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RESOLVENTS ASSOCIATED WITH SEMI-DYNAMICAL SYSTEMS IN DUALITY

BY

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1. Abstract. Semi–dynamical systems were mostly studied on locally compact spaces (see O. Hajek [5] and Bhatia –Szegö [1]) or on measurable spaces (see Gh. Bucur – M. Bezzarga [2]).

Potential theory associated with a semi-dynamical system on an ordered convex cone was initiated in [7]. Notably, the notion of excessive element was introduced and the Riesz' splitting property was established [8], [9].

Potential theory associated with a kernel or a resolvent family of kernels on an (abstract) ordered convex cone was studied by Cornea – Licea [3]. However, the order is supposed to be the "specific" one, i.e.

$$(*) \quad x \leq y \Longrightarrow \exists z \text{ such that } x + z = y$$

The purpose of this paper is to establish the connection of the resolvents of abstract kernels with semi–dynamical systems on algebraic structures, through a "Hille–Yosida" type theorem.

A first problem is to define an integral for functions with values in an abstract cone, such that it commutes with any kernel. As we want to avoid the use of the (*) condition, a different proof, using Bernstein's theorem, is supplied for the coincidence of the excessive elements.

2. Preliminaries. Let X be an ordered convex cone. The linearisability is not supposed: in such a way, we may consider as elements of the set X also numerical (positive) functions. If necessary, the "cancellable" elements $u \in X$ (i.e. $x+u \leq y+u \Longrightarrow x \leq y$); or the archimedian elements ($\inf_n \frac{1}{n}u = 0$) will be considered.

We suppose that X is a lattice, with the property:

$x_1 \le x_2 \le \dots x_n \le \dots \Longrightarrow \exists \bigvee_n x_n$ in X

This situation will be denoted by $x_n \nearrow \bigvee_n x_n$.

A map $V : X \to X$ is called a kernel if it is additive, increasing and continuous from below ($x_n \nearrow x \Longrightarrow Vx_n \nearrow Vx$).

We say that two ordered convex cones X and Y are in duality through:

$$[\cdot, \cdot] : X \times Y \to [0, +\infty]$$

if the following properties hold [7]:

1. $[x + x', y] = [x, y] + [x', y]$; $[x, y + y'] = [x, y] + [x, y']$
2. $x \le x' \Longrightarrow [x, y] \le [x', y]$; $y \le y' \Longrightarrow [x, y] \le [x, y']$
3. $x_n \nearrow x \Longrightarrow [x_n, y] \nearrow [x, y]$; $y_n \nearrow y \Longrightarrow [x, y_n] \nearrow [x, y]$

Example. For each ordered convex cone X, let us consider the set X^* of all applications

$$\mu : X \to [0, +\infty]$$

which have the properties:

1. $\mu(x + x') = \mu(x) + \mu(x')$
2. $\mu(a.x) = a.\mu(x)$, $\forall a \ge 0$
3. $x \le x' \Longrightarrow \mu(x) \le \mu(x')$
4. $x_n \nearrow x \Longrightarrow \mu(x_n) \nearrow \mu(x)$

The set X^* is organized with pointwise addition, scalar multiplication and order becomes an ordered convex cone, in which even $\bigvee_i \mu_i$ exists for increasing families $(\mu_i)_i$ and is in a canonical duality with X through $[x, \mu] := \mu(x)$. The duality will always be strongly separated, i.e. $[x, y] \le [x', y], \forall y \in Y \Longrightarrow x \le x'$, and $[x, y] \le [x, y'], \forall x \in X \Longrightarrow y \le y'$. Then we have a canonical inclusion $X \hookrightarrow Y^*$ defined by $\tilde{x}(y) := [x, y]$.

Each kernel $V : X \to X$ has an adjoint, defined as $V^* : X^* \to X^*$ by $V^*(\mu)(x) := \mu(Vx)$, $\forall \mu \in X^*$ and $x \in X$.

A semi-dynamical system on algebraic structure (s.d.s.s.a. for short) is a couple (X, Φ), where $\Phi : (0, +\infty) \times X \to X$ possesses the properties:

1. $\Phi(s, \Phi(t, x)) = \Phi(s + t, x)$
2. $\Phi(t, x + x') = \Phi(t, x) + \Phi(t, x')$
3. $\Phi(t, \lambda.x) = \lambda.\Phi(t, x)$
4. $x \le x' \Longrightarrow \Phi(t, x) \le \Phi(t, x')$
5. $x_n \nearrow x \Longrightarrow \Phi(t, x_n) \nearrow \Phi(t, x)$, $\forall t > 0$

Each map $x \to \Phi(t, x)$ is a kernel on X.

3. Integrals in convex cones. We define a "weak" integral for functions with values in an ordered convex cone. Using this construction, we will associate with each s.d.s.a.s. a resolvent of abstract kernels, as considered by Cornea–Licea [3]. However, these kernels are naturally defined on the bi–dual of the cone. This issue will be discussed in more detail in a second paper, for the special case of the cone of positive measures on a measurable space.

Let X, Y be in duality. We consider the σ–algebra $\mathcal{X}$ generated by all the maps $x \mapsto [x, y]$, $\forall y \in Y$. $\mathcal{X}$ is the Borel σ–algebra corresponding to the Y–weak topology on X: the coarsest topology in which all the above maps are continuous.

