

ON THE RELATION BETWEEN CURVATURE, DIAMETER
AND VOLUME OF A COMPLETE RIEMANNIAN
MANIFOLD

BY

SI DUC QUANG and NGUYEN DOAN TUAN

Abstract. In this note, we prove that if N is a compact totally geodesic submanifold of a complete Riemannian manifold (M, g) , whose sectional curvature $K \geq k > 0$, then for any point $m \in M$, $d(m, N) \leq \frac{\pi}{2\sqrt{k}}$. In the case $\dim M = 2$ with a Gaussian curvature $K \geq k \geq 0$ and γ has the length ℓ , we get $Vol(M, g) \leq \frac{2\ell}{\sqrt{k}}$ if $k \neq 0$ and $Vol(M, g) \leq 2\ell \text{diam}(M)$ if $k = 0$.

1. Introduction. As well known, one of the most interesting problems in Riemannian geometry is to study a relation between geometrical notions as curvature, diameter and volume. This problem have been studied by many authors for the concrete manifolds. In [1], Y. C. Wong considered this problem for the Grassmann manifolds. W. Klingenberg proved that in a compact simply connected even-dimensional Riemannian manifold with sectional curvature K belonging to $[0, k]$, $k > 0$, the length of any closed geodesic is greater than $\frac{2\pi}{\sqrt{k}}$ (see [2]). The relation between curvature and topology of Riemannian manifolds is exposed in [3], [6].

Generalizing the known results for the sphere S^2 in Euclidian three-space, we obtained the following results.

Theorem 1. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete Riemannian manifold, whose sectional curvature K satisfies $K \geq k > 0$. Let N be a compact totally geodesic submanifold of M , then for any $m \in M$ we have $d(m, N) \leq \frac{\pi}{2\sqrt{k}}$.*

Theorem 2. *Let (M, g) be a complete Riemannian manifold of dimension 2, whose sectional curvature K satisfies $K \geq k \geq 0$, k is constant. Let γ be a closed geodesic in M of length of ℓ . Then we have*

$$\text{Vol}(M) \leq \begin{cases} \frac{2\ell}{\sqrt{k}} & \text{if } k > 0 \\ 2\ell \text{ diam}(M) & \text{if } k = 0. \end{cases}$$

The basic notions used in this article are from [4], [5].

2. Proof of theorem 1. Since N is the compact totally geodesic submanifold of M , then every geodesic of N is also a geodesic of M .

Let m be a point of M , we have to prove that $d(m, N) \leq \frac{\pi}{2\sqrt{k}}$. We assume that $m \notin N$, then $d(m, N) = L > 0$, and $\exists p \in N$ such that $d(m, N) = d(m, p) = L$. By the theorem Hopf-Rinow, there exists the minimal geodesic, parameterized by arc length:

$$c : [0, L] \rightarrow M, \quad c(0) = m, \quad c(L) = p, \quad \|c'(s)\| = 1.$$

Let γ be a geodesic in N , passing through p , then γ is also a geodesic in M . Suppose that γ is parametrized by arc length;

$$\gamma : (-\rho, \rho) \rightarrow M, \quad \rho > 0, \quad \gamma(0) = p, \quad \|\gamma'(t)\| = 1.$$

Let $Y_L = \gamma'(0) \in T_p M$, take a parallel vector field $Y(s)$ along c such that $Y(L) = Y_L$. We have $\|Y(s)\| = \|Y_L\| = 1$ for all $s \in [0, L]$.

Set $X(s) = \sin \frac{\pi s}{2L} Y(s)$, $X(s)$ is a vector field along c , $X(0) = 0$, $X(L) = Y(L) = \gamma'(0)$.

We now consider the variation H of c as follows:

$$\begin{aligned} H : [0, L] \times (-\rho, \rho) &\longrightarrow M \\ (s, t) &\longmapsto H(s, t) = \exp_{c(s)} tX(s). \end{aligned}$$

H is well defined by the completeness of manifold M . We have $H(0, t) = m$.

