for any increasing sequence (ν_n) from $\mathcal{M}_\mu$).

More generally, let $\mathcal{N} \subseteq \mathcal{A}$ a family of "negligible" subsets, i.e.:

$$\emptyset \in \mathcal{N}$$

$$N_1 \subseteq N_2 \in \mathcal{N} \Longrightarrow N_1 \in \mathcal{N}$$

$$(N_n) \text{ arbitrary sequence from } \mathcal{N} \Longrightarrow \bigcup_n N_n \in \mathcal{N}$$

We associate:

$$\mathcal{M}_\mathcal{N} := \{\nu \in \mathcal{M} | \nu(A) = 0, \forall A \in \mathcal{N}\}$$

For a σ–finite measure μ, we have $\mathcal{M}_\mu = \mathcal{M}_\mathcal{N}$, if $\mathcal{N}$ denotes the family of μ–negligible sets.

In general, we have:

$$\{\nu \in \mathcal{M} | \exists f \in \mathcal{F} \text{ such that } \nu = f.\mu\} \subseteq \{\nu \in \mathcal{M} | \nu = \bigvee_n (\nu \wedge n\mu)\} \subseteq$$

$$\subseteq \{\nu \in \mathcal{M} | \mu(A) = 0 \Longrightarrow \nu(A) = 0\}$$

From Radon–Nicodym's theorem, we get the equality for the case of a σ–finite measure μ; if μ is only archimedian ([1]), then the σ–finite measures from the three sets are the same.

For $\nu \in \mathcal{M}_\mu$ (especially for $\nu \leq \mu$) there is a canonical embedding $\mathcal{M}_\nu \hookrightarrow \mathcal{M}_\mu$. Indeed, let $\nu' \in \mathcal{M}_\nu$, hence $\nu' = \bigvee_n \nu' \wedge n\nu$. Then:

$$\bigvee_n \nu' \wedge n\mu = \bigvee_n \left[\bigvee_m \nu' \wedge m\nu\right] \wedge n\mu = \bigvee_{n,m} \nu' \wedge m\nu \wedge n\mu = \bigvee_{m,k} \nu' \wedge m(\nu \wedge k\mu) =$$

$$= \bigvee_m \nu' \wedge m\nu = \nu'$$

Since $\mathcal{M} = \bigcup_{\mu \in \mathcal{M}} \mathcal{M}_\mu$, we obtain that $\mathcal{M} = \text{limind}_\mu \mathcal{M}_\mu$.

We have to consider also some subcones/quotient cones of the cone of positive, measurable functions. We define an equivalence relation $f \sim g$ iff $\mu\{[f \neq g]\} = 0$. Let us denote $\mathcal{F}|_\mu$ the set of equivalence classes and by $\pi_\mu : \mathcal{F} \to \mathcal{F}|_\mu$ the canonical projection. $\mathcal{F}|_\mu$ is naturally organized as a convex ordered cone of positive elements and also a lattice, in which the supremum for every increasing sequence exists; obviously, the canonical projection commutes with these operations.

More generally, we may associate with every part of "negligible" sets $\mathcal{N}$ an equivalence relation, hence a quotient cone $\mathcal{F}_\mathcal{N}$. In this way, we include also the case of a family of measures.

For each $\nu \in \mathcal{M}_\mu$, there exists a canonical, surjective application $\pi_{\mu\nu}$: $\mathcal{F}|_\mu \to \mathcal{F}|_\nu$ (since $\mu(A) = 0 \Rightarrow \nu(A) = 0$). Thus, we get a projective system $(\mathcal{F}|_\mu)$ and $\mathcal{F} =$ limproj ${}_\mu\mathcal{F}|_\mu$ (even for a family of measures μ, for which: $\mu(|f - g|) = 0$, $\forall \mu \Rightarrow f = g$, for example when all the atomic measure belong to the family).

Example. Let $\mu := \sum_{n \in I} \alpha_n \varepsilon_{x_n}$ (with $\alpha_n > 0$, $\forall n \in I$), then $\mathcal{F}|_\mu \equiv [0, +\infty]^I$. In this case, $\mathcal{M}_\mu$ contains exactly the measures of the form: $\nu = \sum_{n \in I} \beta_n \varepsilon_{x_n}$, for $\beta_n \in [0, +\infty)$. Hence $\mathcal{M}_\mu \equiv [0, +\infty)^I$.

If $f \sim g$, then clearly $f.\mu = g.\mu$, hence the correspondence $\mathcal{F}|_\mu \to \mathcal{M}_\mu$, given by $\hat{f} \mapsto f.\mu$ is well defined. Let us suppose that μ is σ–finite. The Radon–Nicodym's theorem shows that this correspondence is surjective. It is in fact an order preserving bijection. Indeed, let $f.\mu \leq g.\mu$. If we denote $A_n := \left[f \geq g + \frac{1}{n}\right]$, then $A := \bigcup_n A_n = [f > g]$. We get thus $\int_{A_n} f d\mu \geq \int_{A_n} g d\mu + \frac{1}{n}\mu(A_n)$. If $\int_{A_n} g d\mu < +\infty$, we get the conclusion $\hat{f} \leq \hat{g}$.