Let $(A, \mathcal{A}, m)$ be a space with a positive measure. Let $\varphi : A \to X$ be a weakly measurable map, meaning that for any $y \in Y$ the map $a \mapsto [\varphi(a), y]$ is $\mathcal{A}$–measurable.

The integral $\int_A [\varphi(a), y] dm(a)$ makes sense, and $y \mapsto \int_A [\varphi(a), y] dm(a)$ obviously defines an element, denoted $I(y)$ from Y^*.

We will be essentially concerned with the case $A = [0, +\infty]$ with the usual borelian structure and the Lebesgue measure; while $\varphi(t) := e^{-\alpha t}\Phi(t, x)$ (for some s.d.s.a.s. Φ on X).

For a decreasing map on $(0, +\infty)$, one can define a "strong" integral (see [3]); also in the case of a positive cone from an ordered Banach space (see Bochner integral [6]).

Definition. The kernel $V : X \to X$ is said to be *compatible with the duality* if for each $y \in Y$ the map $x \mapsto [Vx, y]$ belongs to Y.

This is the same to say that the restriction of the adjoint kernel $V^* : X^* \to X^*$ to Y is a kernel on Y. Of course, this property holds when $Y = X^*$.

Any kernel compatible with the duality will be considered as extended to Y^* as follows: let $u \in Y^*$; we define $V(u)$ as an element of Y^*, hence we have to give a meaning to $V(u)(y)$, for each $y \in Y$. This will be $u\,[V^*(y)]$. This is correct, since $V^*(y) \in Y$ and $u \in Y^*$. It is easy to see that $V(u) \in Y^*$ and this is also a kernel, which extends V from X to Y^*.

Any kernel compatible with the duality is (Y)–weakly measurable: for each $y \in Y$, the map $x \mapsto [Vx, y]$ is measurable.

Hence, if $V : X \to X$ is a (Y)–weakly measurable kernel, then for any weakly measurable map φ, $V \circ \varphi$ is also weakly measurable. In this way, we define an element $J \in Y^*$ through

$$J = \int_A (V \circ \varphi)(a) dm(a)$$

meaning that:

$$J(y) = \int_A [V(\varphi(a)), y] dm(a), \forall y \in Y$$

Lemma 1. *If V is compatible with the duality, then for any measurable φ, the following property holds:*

$$V \left[\int_A \varphi(a) dm(a) \right] = \int_A (V \circ \varphi)(a) dm(a)$$

Proof. Let us compute $V(I)$:

$$V(I)(y) = I\left[V^*(y)\right] = \int_A (V^*(y))(\varphi(a)) dm(a) =$$

$$= \int_A \left[(V \circ \varphi)(a), y\right] dm(a) = J(y)$$

This lemma guarantees the key property for the proof of the main theorem. It is a replacement for the usual Fubini theorem (see [4]). Let us remark that such a property holds for any kernel if the integral is considered in the strong sense.

4. The associated resolvent. Let X, Y be in duality. The s.d.s.a.s. (X, Φ) and (Y, Ψ) are in duality if $[\Phi(t, x), y] = [x, \Psi(t, y)]$, $\forall x \in X$, $y \in Y$, $t > 0$. Let us recall that (X, Φ) and (X^*, Φ^*) are in a canonical duality [8], while Ψ^* is the canonical extension of Φ.

Let us recall [8] that we have defined the supermedian and excessive elements associated with a s.d.s.a.s. (X, Φ):

$$X_s := \{x \in X | \Phi(t, x) \leq x, \forall t > 0\}$$

$$X_e := \{x \in X_s | \bigvee_{t>0} \Phi(t, x) = x\}$$

As usual [3], a resolvent is a family $\mathcal{V} = (V_\alpha)_{\alpha > 0}$ of kernels on X, such that:

$$V_\alpha \circ V_\beta = V_\beta \circ V_\alpha, \forall \alpha, \beta > 0$$

$$V_\alpha = V_\beta + (\beta - \alpha) V_\alpha \circ V_\beta, \forall 0 < \alpha < \beta$$

Analogously, we associate with the resolvent $\mathcal{V}$=$(V_\alpha)_{\alpha>0}$ the $\mathcal{V}$–supermedian and $\mathcal{V}$–excessive elements as:

$$X_s^{\mathcal{V}} := \{x \in X | \alpha V_\alpha x \leq x, \forall \alpha > 0\}$$

$$X_e^{\mathcal{V}} := \{x \in X_s^{\mathcal{V}} | \bigvee_{\alpha>0} \alpha V_\alpha x = x\}$$

If each V_α is compatible with the duality, then the canonical extensions to Y^* form obviously a resolvent.

We show that, under additional assumptions, a resolvent and a s.d.s.a.s. may be associated one another, such that the excessive elements are the same, through the formula

$$V_\alpha x = \int_0^\infty e^{-\alpha t} \Phi(t, x) dt$$

However, if Φ (or $\mathcal{V}$) is on X, then $\mathcal{V}$ (resp. Φ) is on Y^*. Let us recall that, if we accept the compatibility with the duality, then Φ (resp. $\mathcal{V}$) may be canonically extended also to Y^*.