Set $c_t(s) = H(s, t)$, $c_0(s) = H(s, 0) = c(s)$, then

$$H(L, t) = \exp_{c(L)} tX(L) = \exp_p t\gamma'(0) = \gamma(t),$$

$$H * \frac{\partial}{\partial t}(s, 0) = X(s).$$

By construction of the variation H , the length function $L(c_t)$ attain a minimum at $t = 0$. Hence

$$(1) \quad \frac{d}{dt}L(c_t)|_{t=0} = 0$$

and

$$(2) \quad \frac{d^2}{dt^2}(L(c_t))|_{t=0} \geq 0.$$

Using the first variation formula together with remark that c is a geodesic i.e. $\nabla_{c'}c' = 0$, we have

$$(1) \quad \Leftrightarrow \langle X(s), c'(s) \rangle|_0^L - \int_0^L \langle X(s), \nabla_{c'}c'(s) \rangle ds = 0 \\ \Leftrightarrow \langle X(L), c'(L) \rangle = 0 \Leftrightarrow \langle Y(L), c'(L) \rangle = 0.$$

On the other hand

$$\frac{d}{ds}\langle Y(s), c'(s) \rangle = \left\langle \frac{D}{ds}Y(s), c'(s) \right\rangle + \left\langle Y(s), \frac{D}{ds}c'(s) \right\rangle,$$

where $\frac{D}{ds}Y(s) = \nabla_{\frac{\partial}{\partial s}}Y(s)$.

Since $\frac{D}{ds}Y(s) = 0$, $\frac{D}{ds}c'(s) = 0$, hence $\frac{d}{ds}\langle Y(s), c'(s) \rangle = 0$. Thus

$$\langle Y(s), c'(s) \rangle = \langle Y(L), c'(L) \rangle = 0 \quad \forall s \in [0, L]$$

and $\{Y(s), c'(s)\}$ is an orthonormal system.

Set $\bar{X}(s, t) = H * \frac{\partial}{\partial t}(s, t) \Rightarrow \bar{X}(s, 0) = X(s)$.

Using the second variation formula, we have (2) \Leftrightarrow

$$(3) \quad \langle \nabla_{\frac{\partial}{\partial t}}\bar{X}(s, 0), c'(s) \rangle|_0^L + \\ + \int_0^L (|\bar{X}'|^2 - \langle R(\bar{X}, c', c'), \bar{X} \rangle - \langle c', \bar{X}' \rangle^2) ds \geq 0,$$

where $\bar{X}' = \nabla_{\frac{\partial}{\partial s}}\bar{X}(s, 0)$.

We have:

$$\begin{aligned}
+) \bar{X}(s, 0) &= X(s) = \sin\left(\frac{\pi s}{2L}\right)Y(s) \\
\bar{X}'(s, 0) &= X'(s) = \frac{\pi}{2L} \cos\frac{\pi s}{2L}Y(s) + \sin\frac{\pi s}{2L} \frac{D}{ds}Y(s) = \frac{\pi}{2L} \cos\frac{\pi s}{2L}Y(s) \\
\langle c'(s), \bar{X}'(s, 0) \rangle &= \frac{\pi}{2L} \cos\left(\frac{\pi s}{2L}\right) \langle c'(s), Y(s) \rangle = 0 \\
+) \bar{X}(0, t) &= H * \frac{\partial}{\partial t}(0, t) = 0 \text{ (since } H(0, t) = m \forall t) \Rightarrow \nabla_{\frac{\partial}{\partial t}} \bar{X}(0, 0) = 0 \\
+) \bar{X}(L, t) &= H * \frac{\partial}{\partial t}(L, t) = \gamma'(t) \Rightarrow \nabla_{\frac{\partial}{\partial t}} \bar{X}(L, 0) = \frac{D}{dt} \gamma'(t)|_{t=0} = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
(3) &\Leftrightarrow \int_0^L \left[\left(\frac{\pi}{2L}\right)^2 \cos^2\left(\frac{\pi s}{2L}\right) |Y(s)|^2 - \right. \\
&\quad \left. - \sin^2\left(\frac{\pi s}{2L}\right) \langle R(Y(s), c'(s), c(s)), Y(s) \rangle \right] ds \geq 0 \\
&\Leftrightarrow \int_0^L \left[\left(\frac{\pi}{2L}\right)^2 \sin^2\left(\frac{\pi s}{2L}\right) - \sin^2\left(\frac{\pi s}{2L}\right) K(Y(s), c'(s)) \right] ds \geq 0 \\
&\Rightarrow \int_0^L \left[\left(\frac{\pi}{2L}\right)^2 - k \right] \sin^2\frac{\pi s}{2L} ds \geq 0 \Leftrightarrow \left(\frac{\pi}{2L}\right)^2 - k \geq 0 \Rightarrow L \leq \frac{\pi}{2\sqrt{k}}.
\end{aligned}$$