The dual of the cone $\mathcal{F}|_\mu$ is naturally identified with $\mathcal{M}_\mu$, in the case μ σ–finite. Indeed, for $\varphi \in (\mathcal{F}|_\mu)^*$, we define $\hat{\varphi}$ on $\mathcal{F}$ as: $\hat{\varphi}(f) := \varphi(\hat{f})$. Clearly one obtains a measure $\hat{\varphi}$, with the property $\mu(A) = 0 \Longrightarrow \hat{\varphi}(A) = 0$ (since $\hat{\chi_A} = 0$). Hence $\hat{\varphi} \in \mathcal{M}_\mu$.

We have repeatedly used the following known result:

Lemma 1. *Let (μ_i) an increasing family of positive measures. Then:*

$$\left(\bigvee_i \mu_i\right) \wedge \nu = \bigvee_i (\mu_i \wedge \nu)$$

Proof. Let $A \in \mathcal{A}$ and $\varepsilon > 0$. Let us denote $\mu := \bigvee_i \mu_i$. There exists i such that $\mu(A) \leq \mu_i(A) + \varepsilon$. The same relation extends to each $B \in \mathcal{A}$, $B \subseteq A$:

$$\mu(B) + \mu(A \setminus B) \leq \mu_i(B) + \mu_i(A \setminus B) + \varepsilon$$

$$\mu(B) - \mu_i(B) \leq [\mu_i(A \setminus B) - \mu(A \setminus B)] + \varepsilon$$

(the case $\mu(A) = +\infty$ is to be treated separately). Using the formula [1] [t.1.2.9., pg. 52]:

$$(\mu \wedge \nu)(A) \leq \mu(A \cap A_i'') + \nu(A \cap A_i') \leq \mu_i(A \cap A_i'') + \varepsilon + \nu(A \cap A_i') =$$

$$= (\mu_i \wedge \nu)(A) + \varepsilon \leq \varepsilon + \sup_i (\mu_i \wedge \nu)(A)$$

3. Kernels on measures, examples.

Definition. By a kernel on $\mathcal{M}$ we mean an application $V : \mathcal{M} \to \mathcal{M}$ with the following properties:

$$V(\mu + \nu) = V\mu + V\nu$$

$$\mu \leq \nu \Longrightarrow V\mu \leq V\nu$$

$$\mu_n \nearrow \mu \Longrightarrow V\mu_n \nearrow V\mu$$

This last property is obviously equivalent in this case with the next one:

$$V\left(\sum_{n=1}^{\infty} \mu_n\right) = \sum_{n=1}^{\infty} \mu_n$$

and if $\mathcal{A}$ is countably generated, also with this one:

$$\mu_i \nearrow \mu \Longrightarrow V\mu_i \nearrow V\mu$$

As a specific property, we shall consider:

$$(M) \;\; x \mapsto V(\varepsilon_x)(A') \text{ is measurable } \forall A' \in \mathcal{A}$$

(see [3]).

This is exactly the measurability assumption for the kernel V, with respect to the dual pair $[\mathcal{F}, \mathcal{M}]$ (see [5]).

$\mathcal{M}$ may stand also for another ordered convex subcone of measures (for example: σ–finite measures or positive, Radon measures on a compact space).

It will be however assumed that $\mathcal{M}$ is

specifically solid: $\mu, \nu \in \mathcal{M}, \mu \leq \nu \Longrightarrow \nu - \mu \in \mathcal{M}$;

and a σ sup–lattice: for any increasing sequence (μ_n) from $\mathcal{M}$, $\bigvee_n \mu_n \in \mathcal{M}$ (we use the notation $\mu_n \nearrow \mu$, where $\mu := \bigvee_n \mu_n$).

With each true kernel $V : \mathcal{F} \to \mathcal{F}$ (on functions) one associates classically a kernel on measures $\tilde{V} : \mathcal{M} \to \mathcal{M}$ through

$$\tilde{V}(\mu)(f) := \mu\,(Vf)$$

$\tilde{V}$ has the properties of a kernel, including (M).

Conversely, let $V : \mathcal{M} \to \mathcal{M}$ be a kernel on measures with the property (M); then one defines $\hat{V} : \mathcal{F} \to \mathcal{F}$ as: $\hat{V}f(x) := V(\varepsilon_x)(f)$. This definition is correct and the measurability of the function $\hat{V}(f)$ follows from (M).

Hence, starting with a kernel $V : \mathcal{F} \to \mathcal{F}$ on functions, we may construct the kernel $\tilde{V}$ on measures; since it has the property (M), we further construct the kernel $\hat{\tilde{V}}$ on functions and we recover V; indeed:

$$\hat{\tilde{V}}f(x) = \tilde{V}(\varepsilon_x)(f) = \varepsilon_x(Vf) = Vf(x)$$

$(\forall f \in \mathcal{F}, x \in A)$.

However, starting with a kernel $V : \mathcal{M} \to \mathcal{M}$ on measures, possessing (M), we consider the kernel on functions $\tilde{\hat{V}}$, but the equality $\tilde{\hat{V}}(\mu) = V\mu$ holds only for atomic measures μ (and does not extends necessarily for other measures).

This shows that we cannot expect to recover a good adjoint only from the measurability assumption.

By $\mu_i \to \mu$ we understand that $\mu_i(f) \to \mu(f)$, for all $f \in \mathcal{F}$ bounded.