As a replacement for the submarkovian condition, we define a C–submarkovian s.d.s.a.s. if there exists a measurable function $C : (0, +\infty) \to (0, +\infty)$, such that:

$$M(\alpha) := \int_0^\infty e^{-\alpha t} C(t) dt < +\infty, \ \forall \alpha > 0$$

and $D := \{y \in Y | \ \Psi^*(t, x)(y) \leq C(t) x(y), \forall x \in Y^*\}$ is an increasingly dense set in Y.

Especially, this is the case of a weak unit $y_0 \in Y$ such that $\Psi^*(t, x)(y_0) \leq C(t) x(y_0)$, $\forall x \in Y^*$. The submarkovian case corresponds to the duality $[\mathcal{M}, \mathcal{F}]$, y_0 the identically 1 function and $C(t) \equiv 1$.

We get now:

$$V_\alpha x(y) = \int_0^\infty e^{-\alpha t} \Psi^*(t, x)(y) dt \leq \int_0^\infty e^{-\alpha t} C(t) x(y) dt =$$

$$= M(\alpha) x(y), \forall x \in Y^*$$

(the resolvent is M–submarkovian).

Hence, the resolvent equation makes sense for any $x \in Y^*$ and $y \in D$:

$$V_\alpha x(y) = V_{\alpha_0} x(y) + (\alpha_0 - \alpha) V_\alpha V_{\alpha_0} x(y), 0 < \alpha < \alpha_0$$

Since D is increasingly dense, in fact the equality

$$V_\alpha = V_{\alpha_0} + (\alpha_0 - \alpha) V_\alpha V_{\alpha_0}$$

has a meaning, regardless if $\alpha_0 > \alpha$ or $\alpha_0 \leq \alpha$. Now, we obtain successively:

$$V_\alpha = V_{\alpha_0} + (\alpha_0 - \alpha) V_{\alpha_0}^2 + \ldots + (\alpha_0 - \alpha)^n V_{\alpha_0}^{n+1} + (\alpha_0 - \alpha)^{n+1} V_\alpha V_{\alpha_0}^{n+1}$$

Since

$$|(\alpha_0 - \alpha)^{n+1} V_\alpha V_{\alpha_0}^{n+1} x(y)| \leq |\alpha_0 - \alpha|^{n+1} M(\alpha) M^{n+1}(\alpha_0) x(y)$$

it follows that

$$V_\alpha x(y) = \sum_{n=0}^{\infty} (\alpha_0 - \alpha)^n V_{\alpha_0}^{n+1} x(y)$$

holds for $\alpha \in (\alpha_0 - M(\alpha_0), \alpha_0 + M(\alpha_0))$. This formula allows in fact the extension of the family of kernels to (V_z), for $Re\ z > 0$.

It follows that, if $x(y) < +\infty$, then $F(\alpha) := V_\alpha x(y)$ is completely monotone and $F^{(n)}(\alpha) = (-1)^n n! V_\alpha^{n+1} x(y)$. Moreover, the function $G(\alpha) := x(y) - \alpha V_\alpha x(y)$ is also completely monotone, if $x \in Y_s^{*\mathcal{V}}$. We have:

$$G^{(n)}(\alpha) = (-1)^n n! \left(V_\alpha^n x(y) - \alpha V_\alpha^{n+1} x(y) \right)$$

Let us denote by X_c the set of all elements $x \in X$ such that the map $t \mapsto \Phi(t, x)(y)$ is right continuous on $(0, +\infty)$, for each $y \in Y$.

X_c depends on the duality: for example, if we replace Y by Y_e, then $X_c = X$. Let us remark that X_c is a convex subcone and (X_c, Φ) is a s.d.s.s.a. Obviously $X_e \subseteq X_c$.

Proposition 2. *Let (X, Φ), (Y, Ψ) be s.d.s.s.a. in separated duality. We suppose that the map: $t \mapsto x(\Psi(t, y)) = \Psi^*(t, x)(y)$ is measurable (on $(0, +\infty)$), for any $x \in Y^*$ and $y \in Y$. Through the formula:*

$$V_\alpha x(y) := \int_0^\infty e^{-\alpha t} \Psi^*(t, x)(y) dt$$

one associates a resolvent family of kernels $\mathcal{V} := (V_\alpha)_{\alpha > 0}$ on Y^*.

If Φ is C–submarkovian, then $\mathcal{V}$ is M–submarkovian.
The excessive elements are the same, in the following sense:

$$Y_e^* = Y_e^{*\mathcal{V}} \cap Y_c^*$$

and also

$$Y_s^{*\mathcal{V}} \cap Y_c^* \subseteq Y_s^* \subseteq Y_s^{*\mathcal{V}}$$

Proof. We can define, for $x \in Y^*$ and $y \in Y$:

$$(V_\alpha x)(y) := \int_0^\infty e^{-\alpha t}\Psi^*(t,x)(y)dt = \int_0^\infty e^{-\alpha t} x\left(\Psi(t,y)\right) dt$$

Clearly $V_\alpha x \in Y^*$, hence we have defined a map $V_\alpha : Y^* \to Y^*$ which is additive, monotone and continuous in order.