The theorem 1 is proved. \square

3. Proof of theorem 2. In order to prove the theorem 2, the following lemma is necessary:

Lemma. Suppose that $x : (\alpha, \beta) \rightarrow R$ is a differentiable function defined on (α, β) , where $\left(-\frac{\pi}{2} \leq \alpha < 0 < \beta \leq \frac{\pi}{2}\right)$ and satisfies

$$(*) \quad x(0) = 0, \quad x' + x^2 \leq -1.$$

Then

$$\begin{cases} x(t) \leq -\text{tg}(t), & t \in [0, \beta) \\ x(t) \geq -\text{tg}(t), & t \in (\alpha, 0]. \end{cases}$$

Proof. Set $x(t) = -\operatorname{tg} \varphi(t)$, then $\varphi(t) = -\operatorname{arctg} x(t)$, $\varphi(t) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$(*) \Leftrightarrow \varphi(0) = 0 \quad \text{and} \quad -\frac{\varphi'}{\cos^2 \varphi} + \frac{\sin^2 \varphi}{\cos^2 \varphi} \leq -1 \Leftrightarrow \frac{\varphi'}{\cos^2 \varphi} \geq \frac{1}{\cos^2 \varphi} \Leftrightarrow \varphi' \geq 1.$$

Thus $(\varphi(t) - t)$ is the increasing function

$$\Rightarrow \begin{cases} \varphi(t) \geq t & \forall t \in [0, \beta) \\ \varphi(t) \leq t & \forall t \in (\alpha, 0] \end{cases}$$

Since the function $\operatorname{tg} t$ is increasing on (α, β) , we get:

$$\begin{cases} x(t) = -\operatorname{tg} \varphi(t) \leq -\operatorname{tg} t & \forall t \in [0, \beta) \\ x(t) = -\operatorname{tg} \varphi(t) \geq -\operatorname{tg} t & \forall t \in (\alpha, 0]. \end{cases}$$

The lemma is proved. \square

Now we prove the theorem 2.

Suppose that γ is a closed geodesic parameterized by arc length.

$$\gamma : I \rightarrow M, \quad \text{where } [0, \ell] \subset I, \quad \gamma(0) = \gamma(\ell), \quad \gamma'(0) = \gamma'(\ell), \quad \|\gamma'(t)\| = 1$$

$\forall t \in I$, I is interval on R .

Since γ is a closed geodesic, then there exists an unit parallel vector field Y along γ such that $Y(0) = Y(\ell)$ and $\langle Y(0), \gamma'(0) \rangle = 0$.

For $s \in [0, \ell)$, put

$$(4) \quad \begin{aligned} \rho(s) &= \sup \{t \geq 0 \mid t = d(\exp_{\gamma(s)} tY(s), \gamma)\}, \\ \varphi(s) &= \inf \{t \leq 0 \mid -t = d(\exp_{\gamma(s)} tY(s), \gamma)\}. \end{aligned}$$

We will prove that $\rho(s)$ is upper semicontinuous and $\varphi(s)$ is lower semicontinuous. We consider $s_n \rightarrow s$, set $t_0 = \overline{\lim}_{n \rightarrow \infty} \rho(s_n)$, we claim that $t_0 \leq \rho(s)$.