Proposition 2. *(i) Let V be a sub–markovian kernel on functions. Then $\mu_i \to \mu \Longrightarrow \tilde{V}\mu_i \to \tilde{V}\mu$.*

(ii) Let V be a kernel on measures, with the property (M) and such that:

$$\mu_i \to \mu \Longrightarrow V\mu_i \to V\mu$$

Then $\tilde{\hat{V}}\mu = V\mu$ for any σ–finite measure μ.

Proof. (i) For each bounded function f we have:

$$\tilde{V}\mu_i(f) = \mu_i(Vf) \to \mu(Vf) = \tilde{V}\mu(f)$$

since Vf is also bounded.

(ii) We have already remarked that the equality $\tilde{\tilde{V}}(\mu) = V(\mu)$ holds for all atomic measures μ. For each finite measure μ, there exists a net of atomic measures μ_i such that $\mu_i \to \mu$.

Let us define the ordered set I as follows: $i \in I$ is given by: a natural number N; a finite partition $(A_n)_{1\leq n\leq N}$ of A with elements from $\mathcal{A}$, together with a choice of points $x_n \in A_n$, $\forall 1 \leq n \leq N$; if we define $i \leq j$ if every part from i is partitioned with elements from j, I becomes increasing: given $i, j \in I$, we define k as the partition with the sets $A_n \cap B_m$.

For each finite measure μ and any choice of the points $x_n \in A_n$, we define the net of atomic measures:

$$\mu_i := \sum_{n=1}^{N} \mu(A_n).\varepsilon_{x_n}$$

This net converges to μ: let $f \in \mathcal{F}$ be bounded and let $\varepsilon > 0$. We find a partition $i \in I$ for which $\forall n\ x, y \in A_n \Longrightarrow |f(x) - f(y)| < \varepsilon$. For any finer partition j we have:

$$|\mu_j(f) - \mu(f)| = \left| \sum_{m=1}^{M} \mu(B_m) f(y_m) - \sum_{m=1}^{M} \int_{B_m} f.d\mu \right| \leq$$

$$\leq \sum_{m=1}^{M} \int_{B_m} |f(y_m) - f(x)|.\mu(dx) < \varepsilon.\mu(1)$$

(see [3] [ch. XI, 8, pg.275]).

Hence $\tilde{\tilde{V}}\mu_i = V\mu_i$ implies, by hypothesis, the asserted equality for finite measures. The extension to σ–finite measures is immediate.

Remark. We may consider functions $f \in \mathcal{F}$, which are finite, except for a countable set; in this case, we have to consider countable partitions.

The sub–markovian condition may also be weakened to: f finite $\Longrightarrow$ $V\varepsilon_x(f) < +\infty$ (except for a countable set).

In order to study the absolutely continuous kernels, we make the following remarks.

Let $V : \mathcal{F} \to \mathcal{F}$ be a kernel. For each measure μ, it naturally defines a kernel $\mathcal{F}|_{V\mu} \to \mathcal{F}|_{\mu}$: if $f = g$ $V\mu$–a.e., then $V\mu(f-g) = 0 \Leftrightarrow \mu(Vf-Vg) = 0 \Leftrightarrow Vf = Vg$, μ–a.e.

It also defines a kernel on measures: $\mathcal{M}_{V\mu} \to \mathcal{M}_{\mu}$. Indeed, let $\nu \in \mathcal{M}_{V\mu}$, hence $\nu = f.(V\mu)$, where $f \in \mathcal{F}|_{V\mu}$. Then $(Vf).\mu \in \mathcal{M}_{\mu}$.

On the other hand, each kernel on measures $V : \mathcal{M} \to \mathcal{M}$ acts (by restriction) as $\mathcal{M}_{\mu} \to \mathcal{M}_{V\mu}$. Indeed, for $\nu \in \mathcal{M}_{\mu}$, from $\nu = \bigvee_n (\nu \wedge n\mu)$ we get:

$$V\nu \geq V\nu \wedge n.V\mu \geq \bigvee_n (\nu \wedge n\mu)$$

hence

$$V\nu \geq \bigvee_n (V\nu \wedge n.V\mu) \geq \bigvee_n V(\nu \wedge n\mu) = V\left(\bigvee_n (\nu \wedge n\mu\right) = V\nu$$

proving thus that $V\nu \in \mathcal{M}_{V\mu}$.

We have thus the following commuting diagrams:

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{V} & \mathcal{M} \\
\uparrow & & \uparrow \\
\mathcal{M}_{\mu} & \xrightarrow{V} & \mathcal{M}_{\mathcal{V}\mu} \\
\uparrow & & \uparrow \\
\mathcal{M}_{\nu} & \xrightarrow{V} & \mathcal{M}_{\mathcal{V}\nu}
\end{array}$$

(where $\nu \leq \mu$). However, $\mathcal{M}$ is **not** necessarily the inductive limit of the cones $\mathcal{M}_{V\mu}$.

If μ is V–excessive (i.e. $V\mu \leq \mu$), then composing with the canonical inclusion $\mathcal{M}_{V\mu} \to \mathcal{M}_{\mu}$, we obtain a kernel $\mathcal{M}_{\mu} \to \mathcal{M}_{\mu}$ (and correspondingly, a "pseudo–kernel" $\mathcal{F}|_{\mu} \to \mathcal{F}|_{\mu}$).