Let us prove that the family of kernels $(V_\alpha)_{\alpha>0}$ is a resolvent on Y^*. The usual computation works:

$$V_\alpha(V_\beta x)(y) = \int_0^\infty e^{-\alpha t}\Psi^*(t, V_\beta(x))(y)dt = \int_0^\infty e^{-\alpha t}(V_\beta x)(\Psi(t,y))dt =$$

$$= \int_0^\infty e^{-\alpha t}\left(\int_0^\infty e^{-\beta s}(\Psi^*(s,x))(\Psi(t,y))ds\right) dt =$$

$$= \int_0^\infty e^{-\alpha t}\left(\int_0^\infty e^{-\beta s}x(\Psi(s,\Psi(t,y))ds\right) dt =$$

$$= \int_0^\infty e^{-\alpha t}\left(\int_0^\infty e^{-\beta s}x(\Psi(s+t,y)ds\right) dt =$$

$$= \int_0^\infty \int_0^\infty e^{-\alpha t-\beta s}\Psi^*(s+t,x)(y)dsdt =$$

$$= \int_0^\infty \left[\int_t^\infty e^{-\alpha t-\beta u+\beta t}\Psi^*(u,x)(y)du\right] dt =$$

$$= \int_0^\infty e^{(\beta-\alpha)t}\left[\int_t^\infty e^{-\beta u}\Psi^*(u,x)(y)du\right] dt =$$

$$= \int_0^\infty e^{\beta u}\Psi^*(u,x)(y)\left(\int_0^u e^{(\beta-\alpha)t}dt\right) du =$$

(for $\alpha \neq \beta$)

$$= \frac{1}{\beta-\alpha}\int_0^\infty \left((e^{(\beta-\alpha)u}-1\right) e^{-\beta u}\Psi^*(u,x)(y)du =$$

$$= \frac{1}{\beta-\alpha}\int_0^\infty \left(e^{-\alpha u}-e^{-\beta u}\right)\Psi^*(u,x)(y)du$$

We may suppose that $\beta > \alpha$; if $V_\beta x(y) < +\infty$, then adding this term to both sides of the equality we obtain:

$$V_\alpha(V_\beta x)(y) + \frac{1}{\beta-\alpha}V_\beta x(y) = \frac{1}{\beta-\alpha}V_\alpha x(y)$$

As such, we obtain without any further assumption, the formula:

$$V_\alpha x(y) = V_\beta x + (\beta-\alpha)(V_\alpha \circ V_\beta)(x)(y)$$

for $\beta > \alpha$. If $V_\beta x(y) = +\infty$ then we have $V_\alpha x(y) = +\infty$, hence the equality also holds. Let us remark that we have also proved that $V_\alpha \circ V_\beta = V_\beta \circ V_\alpha$ in all cases, since (for $\alpha \neq \beta$):

$$V_\alpha(V_\beta x)(y) = \int_0^\infty \frac{e^{-\alpha u}-e^{-\beta u}}{\beta-\alpha}\Psi^*(u,x)(y)du$$

We prove now that $Y_e^{*\mathcal{V}} \cap Y_c^* \equiv Y_e^*$.

If $x \in Y^*$, is such that $\Psi^*(t,x) \leq x$, $\forall t > 0$, then, for each $y \in Y$:

$$V_\alpha x(y) = \int_0^\infty e^{-\alpha t}\Psi^*(t,x)(y)dt \leq \int_0^\infty e^{-\alpha t}x(y)dt = \frac{1}{\alpha}x(y)$$

hence $\alpha V_\alpha x \leq x$, $\forall \alpha > 0$.

If $x = \bigvee_{t>0}\Psi^*(t,x)$, then:

$$\sup_{\alpha>0}\alpha V_\alpha x(y) = \sup_{\alpha>0}\alpha\int_0^\infty e^{-\alpha t}\Psi^*(t,x)(y)dt = \sup_{\alpha>0}\int_0^\infty e^{-u}\Psi^*(\frac{u}{\alpha},x)(y)du =$$

$$= \int_0^\infty e^{-u}\left[\sup_{\alpha>0}\Psi^*(\frac{u}{\alpha},x)(y)\right]du = \int_0^\infty e^{-u}x(y)du = x(y)$$

Let now $x \in Y_c^*$ be such that $\alpha V_\alpha x \leq x$, $\forall \alpha > 0$. For $y \in Y$ such that $x(y) < +\infty$ we get:

$$x(y) - \alpha V_\alpha x(y) = \alpha\int_0^\infty e^{-\alpha t}\left(x(y) - \Psi^*(t,x)(y)\right)dt \geq 0$$

Hence, using Bernstein's theorem and the right continuity, we obtain:

$$\Psi^*(t,x)(y) \leq x(y)$$

(the inequality being obvious for $x(y) = +\infty$).

Thus $u := \bigvee_{t>0} \Psi^*(t,x)$ exists. If moreover $\bigvee_{\alpha>0} \alpha V_\alpha x = x$ we have:

$$\Psi^*(s,u) = \Psi^* \left(s, \bigvee_{t>0} \Psi^*(t,x) \right) = \bigvee_{t>0} \Psi^*(s+t,x) = \Psi^*(s,x),\ \forall s > 0$$

(using again the right continuity). We get $V_\alpha u = V_\alpha x$, $\forall \alpha > 0$. Since $\Psi^*(t,u) \leq u$, we obtain, as above, that $\alpha V_\alpha u \nearrow u$, hence $u = x$.