In fact, since

$$t_0 = \overline{\lim}_{n \rightarrow \infty} \rho(s_n), \quad \text{then } \exists \{s_{n_k}\} \subset \{s_n\} \mid \rho(s_{n_k}) \rightarrow t_0.$$

Put

$$\begin{aligned} t_k &= \max \left\{0, \rho(s_{n_k}) - \frac{1}{k}\right\}, \quad t_k \rightarrow t_0 \Rightarrow \\ &\Rightarrow \exp_{\gamma(s_{n_k})} t_k Y(s_{n_k}) \rightarrow \exp_{\gamma(s)} t_0 Y(s) \end{aligned}$$

and

$$\begin{aligned} d(\exp_{\gamma(s)} t_0 Y(s), \gamma) &= \lim_{k \rightarrow \infty} (\exp_{\gamma(s_{n_k})} t_k Y(s_{n_k}), \gamma) = \\ &= \lim_{k \rightarrow \infty} t_k = t_0 \Rightarrow \rho(s) \geq t_0 \end{aligned}$$

Thus $\rho(s)$ is upper semicontinuous. Similarly, $\varphi(s)$ is lower semicontinuous. So that $\rho(s)$ and $\varphi(s)$ are the measurable functions.

We have:

$$A = \{(s, t) | 0 \leq s < \ell, \varphi(s) < t < \rho(s)\} \text{ is measurable set and } \text{mes } B = 0,$$

where $B = \{(s, t) | 0 \leq s < \ell, \varphi(s) = 0 \text{ or } \rho(s) = t\}$.

Consider the sets:

$$\begin{aligned} C &= \{\exp_{\gamma(s)} t Y(s) | (s, t) \in A\}, \\ D &= \{\exp_{\gamma(s)} t Y(s) | (s, t) \in B\}. \end{aligned}$$

It is clear that C, D are the images respectively of A and B under a continuous mapping. Thus C and D are measurable and $\text{mes}(D) = 0$.

We will prove that $M = C \cup D$ and $C \cap D = \emptyset$.

In fact, since γ is closed, hence γ compact.

It implies that for arbitrary $p \in M \exists q \in \gamma | r = d(p, q) = d(p, \gamma), q = \gamma(s)$.

Suppose that c is a minimal geodesic length-parameterized joining p and q

$$c : [c, r] \rightarrow M, \quad c(0) = q, \quad c(r) = p.$$

X is the parallel vector field along c such that $X(0) = \gamma'(s)$. We consider the variation

$$H : [0, r] \times (-\varepsilon, \varepsilon) \rightarrow M, H(u, v) = \exp_{c(u)} \left(v \cos \frac{\pi}{2r} u \right) X(u).$$

Put $c_v(u) = H(u, v)$, it is clear that $c_0(u) = c(u), H(0, v) = \exp_q(v\gamma'(s)) \in \gamma$, so the function $L(v)$ attain a minimum at $v = 0$.

Using the first variation formula, we have

$$\begin{aligned} \frac{d}{dv} (L(c_v)) \Big|_{v=0} &= 0 \\ \Leftrightarrow \langle \cos \frac{\pi u}{2r} X(u), c'(u) \rangle \Big|_0^r - \int_0^r \langle \cos \frac{\pi u}{2r} X(u), \nabla_{c'} c'(u) \rangle du &= 0 \\ \Leftrightarrow \langle X(0), c'(0) \rangle = 0 \Leftrightarrow \langle \gamma'(0), c'(0) \rangle = 0 \Leftrightarrow c'(0) &= \pm Y(s). \end{aligned}$$

Without generality we can suppose that $c'(0) = Y(s)$. Hence,

$$p = c(r) = \exp_{c(0)} r c'(0) = \exp_{\gamma(s)} r Y(s) \quad \text{and} \quad d(p, \gamma) = r.$$

By the definition of $\varphi(s)$, $\rho(s)$, we have $\varphi(s) \leq r \leq \rho(s)$.

From that, $(s, r) \in A \cup B$ and $p \in C \cup D$. Thus $M = C \cup D$.

In order to prove $C \cap D = \emptyset$, we suppose that there exists $p_1 \in C \cap D$. Then $\exists s_1 \neq s_2$ such that

$$p_1 = \exp_{\gamma(s_1)} t_1 Y(s_1) = \exp_{\gamma(s_2)} t_2 Y(s_2),$$

where $\varphi(s_1) < t_1 < \rho(s_1)$, $t_2 = \varphi(s_2)$ or $t_2 = \rho(s_2)$.