We obtain again commuting diagrams:

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{V} & \mathcal{M} \\
\uparrow & & \uparrow \\
\mathcal{M}_{\mu} & \xrightarrow{V} & \mathcal{M}_{\mu} \\
\uparrow & & \uparrow \\
\mathcal{M}_{\nu} & \xrightarrow{V} & \mathcal{M}_{\nu}
\end{array}$$

This time again, $\mathcal{M}$ is **not** necessarily the inductive limit of the cones $\mathcal{M}_\mu$, as μ goes only through the V–excessive measures.

Let $V : \mathcal{F} \to \mathcal{F}$ be a kernel. We call V absolutely continuous with respect to the (positive) measure μ if: $f \in \mathcal{F}$ and $\mu(f) = 0 \Rightarrow Vf \equiv 0$.

Proposition 3. *The following properties are equivalent:*
(i) V is absolutely continuous with respect to μ;
(ii) $V(\mathcal{M}) \subseteq \mathcal{M}_\mu$.
The following properties are equivalent:
a) V is absolutely continuous;
b) $V(\mathcal{M})$ has a weak unit.

Proof. (i) $\Rightarrow$ (ii). Let $f \in \mathcal{F}$ be such that $\mu(f)$; for any $\nu \in \mathcal{M}$, we have $(V\nu)(f) = \nu(Vf) = 0$, hence $V\nu \in \mathcal{M}_\mu$.

(ii) $\Rightarrow$(i) If $\mu(f) = 0$, we have $\nu(f) = 0$, for any $\nu \in \mathcal{M}_\mu$. In particular, for each $x \in A$ we have $V\varepsilon_x \in \mathcal{M}_\mu$, hence $V(f)(x) = 0$.

a)$\Rightarrow$ b). Let V be absolutely continuous with respect to μ. Let us denote $\mu_0 := \bigvee \nu$, for $\nu \in V(\mathcal{M})$ and $\nu \leq \mu$. If $\mathcal{A}$ is separable, then $\mu_0 \in V(\mathcal{M})$ and μ_0 is a weak unit.

b) $\Rightarrow$ a) Clearly, if μ_0 is a weak unit in $V(\mathcal{M})$, then V is absolutely continuous with respect to μ_0.

Example (see [2]). Let $\mu \in \mathcal{M}$ and define:

$$B_\mu(\nu) := \bigvee_n (\nu \wedge n.\mu)$$

($B_\mu \nu$ is the μ–absolutely continuous part ν).

Proposition 4. *B_μ is a kernel on measures.*

Proof. The following description for B_μ holds:

$$B_\mu(\nu) = \bigvee \{\mu' | \mu' \leq \nu, \mu' \text{ absolutely continuous with respect to } \mu\}$$

Indeed, let us denote by m the measure in the second member. Clearly $B_\mu(\nu) \leq m$. On the other hand:

$$\mu' = \bigvee_n (\mu' \wedge n.\mu) \leq \bigvee_n (\nu \wedge n.\mu) = B_\mu(\nu)$$

hence $m \leq B_\mu(\nu)$.

Now we have:

$B_\mu(\nu + \nu') = B_\mu(\nu) + B_\mu(\nu')$.

Indeed, for $\mu' \leq \nu$, $\mu'' \leq \nu'$ such that μ' and μ'' are μ–absolutely continuous, we have $\mu' + \nu' \leq \mu + \nu$ and $\mu' + \nu'$ is μ–absolutely continuous. Hence $B_\mu(\mu + \nu) \geq B_\mu \mu + B_\mu \nu$.

Conversely, let $\omega' \leq \mu + \nu$, ω' μ–absolutely continuous. Hence $\omega' = \mu' + \nu'$ with $\mu' \leq \mu$, $\nu' \leq \nu$. It follows that μ', ν' are μ–absolutely continuous.

Clearly

$\nu \leq \nu' \Longrightarrow B_\mu(\nu) \leq B_\mu \nu'$;

Hence, if $\nu_i \nearrow \nu$, then $B_\mu(\nu_i)$ is increasing and $\bigvee_i B_\mu(\nu_i) \leq B_\mu\left(\bigvee_i \nu_i\right)$. Conversely, let $\nu' \leq \nu$ be such that ν' is μ–absolutely continuous. For each i: $\nu' \wedge \nu_i \leq \nu_i$ and $\nu' \wedge \nu_i$ is μ–absolutely continuous. Hence $\nu' \wedge \nu_i \leq B_\mu(\nu_i)$, $\forall i$. But:

$$\nu' = \nu' \wedge \nu = \nu' \wedge \left(\bigvee_i \nu_i\right) = \bigvee_i (\nu \wedge \nu_i) \leq \bigvee_i B_\mu(\nu_i)$$

hence $B_\mu(\nu) \leq \bigvee_i B_\mu(\nu_i)$.

Moreover, B_μ has the property (M), at least for σ–finite μ: in this case, the set $\{x | \mu(\{x\}) > 0\}$ is at most countable.

Properties. (i) B_μ comes from a kernel on functions iff μ is atomic.

(ii) $B_\mu \nu = \nu \Longleftrightarrow \nu$ is μ–absolutely continuous.