Remark. a) Under the assumption

$$\Psi^*(t,x) = \Psi^*(t,y) \Longrightarrow x = y$$

we have also: $Y_s^{*\mathcal{V}} \cap Y_c^* = Y_e^{*\mathcal{V}} \cap Y_c^*$. Indeed, as supremums commute, the map $t \mapsto \Psi^*(t,u)(y)$ is also right continuous:

$$\sup_{t \searrow t_0} \Psi^*(t,u)(y) = \sup_{t \searrow t_0} \sup_{s \to +\infty} \Psi^*(t+s,x)(y) = \sup_{s \to +\infty} \sup_{t \searrow t_0} \Psi^*(t+s,x)(y) =$$

$$= \sup_{s \to +\infty} \Psi^*(t+s,x)(y) = \Psi^*(t,u)(y)$$

Hence, from $V_\alpha u = V_\alpha x$, $\forall \alpha > 0$, we obtain $\Psi^*(t,u) = \Psi^*(t,x)$, $\forall t > 0$.

b) In order to define the dual resolvent $W_\alpha : X^* \to X^*$, through the formula:

$$(W_\alpha y)(x) := \int_0^\infty e^{-\alpha t} \Phi^*(t,y)(x) dt$$

we have to admit that the map: $t \mapsto y(\Phi(t,x)) = \Phi^*(t,y)(x)$ is measurable (on $(0,+\infty)$), for any $y \in X^*$ and $x \in X$.

The following weak form of duality holds: $V_\alpha x(y) = W_\alpha y(x)$, $\forall \alpha > 0$, only for $x \in X$ and $y \in Y$.

5. The associated semi–dynamical system.

Theorem 3. *Let X and Y be in separated duality; let $\mathcal{V} := (V_\alpha)_{\alpha>0}$ be a M–submarkovian resolvent family of kernels on X, compatible with the duality $[X,Y]$, where $M(\alpha) = \int_0^\infty e^{-\alpha t} C(t) dt$, for some right continuous C.*

We assume the following Ray–type density condition: the canonical map restricts to an order preserving (i.e. $\mu \leq \mu'$ on $Y_s \Longrightarrow \mu \leq \mu'$ on Y) bijection between specifically solid subcones from Y^* and $(Y_s)^*$

Also the following finiteness property is supposed:

$$\forall x \in Y^* \text{ and } y \in Y_s \exists y_n \in Y_s \text{ such that } y_n \nearrow y \text{ and } x(y_n) < +\infty$$

Then one associates a C–submarkovian s.d.s.s.a. Φ on Y^* such that $\forall x \in Y^*$:

$$V_\alpha x = \int_0^\infty e^{-\alpha t}\Phi(t,x)dt$$

The excessive elements (with respect to the resolvent and to Φ) are the same $Y_e^* = Y_e^{*\mathcal{V}}$.

Proof. We may suppose that V_0 exists and is a bounded kernel, in the sense that there exists $M > 0$ such that $V_0x(y) \leq M.x(y), \forall y \in Y, \forall x \in Y^*$.

We extend as usual each V_α from X to Y^*, since it is compatible with the duality. Using Bernstein's theorem, for each $x \in Y^*$ and $y \in Y$ such that $x(y) < +\infty$, there exists a unique, positive bounded measure $m_{x,y}$ on $[0, +\infty)$, such that:

$$V_\alpha x(y) = \int_0^\infty e^{-\alpha t} dm_{x,y}(t),\ \forall \alpha > 0$$

Using again Bernstein's theorem, for each $x \in Y^*$ and $y \in Y_s$ such that $x(y) < +\infty$, there exists a unique positive bounded measure $n_{x,y}$ on $[0, +\infty)$ such that

$$x(y) - \alpha V_\alpha x(y) = \int_0^\infty e^{-\alpha t} dn_{x,y}(t),\ \forall \alpha > 0$$

Since $\alpha V_\alpha x(y) \to 0$ as $\alpha \to 0$, we obtain: $x(y) = n_{x,y}([0, +\infty])$. The following properties hold obviously, for $x \in Y^*$, $y \in Y_s$ with $x(y) < +\infty$:

$$n_{x,0} = 0$$

$$n_{x,y+y'} = n_{x,y} + n_{x,y'}$$

Combining the two representations, we get:

$$V_\alpha x(y) = \int_0^\infty e^{-\alpha t} dm_{x,y}(t) = \int_0^\infty \frac{1 - e^{-\alpha t}}{\alpha} dn_{x,y}(t) =$$

$$= \int_0^\infty \left[\int_0^t e^{-\alpha\tau} d\tau \right] dn_{x,y}(t) = \int_0^\infty e^{-\alpha\tau} \left[\int_\tau^\infty dn_{x,y}(t) \right] d\tau$$