Choose a number t_3 such that

$$\begin{aligned} t_1 < t_3 < \rho(s_1) & \quad \text{if } t_1 \geq 0 \text{ and} \\ \varphi(s_1) < t_3 < t_1 & \quad \text{if } t_1 < 0. \end{aligned}$$

Put $q_1 = \exp_{\gamma(s_1)} t_3 Y(s_1)$. We get

$$\begin{aligned} d(q_1, \gamma) &= d(q_1, p_1) + d(p_1, \gamma(s_1)) = d(p_1, q_1) + d(p_1, \gamma) \\ &= d(p_1, q_1) + d(p_1, \gamma(s_2)) > d(q_1, \gamma(s_2)). \end{aligned}$$

This contradiction prove our assertion.

Consider the variation $\overline{H} : I \times R \rightarrow M$ defined by $\overline{H}(s, t) = \exp_{\gamma(s)} t Y(s)$. Put $H = \overline{H}|_A$. It is clear that $H(A) = C$ by definition of the set C . We now prove that H is injective. In fact, suppose (inversely) that H isn't injective, then $\exists (s_1, t_1) \neq (s_2, t_2)$, $(s_i, t_i) \in A$, $i = 1, 2$ such that $H(s_1, t_1) = H(s_2, t_2) = q$, hence $d(q, \gamma) = |t_1| = |t_2| = t_0$ by (4). There are two cases:

Case 1. If $s_1 = s_2$, $t_2 = -t_1 > 0$. Then there exists $t_3 : 0 < t_2 < t_3 < \rho(s_1)$. Consider the geodesic $c(t) = \exp_{\gamma(s_1)} t Y(s_1)$, we have $c(t_1) = c(t_2)$. By a consequence of the theorem Hopf-Rinow (see [6], p. 100), the geodesic c is no more minimal on the interval $[0, t_3]$. This is a contradiction.

Case 2. If $s_1 \neq s_2$, we can suppose $t_2 = t_0 > 0$, for all $0 < t_2 < t_3 < \rho(s_2)$

$p = \exp_{\gamma(s_2)} t_3 Y(s_2)$ we have

$$\begin{aligned} d(p, \gamma) &\leq d(p, \gamma(s_1)) < d(p, q) + d(q, \gamma(s_1)) \\ &= d(p, q) + d(q, \gamma(s_2)) = d(p, \gamma). \end{aligned}$$

This is a contradiction.

Thus H is bijective from A on C . Moreover, H is the homeomorphism.

Let μ be the canonical measure on M . We have

$$(5) \quad \begin{aligned} Vol(M) &= \int_M \mu = \int_{C \cup D} \mu = \int_C \mu + \int_D \mu = \int_C \mu \\ &= \int_{H^{-1}(C)} Gr\left(\frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t}\right)^{1/2} ds dt = \int_0^\ell \left(\int_{\varphi(s)}^{r(s)} Gr\left(\frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t}\right)^{1/2} dt \right) ds. \end{aligned}$$

+) Since $H(s, \cdot)$ is a geodesic, then

$$\nabla_{\frac{\partial}{\partial t}} \frac{\tilde{\partial}}{\partial t} = 0 \Rightarrow \left| \frac{\tilde{\partial}}{\partial t}(s, t) \right| = |Y(s)| = 1.$$

+) Because

$$\begin{aligned} \frac{d}{dt} \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t} \right\rangle &= \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t} \right\rangle = \left\langle \nabla_{\frac{\partial}{\partial s}} \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial t} \right\rangle \\ &= \frac{1}{2} \frac{d}{ds} \left\langle \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial t} \right\rangle = 0, \end{aligned}$$

we have

$$\left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t} \right\rangle(s, t) = \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t} \right\rangle(s, 0) = \langle \gamma'(s), Y(s) \rangle = 0.$$

Thus

$$Gr\left(\frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t}\right) = \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s} \right\rangle \cdot \left\langle \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial t} \right\rangle - \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t} \right\rangle^2 = \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s} \right\rangle.$$

Put $f = \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s} \right\rangle$. We will estimate the function $f(s, t)$.