(iii) $B_\mu \leq I$

(iv) $B_\mu^2 = B_\mu$

(v) $B_\mu(\nu \wedge \nu') = \nu \wedge B_\mu(\nu') = B_\mu(\nu) \wedge \nu'$

(vi) $B_\mu + B_{\mu'} = B_{\mu+\mu'} + B_{\mu\wedge\mu'}$ (particularly, if $\mu \wedge \mu' = 0$ then $B_{\mu+\mu'} = B_\mu + B_{\mu'}$).

(vii) $\mu \leq \nu \Longrightarrow B_\mu \leq B_\nu$

(viii) $\mu_n \nearrow \mu \Longrightarrow B_{\mu_n} \nearrow B_\mu$

(ix) $B_\mu \circ B_\nu = B_{\mu\wedge\nu}$

(x) The kernels V on functions, which are μ–absolutely continuous, are characterized as: $B_\mu \circ V = V$.

Proof. (v) Indeed:

$\tilde{B_\mu}(f)(x) = B_\mu(\varepsilon_x)(f) = \mu(\{x\}) f(x)$

hence, if we denote by ν the atomic part of μ, we have: $\hat{\tilde{B}}_\mu = B_\nu$.

(vi) Let us denote:

$\mu_n := \nu \wedge n.\mu + \nu \wedge n.\mu' = 2\nu \wedge (\nu + n.\mu') \wedge (\nu + n.\mu) \wedge n.(\mu + \mu')$

and:

$\mu'_n = \nu \wedge n.(\mu+\mu') + \nu \wedge n.\mu \wedge n.\mu' = 2\nu \wedge (\nu + n.\mu) \wedge (\nu + n.\mu') \wedge (2n.\mu + n.\mu') \wedge (n.\mu + 2n.\mu')$

We have $\mu_n \leq \mu'_n$ and $\mu'_n \leq \mu_{2n}$, hence the conclusion.

We shall consider next the weak integrals of functions with values in a cone of (positive) measures. The draw–back of the weak integral is that the bi–dual is not identified with an explicit cone.

For kernels on functions, the commutation with the weak integral holds:

$$\left[\int_A (V \circ \varphi)(a) d\mu(a)\right](f) = \int_A (V \circ \varphi)(a)(f) d\mu(a) = \int_A \varphi(a)[V(f)] d\mu(a) =$$

$$= \left[\int_A \varphi(a) d\mu(a)\right](Vf) = V\left[\int_A \varphi(a) d\mu(a)\right](f)$$

Let us formulate a sufficient condition for commutation in the case of kernels on measures. We define two maps:

$R : \mathcal{M}^{**} \to \mathcal{M}$ as $(R\alpha)(f) := \alpha(\hat{f})$, where $\alpha \in \mathcal{M}$, $f \in \mathcal{F}$ and $\hat{f} \in \mathcal{M}^*$ being given by $\hat{f}(\nu) := \nu(f)$, $\forall \nu \in \mathcal{M}$;

$I : \mathcal{M} \to \mathcal{M}^{**}$ as $I(\mu)(\alpha) := \alpha(\mu)$, $\forall \alpha \in \mathcal{M}^*$.

We have $R \circ I = 1_{\mathcal{M}}$, but $I \circ R \neq 1_{\mathcal{M}^{**}}$. Also, for any kernel $V : \mathcal{M} \to \mathcal{M}$ we have $R \circ V^{**} \circ I = V$. However, $R \circ V^{**} = V \circ R$ seems to be an additional assumption on V and implies the commutation with the integral. The kernels on functions do satisfy this relation:

$$R \circ V^{**}(\alpha)(f) = V^{**}(\alpha)(\hat{f}) = \alpha(V^* \hat{f}) = (V \circ R)(\alpha)(f), \forall \alpha \in \mathcal{M}^{**}, f \in \mathcal{F}$$

As a conclusion, the condition to commute with the integral may be replaced in the following situations:

a) with the sufficient condition $R \circ V^{**} = V \circ R$. In particular, one may consider the semi–group/resolvent on the bidual $\mathcal{M}^{**}$;

let us remark that such a condition may be considered not only for $\mathcal{M}$, but for any dual.

b) one may restrict the semi–group/resolvent to the cone $\mathcal{M}_{\mathcal{P}}$ / $\mathcal{M}_{\mathcal{V}}$, which provides directly with measures and the commutation holds.

4. Excessive functions and measures with respect to a resolvent or a semi–group on measures. With a semi–group $\mathcal{P} = (P_t)_{t>0}$ of kernels on measures, we associate the following cones of (positive, numerical, measurable) functions:

$\mathcal{S}^1_{\mathcal{P}}$: all the functions f, for which $t \mapsto P_t\mu(f)$ is decreasing, $\forall \mu$;
$\mathcal{S}^2_{\mathcal{P}}$: all the functions f, for which $t \mapsto P_t\varepsilon_x(f)$ is decreasing, $\forall x$;
$\mathcal{S}^3_{\mathcal{P}}$: all the functions f, for which $P_t\mu(f) \leq \mu(f)$, $\forall \mu$, $\forall t > 0$;
$\mathcal{S}^4_{\mathcal{P}}$: all the functions f, for which $P_t\varepsilon_x(f) \leq f(x)$, $\forall x$, $\forall t > 0$.