We show next that through the formula: $\Phi(t,x)(y) := n_{x,y}((t,+\infty))$ we define $\Phi(t,x)$ as an element from $(Y_s)^*$, for $x \in Y^*$. Indeed, as for each $x \in Y^*$ and $y \in Y_s$ there exists an increasing sequence $y_n \in Y_s$ such that $y_n \nearrow y$ and $x(y_n) < +\infty$, then we define $\Phi(t,x)(y)$ (for the case $x(y) = +\infty$) as $f(t) := \sup_n \Phi(t,x)(y_n)$. $t \mapsto \Phi(t,x)(y_n)$ is decreasing and right continuous on $(0,+\infty)$, hence l.s.c.; now f is decreasing and l. s. c., hence right continuous. Moreover, the following relation holds:

$$V_\alpha x(y) = \int_0^\infty e^{-\alpha t} f(t) dt$$

Hence f is independent of the sequence (y_n), so $\Phi(t,x)(y)$ is well defined. In the usual way, we get the following properties:

$$V_\alpha x(y) = \int_0^\infty e^{-\alpha t} \Phi(t,x)(y) dt$$

$t \mapsto \Phi(t,x)(y)$ is right continuous on $(0,+\infty)$ and decreasing, hence l.s.c.

$$\Phi(t,x)(y+y') = \Phi(t,x)(y) + \Phi(t,x)(y')$$

$$y \le y' \Longrightarrow \Phi(t,x)(y) \le \Phi(t,x)(y')$$

$$y_n \nearrow y \Longrightarrow \Phi(t,x)(y_n) \nearrow \Phi(t,x)(y)$$

Hence $\Phi(t,x) \in (Y_s)^*$.

If $y \le y'$ and denote $f(\alpha) := V_\alpha x(y') - V_\alpha x(y)$ then

$$f^{(n)}(\alpha) = (-1)^n n! \left[V_\alpha^{n+1} x(y') - V_\alpha^{n+1} x(y) \right]$$

Hence, f is completely monotone. On the other hand:

$$f(\alpha) = \int_0^\infty e^{-\alpha t} \left[\Phi(t,x)(y') - \Phi(t,x)(y) \right] dt$$

hence using Bernstein's theorem (combined with the right continuity), we get the property:

$$y \le y' \Longrightarrow \Phi(t,x)(y) \le \Phi(t,x)(y')$$

In the same manner, we obtain the property:

$$y_n \nearrow y \Longrightarrow \Phi(t,x)(y_n) \nearrow \Phi(t,x)(y)$$

Indeed, if we consider $f(t) := \sup_n \Phi(t,x)(y_n)$, then

$$V_\alpha x(y) - V_\alpha x(y_n) = \int_0^\infty e^{-\alpha t} \left[\Phi(t,x)(y) - \Phi(t,x)(y_n)\right] dt$$

hence

$$0 = \int_0^\infty e^{-\alpha t} \left[\Phi(t,x)(y) - f(t)\right] dt$$

As both functions are right continuous, we get the desired conclusion. By the monotony in t, the last relation holds also for increasing families $(y_i)_i$.

For the time being, we have defined a map

$$\Phi(t,\cdot) : Y^* \to (Y_s)^*$$

We use now the "density" condition to ensure that a specifically solid part from $(Y_s)^*$ extend to Y^*.

Since the properties:

$$n_{0,y} = 0$$

$$n_{(x+x'),y} = n_{x,y} + n_{x',y}$$

$$x \le x' \Longrightarrow n_{x,y} \le n_{x',y}$$

$$x_n \nearrow x \Longrightarrow n_{x_n,y} \nearrow n_{x,y}$$

obviously hold, $\Phi(t,x)$ is in fact a kernel on Y^*.

One verifies the s.d.s.a.s. properties:

$$\Phi(t, x + x') = \Phi(t,x) + \Phi(t,x')$$

and, using Bernstein's theorem:

$$x \le x' \Longrightarrow \Phi(t,x) \le \Phi(t,x')$$

$$x_n \nearrow x \Longrightarrow \Phi(t,x_n) \nearrow \Phi(t,x)$$

Before proving the semi–group property $\Phi(t,\Phi(s,x))(y) = \Phi(t+s,x)(y)$, we have to verify the property: $\Phi(t, V_\alpha x)(y) = V_\alpha \Phi(t,x)(y)$. Since both functions $t \mapsto \Phi(t, V_\alpha x)(y)$ and $t \mapsto V_\alpha(\Phi(t,x))(y)$ are decreasing and right continuous, (since V_α is a kernel), taking the Laplace transform of both members, the equality becomes:

$$\int_0^\infty e^{-\beta t}\Phi(t, V_\alpha x)(y)dt = \int_0^\infty e^{-\beta t}V_\alpha(\Phi(t,x)(y))dt$$

The left hand side equals $V_\beta(V_\alpha x)(y)$, while the right hand side may be transformed, using lemma 1:

$$V_\alpha\left(\int_0^\infty e^{-\beta t}\Phi(t,x)(y)dt\right) = V_\alpha(V_\beta x)(y)$$

Taking the Laplace transform in the variable t in the relation to be proved $\Phi(t,\Phi(s,x))(y) = \Phi(t+s,x)(y)$, we obtain the equality:

$$\int_0^\infty e^{-\alpha t}\Phi(t,(\Phi(s,x))(y)dt = \int_0^\infty e^{-\alpha t}\Phi(t+s,x)(y)dt$$

since both members are right continuous in t, for $x \in Y^*$ and $y \in Y_s$. The left hand side equals $V_\alpha(\Phi(s,x))(y)$; taking again the Laplace transform in the variable s, the right hand side is further transformed as:

$$\int_0^\infty e^{-\beta s}\int_0^\infty e^{-\alpha t}\Phi(t+s,x)(y)dtds = \int_0^\infty\int_0^\infty e^{-\beta s-\alpha t}\Phi(t+s,x)(y)dtds =$$

$$= \int_0^\infty\left(\int_s^\infty e^{-\beta s-\alpha u+\alpha s}\Phi(u,x)(y)du\right)ds =$$

$$= \int_0^\infty e^{(\alpha-\beta)s}\left(\int_s^\infty e^{-\alpha u}\Phi(u,x)(y)du\right)ds =$$