From $K \geq k$ we have

$$\left\langle \bar{R}\left(\frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial s}\right), \frac{\tilde{\partial}}{\partial s} \right\rangle \geq k \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s} \right\rangle = kf,$$

hence

$$(6) \quad -\left\langle \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial s} \right\rangle \geq kf.$$

Furthermore, since

$$\left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t} \right\rangle = \left\langle \nabla_{\frac{\partial}{\partial s}} \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial t} \right\rangle = \frac{1}{2} \frac{d}{ds} \left\langle \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial t} \right\rangle = 0$$

then

$$(7) \quad \nabla_{\frac{\partial}{\partial t}} \frac{\tilde{\partial}}{\partial s} = \bar{x}(s, t) \frac{\tilde{\partial}}{\partial s}.$$

Thus

$$(8) \quad \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\tilde{\partial}}{\partial t} = \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} \frac{\tilde{\partial}}{\partial s} = \bar{x}'_t(s, t) \frac{\tilde{\partial}}{\partial s} + \bar{x}^2(s, t) \frac{\tilde{\partial}}{\partial s} = (\bar{x}'_t + \bar{x}^2) \frac{\tilde{\partial}}{\partial s}.$$

On the other hand, from (7) we get:

$$\bar{x}(s, 0) = \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s} \right\rangle(s, 0) = \frac{1}{2} \frac{d}{ds} \left\langle \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial s} \right\rangle(s, 0) = 0,$$

$$(9) \quad f'_t = \frac{d}{dt} \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s} \right\rangle = 2 \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s} \right\rangle = 2\bar{x}f.$$

From (6) and (8), we obtain

$$(10) \quad -(\bar{x}'_t + \bar{x}^2)f \geq kf \Leftrightarrow \bar{x}'_t + \bar{x}^2 \leq -k.$$

We now consider two cases.

a) In the case $k > 0$, set

$$\begin{aligned} x(s, t) &= \frac{1}{\sqrt{k}} \bar{x} \left(s, \frac{t}{\sqrt{k}} \right), \\ x'_t(s, t) &= \frac{1}{k} \bar{x}'_t \left(s, \frac{t}{\sqrt{k}} \right). \end{aligned}$$

So $kx'_t + kx^2 \leq -k \Leftrightarrow x'_t + x^2 \leq -1$.

Theorem 1 says that

$$-\frac{\pi}{2\sqrt{k}} \leq \varphi(s) \leq 0 \leq \rho(s) \leq \frac{\pi}{2\sqrt{k}} \Rightarrow -\frac{\pi}{2} \leq \sqrt{k}\varphi(s) \leq 0 \leq \sqrt{k}\rho(s) \leq \frac{\pi}{2}.$$

Using lemma 1, we get

$$\begin{cases} x(s, t) \leq -tgt & s \in [0, \ell), 0 \leq t < \sqrt{k}\rho(s) \\ x(s, t) \geq -tgt & s \in [0, \ell), \sqrt{k}\rho(s) < t \leq 0 \end{cases}$$

$$\Rightarrow \begin{cases} \bar{x}(s, t) \leq -\sqrt{kt} \operatorname{tg}(\sqrt{kt}) & s \in [0, \ell), 0 \leq t < \rho(s) \\ \bar{x}(s, t) \geq -\sqrt{kt} \operatorname{tg}(\sqrt{kt}) & s \in [0, \ell), \varphi(s) < t \leq 0. \end{cases}$$

From (9), we have $\frac{f'_t}{f} = 2\bar{x}$.

$$\begin{aligned} +) \text{ for } t \geq 0 &\Rightarrow \frac{f'_t}{f} \leq -\sqrt{k} \cdot 2\operatorname{tg}(\sqrt{kt}) \Leftrightarrow (\ln|f|)'_t \leq (\ln(\cos^2(\sqrt{kt})))' \\ &\Rightarrow [\ln|f| - \ln(\cos^2(\sqrt{kt}))]'_t \leq 0 \Rightarrow \left(\ln \frac{|f(s, t)|}{\cos^2(\sqrt{kt})} \right)'_t \leq 0 \\ &\Rightarrow \ln \frac{|f(s, t)|}{\cos^2(\sqrt{kt})} \leq \ln |f(s, 0)| \\ &\Rightarrow \frac{|f(s, t)|}{\cos^2(\sqrt{kt})} \leq |f(s, 0)| = \langle \gamma'(s), \gamma'(s) \rangle = 1 \\ &\Rightarrow |f| \leq \cos^2(\sqrt{kt}). \end{aligned}$$

$$\begin{aligned} +) \text{ for } t \leq 0 &\Rightarrow \frac{f'_t}{f} \geq -\sqrt{k} \cdot 2\operatorname{tg}(\sqrt{kt}) \Leftrightarrow (\ln|f|)'_t \geq (\ln \cos^2(\sqrt{kt}))' \\ &\Rightarrow \frac{|f|}{\cos^2 \sqrt{kt}} \leq |f(s, 0)| = 1 \Rightarrow |f| \leq \cos^2(\sqrt{kt}). \end{aligned}$$