As usual, for each $\varepsilon > 0$, we define a new semi–group $\mathcal{P}^\varepsilon$ as: $P^\varepsilon_t = e^{-\varepsilon t}P_t$.

The following relations hold: $\mathcal{S}^1_{\mathcal{P}} = \bigcap_{\varepsilon>0} \mathcal{S}^1_{\mathcal{P}^\varepsilon}$. Indeed, for $t < s$ we have:

$$P_t\mu(f) \geq P_s\mu(f) \Longrightarrow e^{-\varepsilon t}P_t\mu(f) \geq e^{-\varepsilon t}P_s\mu(f) \geq e^{-\varepsilon s}P_s\mu(f)$$

Conversely, from $e^{-\varepsilon t}P_t\mu(f) \geq e^{-\varepsilon s}P_s\mu(f)$ we get the conclusion by making $\varepsilon \searrow 0$. As a special case, we obtain: $\mathcal{S}^2_{\mathcal{P}} = \bigcap_{\varepsilon>0} \mathcal{S}^2_{\mathcal{P}^\varepsilon}$.

Also $\mathcal{S}^1_{\mathcal{P}} \subseteq \mathcal{S}^2_{\mathcal{P}}$ and $\mathcal{S}^1_{\mathcal{P}^\varepsilon} \subseteq \mathcal{S}^2_{\mathcal{P}^\varepsilon}$, with equality when we have true kernels (on functions). In the same way:

$$\mathcal{S}^3_{\mathcal{P}} = \bigcap_{\varepsilon>0} \mathcal{S}^3_{\mathcal{P}^\varepsilon}$$

$$\mathcal{S}^4_{\mathcal{P}} = \bigcap_{\varepsilon>0} \mathcal{S}^4_{\mathcal{P}^\varepsilon}$$

Also $\mathcal{S}^3_{\mathcal{P}} \subseteq \mathcal{S}^4_{\mathcal{P}}$ and $\mathcal{S}^3_{\mathcal{P}^\varepsilon} \subseteq \mathcal{S}^4_{\mathcal{P}^\varepsilon}$, with equality when we have true kernels (on functions).

Clearly, if $0 < \varepsilon < \varepsilon'$ then: $\mathcal{S}^i_{\mathcal{P}^\varepsilon} \subseteq \mathcal{S}^i_{\mathcal{P}^{\varepsilon'}}$ for $i = 1, .., 4$.

Finally, $\mathcal{S}^3_{\mathcal{P}} \subseteq \mathcal{S}^1_{\mathcal{P}}$ and $\mathcal{S}^3_{\mathcal{P}^\varepsilon} \subseteq \mathcal{S}^1_{\mathcal{P}^\varepsilon}$.

Indeed, if $t < s$ and $P_\tau\mu(f) \leq \mu(f)$, $\forall \tau > 0$, μ, let us denote: $s = t + h$. Then: $P_s\mu(f) = P_h\left[P_t\mu\right](f) \leq P_t\mu(f)$.

In the same way, with a resolvent $\mathcal{V} = (V_\alpha)_{\alpha>0}$ of kernels on measures, we associate the following cones of (positive, numerical, measurable) functions:

$\mathcal{S}^1_{\mathcal{V}}$: all the functions f, for which $\alpha \mapsto \alpha V_\alpha\mu(f)$ is increasing, $\forall \mu$;
$\mathcal{S}^2_{\mathcal{V}}$: all the functions f, for which $\alpha \mapsto \alpha V_\alpha\varepsilon_x(f)$ is increasing, $\forall x$;
$\mathcal{S}^3_{\mathcal{V}}$: all the functions f, for which $\alpha V_\alpha\mu(f) \leq \mu(f)$, $\forall \mu$, $\forall \alpha > 0$;
$\mathcal{S}^4_{\mathcal{V}}$: all the functions f, for which $\alpha V_\alpha\varepsilon_x(f) \leq f(x)$, $\forall x$, $\forall \alpha > 0$.

As usual, for each $\varepsilon > 0$, we define a new resolvent $\mathcal{V}^\varepsilon$ as: $V^\varepsilon_\alpha = V_{\alpha+\varepsilon}$.

The following relations hold: $\mathcal{S}^1_{\mathcal{V}} = \bigcap_{\varepsilon>0} \mathcal{S}^1_{\mathcal{V}^\varepsilon}$. Indeed, we have:

$$(\alpha+\varepsilon)V_{\alpha+\varepsilon}\mu(f) \leq (\beta+\varepsilon)V_{\beta+\varepsilon}\mu(f)$$

$$\alpha V_{\alpha+\varepsilon}\mu(f) + \varepsilon\left[V_{\alpha+\varepsilon}\mu(f) - V_{\beta+\varepsilon}\mu(f)\right] \leq \beta V_{\beta+\varepsilon}\mu(f)$$

(the case $V_{\beta+\varepsilon}\mu(f) = +\infty$ being obvious).

Next, let us suppose that $\alpha V_{\alpha+\varepsilon}\mu(f) \leq \beta V_{\beta+\varepsilon}\mu(f)$ for $0 < \alpha < \beta$. For any $0 < \varepsilon < \alpha$ we have:

$$(\alpha-\varepsilon)V_{\alpha}\mu(f) \leq \beta V_{\beta+\varepsilon}\mu(f) \leq \beta V_{\beta}\mu(f)$$

(again the case $V_\beta\mu(f) = +\infty$ is obvious). The conclusion now follows letting $\varepsilon \searrow 0$.