(for $\alpha \neq \beta$):

$$\int_0^\infty \frac{1}{\alpha-\beta}(e^{(\alpha-\beta)u}-1)e^{-\alpha u}\Phi(u,x)(y)du = \frac{1}{\alpha-\beta}\left(V_\beta x(y) - V_\alpha x(y)\right)$$

hence the equality is proved, since both members are right continuous with respect to s.

As there is a slight difference (i.e. Φ is defined on Y^* and not on X), we have to check once again that the excessive elements are the same.

The computations are formally the same: if $x \in Y^*$ is such that $\Phi(t,x) \leq x$, $\forall t > 0$, then $\forall y \in Y_s$ we have:

$$V_\alpha x(y) = \int_0^\infty e^{-\alpha t}\Phi(t,x)(y)dt \leq \int_0^\infty e^{-\alpha t}x(y)dt \leq \frac{1}{\alpha}x(y)$$

hence $\alpha V_\alpha x \le x$. If $x = \bigvee_{t>0} \Phi(t,x)$, then $\forall y \in Y_s$ we have:

$$\sup_{\alpha>0} \alpha V_\alpha x(y) = \sup_{\alpha>0} \alpha \int_0^\infty e^{-\alpha t}\Phi(t,x)(y)dt = \sup_{\alpha>0} \int_0^\infty e^{-u}\Phi(\frac{u}{\alpha},x)(y)du =$$

$$= \int_0^\infty e^{-u}\left[\sup_{\alpha>0} \Phi(\frac{u}{\alpha},x)(y)\right]du = \int_0^\infty e^{-u}x(y)du = x(y)$$

Let now $x \in Y^*$ be such that $\alpha V_\alpha x \le x$, $\forall \alpha > 0$. For $y \in Y_s$ such that $x(y) < +\infty$ we get:

$$x(y) - \alpha V_\alpha x(y) = \alpha \int_0^\infty e^{-\alpha t}\left(x(y) - \Phi(t,x)(y)\right)dt \ge 0$$

Hence, using Bernstein's theorem, the right continuity and the Ray–type condition, we obtain:

$$\Phi(t,x)(y) \le x(y)$$

Thus $u := \bigvee_{t>0} \Phi(t,x)$ exists. If moreover $\bigvee_{\alpha>0} \alpha V_\alpha x = x$, then we have:

$$\Phi(s,u) = \Phi\left(s, \bigvee_{t>0} \Phi(t,x)\right) = \bigvee_{t>0} \Phi(s+t,x) = \Phi(s,x),\ \forall s > 0$$

(using the right continuity). We get $\alpha V_\alpha u = \alpha V_\alpha x$, $\forall \alpha > 0$. Since $\Phi(t,u) \le u$, we obtain that $\alpha V_\alpha u \nearrow u$, hence $u = x$.

Remark. a) The measure $n_{x,y}$ also exists in the dual situation: $x(y) < +\infty$ and $\alpha V_\alpha x \le x$. As a dual proof, we may replace the Ray–type density condition by the dual one:

"$Y_s^* - Y_s^*$ is dense in Y^*". Then $\Phi(t,x)$ is defined for each $x \in Y_s^*$ as an element from Y^* (with the usual finiteness condition: $\{y \in Y^* \mid x(y) < +\infty\}$ is increasingly dense).

However, as the example of the cone of positive measures shows, it doesn't seems reasonable to suppose such a density; in other words, Φ would be defined on a "thin" cone (a typical example being the absolutely continuous measures with respect to a fixed one). In the case when X is a cone of positive, continuous functions on a compact space K, this assumption comes exactly to the density of $Y_s - Y_s$ in $\mathcal{C}(K)$.

If for each $x \in Y^*$ there exists an increasing sequence $x_n \in Y^*$ such that $x_n \nearrow x$ and $x_n(y) < +\infty$, $\forall y \in Y_s$, then we define $\Phi(t,x)(y) := \Phi(t,x_n)(y)$. Again, there is a decreasing and l.s.c. function in $t \in (0,+\infty)$, hence

$$V_\alpha x(y) = \int_0^\infty e^{-\alpha t} \Phi(t,x)(y) dt$$

show that the definition is indeed independent of the sequence (x_n).

b) Why a duality? Except a greater generality, there are several arguments. We have to consider a general duality situation $[X,Y]$, and not limit ourselves to the dual X^*, as it may contain some "non–measurable" elements (see the assumption from prop. 2); or some finiteness condition should be imposed on the elements from the dual.