So for all t such that $\varphi(s) < t < \rho(s)$ and for $s \in [0, \ell)$, we get $|f(s, t)| \leq \cos^2(\sqrt{kt})$.

(11)

$$\begin{aligned} \text{Thus } Vol(M) &= \int_0^\ell \left(\int_{\varphi(s)}^{\rho(s)} Gr \left(\frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t} \right)^{1/2} ds \right) dt = \int_0^\ell \left(\int_{\varphi(s)}^{\rho(s)} \sqrt{f} dt \right) ds \\ &\leq \int_0^\ell \left(\int_{\varphi(s)}^{\rho(s)} \cos \sqrt{kt} dt \right) ds \leq \int_0^\ell \left(\int_{-\frac{\pi}{2\sqrt{k}}}^{\frac{\pi}{2\sqrt{k}}} \cos \sqrt{kt} dt \right) ds = \frac{2\ell}{\sqrt{k}} \end{aligned}$$

b) In the case $k = 0$,

From (10)

$$\begin{aligned} \Rightarrow \bar{x}'_t + \bar{x}^2 &\leq 0 \quad \Rightarrow \bar{x}'_t \leq 0 \\ &\Rightarrow \begin{cases} \bar{x}(s, t) \leq \bar{x}(s, 0) = 0 & \forall 0 \leq t \leq \rho(s) \\ \bar{x}(s, t) \geq \bar{x}(s, 0) = 0 & \forall \varphi(s) < t \leq 0 \end{cases} \\ &\Rightarrow \begin{cases} f'_t \leq 0 & \forall 0 \leq t < \rho(s) \\ f'_t \geq 0 & \forall \varphi(s) < t \leq 0 \end{cases} \\ &\Rightarrow \begin{cases} f(s, t) \leq f(s, 0) = 1 & \forall 0 \leq t < \rho(s) \\ f(s, t) \leq f(s, 0) = 1 & \forall \varphi(s) < t \leq 0 \end{cases} \end{aligned}$$

$$(12) \quad \Rightarrow \text{Vol}(M) = \int_0^\ell \left(\int_{\varphi(s)}^{\rho(s)} dt \right) ds \leq \int_0^\ell 2\ell \text{diam}(M) ds = 2\ell \text{diam}(M).$$

Thus

$$\text{Vol}(M) \leq \begin{cases} \frac{2\ell}{\sqrt{k}} & \text{if } k > 0 \\ 2\ell \text{diam } M & \text{if } k = 0. \end{cases}$$

and the theorem 2 is proved. \square

REFERENCES

1. WONG, Y.C. – *Differential geometry of Grassman manifolds*, Pro. Nat. Acad. Sci. USA 1967 57p. 589-594.
2. CHEEGER, J. and EBIN, D.G. – *Comparison Theorems in Riemannian Geometry*, North-Holland, Amsterdam, 1975.
3. KAZDAN, JERRY L. – *Prescribing the curvature of a Riemannian manifold*, Conference Board of Mathematical Sciences, AMS, 1985.
4. KOBAYASHI, S. and NOMIZU, K. – *Foundations of differential geometry - inter-science*, publishers New York - London Vol 1, 1963; Vol 2, 1969.

5. GROMOLL, D., KLINGENBERG, W. and MEYER, W. – *Riemansche Geometrie in Grossen*, Lecture notes in Math. Springer, Berlin-Heidelberg 1968.
6. GALLOT, S., HULIN, D., LAFONTAINE, J. – *Riemanian Geometry*, Springer Verlag 1993.

Received: 1.VI.2003

*Hanoi University of education,
(Cau giay - Hanoi - Vietnam),
Tung-mais@fpt.vn*