As a special case, we obtain: $\mathcal{S}^2_{\mathcal{V}} = \bigcap_{\varepsilon>0} \mathcal{S}^2_{\mathcal{V}^\varepsilon}$. Also $\mathcal{S}^1_{\mathcal{V}} \subseteq \mathcal{S}^2_{\mathcal{V}}$ and $\mathcal{S}^1_{\mathcal{V}^\varepsilon} \subseteq \mathcal{S}^2_{\mathcal{V}^\varepsilon}$, with equality when we have true kernels (on functions).

In the same way:

$$\mathcal{S}^3_{\mathcal{V}} = \bigcap_{\varepsilon>0} \mathcal{S}^3_{\mathcal{V}^\varepsilon}$$

$$\mathcal{S}^4_{\mathcal{V}} = \bigcap_{\varepsilon>0} \mathcal{S}^4_{\mathcal{V}^\varepsilon}$$

Also $\mathcal{S}^3_{\mathcal{V}} \subseteq \mathcal{S}^4_{\mathcal{V}}$ and $\mathcal{S}^3_{\mathcal{V}^\varepsilon} \subseteq \mathcal{S}^4_{\mathcal{V}^\varepsilon}$, with equality when we have true kernels (on functions).

Clearly, if $0 < \varepsilon < \varepsilon'$ then: $\mathcal{S}^i_{\mathcal{V}^\varepsilon} \subseteq \mathcal{S}^i_{\mathcal{V}^{\varepsilon'}}$ for $i = 1, .., 4$.

Finally, $\mathcal{S}^3_{\mathcal{V}} \subseteq \mathcal{S}^1_{\mathcal{V}}$ and $\mathcal{S}^3_{\mathcal{V}^\varepsilon} \subseteq \mathcal{S}^1_{\mathcal{V}^\varepsilon}$.

Indeed, let $\alpha < \beta$. We have:

$$\alpha V_\alpha\mu(f) = \beta V_\beta\mu(f) + (\beta-\alpha)\left[\alpha V_\alpha(V_\beta\mu)(f) - V_\beta\mu(f)\right] \leq V_\beta\mu(f)$$

The following inclusions hold: $\mathcal{S}^i_{\mathcal{P}} \subseteq \mathcal{S}^i_{\mathcal{V}}$; $\mathcal{S}^i_{\mathcal{P}^\varepsilon} \subseteq \mathcal{S}^i_{\mathcal{V}^\varepsilon}$, for $i = 1, ..4$.

Let $\alpha < \beta$, hence $\beta = c\alpha$, with $c > 1$. We have then:

$$\alpha V_\alpha\mu(f) = \alpha\int_0^\infty e^{-\alpha t}P_t\mu(f)dt = \alpha\int_0^\infty e^{-\beta s}P_{cs}\mu(f)cds =$$

$$= \beta\int_0^\infty e^{-\beta s}P_{cs}\mu(f)ds \leq \beta\int_0^\infty e^{-\beta s}P_s\mu(f)ds = \beta V_\beta\mu(f)$$

With each resolvent $\mathcal{V}$, one can associate:

the $\mathcal{V}$–excessive measures: $\mu \in \mathcal{M}_{\mathcal{V}}$ if $\alpha V_\alpha\mu \leq \mu, \forall\alpha > 0$;

and also:

the $\mathcal{V}$–supramedian functions: $f \in \mathcal{F}_{\mathcal{V}}$ if $\alpha V_\alpha \mu(f) \leq \mu(f), \forall \alpha > 0, \forall \mu \in \mathcal{M}$.

Clearly, if V_α are kernels on functions, these are the usual definitions.

Proposition 6. *If $\mathcal{R}$ denotes the set of $f \in \mathcal{F}$ such that $t \mapsto P_t\mu(f)$ is right continuous on $(0, +\infty)$, for each μ, then $\mathcal{S}^1_{\mathcal{V}} \cap \mathcal{R} \subseteq \mathcal{S}^1_{\mathcal{P}}$.*

Proof. We have to prove that, if $t \mapsto P_t\mu(f)$ is right continuous on $(0, +\infty)$, and $\alpha \mapsto \alpha V_\alpha \mu(f)$ is increasing for each μ, then $t \mapsto P_t\mu(f)$ is decreasing on $(0, +\infty)$, for each μ.

Let $c > 1$ be fixed. Since we have $\alpha V_\alpha \mu(f) \leq \beta V_\beta \mu(f)$ (where $\beta := c\alpha$), we obtain: $\int_0^\infty e^{-\alpha s} \left[P_{s/c}\mu(f) - P_s\mu(f)\right] ds \geq 0$.