Starting with (X,Φ), (Y,Ψ) in duality, we construct (under dual hypotheses): $\mathcal{V} = (V_\alpha)$ on Y^*

$$V_\alpha u(y) = \int_0^\infty e^{-\alpha t} u\left(\Psi(t,y)\right) dt, \quad u \in Y^*, y \in Y$$

and

$$W_\alpha v(x) = \int_0^\infty e^{-\alpha t} v\left(\Phi(t,x)\right) dt, \quad v \in X^*, x \in X$$

When we try to associate (as in th. 3) another s.d.s.a.s., we need to know that the kernels in $\mathcal{V}$ are compatible with some duality $[Y^*,Z]$, i.e. to have $\mathcal{W}' = (W'_\alpha)$ such that:

$$[V_\alpha u, z] = [u, W'_\alpha z], \quad u \in Y^*, z \in Z$$

The natural choice would be $Z = X^*$ (and $\mathcal{W}' = \mathcal{W}$). However, there is no canonical duality between Y^* and X^*. Moreover, if we suppose that such a duality exists, is strongly separated and induces the duality $[X,Y]$, then we have $X^* \hookrightarrow Y^{**}$ and $Y^* \hookrightarrow X^{**}$. These maps will be the inverses of the canonical mappings $Y^{**} \to X^*$ and $X^{**} \to Y^*$.

This shows the importance of finding conditions under which the associated $\mathcal{V}$ (resp. Φ) are on X and not on Y^*.

c) In the proof of th. 3, V_0 may be supposed bounded: $V_0 x(y) \le M x(y)$, $\forall x \in Y^*$, $y \in Y$.

Indeed, for each $\beta > 0$ let us define $\Psi^\beta(t,x) := e^{-\beta t}\Psi(t,x)$. Obviously, Ψ^β is a s.d.s.a.s., whose associated resolvent is $\mathcal{V}^\beta = (V_{\alpha+\beta})_{\alpha>0}$, since:

$$\int_0^\infty e^{-\alpha t} \Psi^\beta(t,x)(y) dt = \int_0^\infty e^{-(\alpha+\beta)t} \Psi(t,x)(y) dt = V_{\alpha+\beta} x(y)$$

Now, with self–explaining notations:

$$X_s = \bigcap_{\beta>0} X_s^\beta \quad ; \quad X_e = \bigcap_{\beta>0} X_e^\beta$$

are obvious, while for:

$$X_s^\mathcal{V} = \bigcap_{\beta>0} X_s^{\mathcal{V}\beta} \quad ; \quad X_e^\mathcal{V} = \bigcap_{\beta>0} X_e^{\mathcal{V}\beta}$$

to hold, we need to consider only archimedian elements.

Now, we can replace $\mathcal{V}$ by $\mathcal{V}^\beta$, for each $\beta > 0$. We get thus a s.d.s.a.s. Ψ^β, for which:

$$V_{\alpha+\beta}x = \int_0^\infty e^{-\alpha t}\Psi^\beta(t,x)dt$$

Let $\beta' > \beta$. Then:

$$V_{\alpha+\beta'}x = \int_0^\infty e^{-\alpha t}\Psi^{\beta'}(t,x)dt =$$

$$= V_{(\alpha+h)+\beta}x = \int_0^\infty e^{-(\alpha+h)t}\Psi^\beta(t,x)dt = \int_0^\infty e^{-\alpha t}e^{-ht}\Psi^\beta(t,x)dt$$

Hence, under a right continuity assumption on x, we get:

$$\Psi^{\beta'}(t,x) = e^{-(\beta'-\beta)t}\Psi^\beta(t,x), \text{ so that } \Psi(t,x) := e^{\beta t}\Psi^\beta(t,x)$$

is independent of β.

The results above allow to translate the usual proof for the coincidence of the excessive elements ([4]). It is necessary however to use the assumption (*) in order to prove the version of Hunt's theorem: $\forall x \in X_e^\mathcal{V}\ \exists x_n$ such that $V_0x_n \nearrow x$

REFERENCES

1. Bhatia, Szegö – *Local semi–dynamical systems*, Lecture Notes in Mathematics, vol. 90, Springer–Verlag, Berlin 1990.

2. Bucur, Gh., Bezzarga, M. – *Théorie du potentiel pour les systèmes semi-dynamiques,* Rev. Roum. Math. Pures Appl. t.XXXIX,no.5 (1994),pg.439–456.
3. Cornea, A., Licea, G. – *Order and Potential. Resolvent Families of Kernels*, Lecture Notes in Math. vol. 494, Springer Verlag 1975.
4. Dellacherie, C., Meyer, P.-A. – *Probabilités et Potentiel*, Act. Sci. Ind. Hermann, Paris 1987.
5. Hajek, O. – *Dynamical systems in the plane*, Academic Press
6. Hille, E., Phillips, R.S. – *Functional Analysis and Semi–groups*, A.M.S., Providence, 1957.
7. Popa, E. – *Semi–dynamical systems on algebraic structures,* An. şt. Univ. Iaşi t. XLIV, sIa, Matematica, (1998), Supliment, pg. 585–596.
8. Popa, E. – *Excessive elements in semi–dynamical systems on algebraic structures,* An. şt. Univ. Iaşi t. XLV, sIa, Matematica, (1999), f.2, pg. 367–378.
9. Popa, E. – *On Riesz' splitting property,* Libertas Math. vol. XXII (2002), pg. 79–96.
10. Popa, E. – *Semi–groups of kernels on cones of positive measures,* Int. Congress of Math. Beijing 2002, Abstracts of short communications and poster sessions, Higher Ed. Press, pg. 130–131.

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