Now, the function $\alpha \mapsto cV_{c\alpha}\mu(f) - V_\alpha\mu(f)$ is completely monotone, as its n–th order derivative equals $(-1)^n \left[c^{n+1}V^{n+1}_{c\alpha}\mu(f) - V^{n+1}_\alpha\mu(f)\right]$; a simple induction shows that:

$$(\beta V_\beta)^{n+1}\mu(f) = (\beta V_\beta)^n \left[\beta V_\beta \mu\right](f) \geq (\alpha V_\alpha)^n \left[\beta V_\beta \mu\right](f) =$$

$$= \beta V_\beta \left[(\alpha V_\alpha)^n \mu\right](f) \geq (\alpha V_\alpha)\left[(\alpha V_\alpha)^n \mu\right](f) = (\alpha V^{n+1}_\alpha \mu(f)$$

hence $c^{n+1}V^{n+1}_{c\alpha}\mu(f) \geq V^{n+1}_\alpha \mu(f)$.

From Bernstein's theorem, we obtain that $P_{s/c}\mu(f) \geq P_s\mu(f)$, a.e. in s. From the right continuity, the conclusion follows.

5. The associated semi–dynamical system. In this part, the general measurable space $(A, \mathcal{A})$ is replaced by $(K, \mathcal{B})$, where K is a compact space and $\mathcal{B}$ is the Borel σ–algebra.

Theorem 7. *Let $\mathcal{V}$ be a resolvent of kernels on (positive, Radon) measures on $(K, \mathcal{B})$, such that:*

$1 \in \mathcal{S}^3_{\mathcal{V}}$ *(equivalent with: $\alpha V_\alpha \mu(1) \leq \mu(1)$, $\forall \alpha > 0$);*

$\mathcal{S}^{3c}_{\mathcal{V}}$ *separates K;*

for each $\alpha > 0$ the condition $R \circ V^{**}_\alpha = V_\alpha \circ R$ holds.

Then there exists a semi–group of kernels on (positive, Radon) measures on $(K, \mathcal{B})$, such that:

$$V_\alpha \mu = \int_0^\infty e^{-\alpha t} \Phi(t, \mu) dt$$

Proof. We give only the different arguments with respect to the proof form [5]. There exists a unique, positive Radon measure $m_{\mu,f}$ on $[0,+\infty]$, such that:

$$V_\alpha \mu(f) = \int_0^\infty e^{-\alpha t} dm_{\mu,f}(t)$$

Since the function $\alpha \mapsto \mu(f) - \alpha V_\alpha \mu(f)$ is completely monotone for $f \in \mathcal{S}^3_{\mathcal{V}}$, from Bernstein's theorem, there exists a unique positive measure $n_{\mu,f}$ on $[0,+\infty]$ such that

$$\mu(f) - \alpha V_\alpha \mu(f) = \int_0^\infty e^{-\alpha t} dn_{\mu,f}(t)$$

For each $t > 0$, $\mu \in X$ and $f \in \mathcal{S}^3_{\mathcal{V}}$ we define:

$$\Phi(t,\mu)(f) := n_{\mu,f}\left((t,+\infty)\right)$$

The function $t \mapsto \Phi(t,\mu)(f)$ is right continuous on $[0,+\infty)$. Using the hypothesis, $\Phi(t,\mu)$ is in fact a (positive, Radon) measure: indeed, $\Phi(t,\mu)$ extends by linearity to $\mathcal{S}^{3c}_{\mathcal{V}} - \mathcal{S}^{3c}_{\mathcal{V}}$ as a positive linear functional; the density in $\mathcal{C}(K)$ gives the measure. The right continuity is preserved under linear and uniform extensions, hence $t \mapsto \Phi(t,\mu)(f)$ is right continuous on $(0,+\infty)$, for all $f \in \mathcal{C}(K)$ and all positive Radon measures μ.

Also, the relation:

$$V_\alpha \mu(f) = \int_0^\infty e^{-\alpha t} \Phi(t,\mu)(f) dt$$

holds $\forall \mu$, $\forall f \in \mathcal{C}(K)$. $\Phi(t,\cdot)$ is in fact a kernel on (positive, Radon) measures.

The rest of the proof is the same.

Remark. Using the fact that the function $\alpha \mapsto \mu(f) - \alpha V_\alpha \mu(f)$ is also completely monotone for $\mu \in \mathcal{M}_{\mathcal{V}}$, we may replace the hypothesis:

"$\mathcal{S}^{3c}_{\mathcal{V}}$ separates K"

by a density of $\mathcal{M}_{\mathcal{V}} - \mathcal{M}_{\mathcal{V}}$ in $\mathcal{M}$; but this conditions does not seems reasonable.

REFERENCES

1. BOBOC, N., BUCUR, GH. – *Măsură şi capacitate*, Ed. şt. şi encicl., Bucureşti, 1985.
2. CORNEA, A., LICEA, G. – *Order and Potential. Resolvent Families of Kernels*, Lecture Notes in Math. vol. 494, Springer Verlag 1975.
3. DELLACHERIE, C., MEYER, P.-A. – *Probabilités et Potentiel*, Act. Sci. Ind. Hermann,Paris 1987.
4. POPA, E. – *Semi–dynamical systems on algebraic structures,* An. şt. Univ. Iaşi t. XLIV, sIa, Matematica, (1998), Supliment, pg. 585–596
5. POPA, E. – *Resolvents associated with semi-dynamical systems in duality*, to appear in An. şt. Univ. Iaşi t.
6. POPA, E. – *Semi–groups of kernels on cones of positive measures,* Int. Congress of Math. Beijing 2002.