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A REPRESENTATION THEOREM OF $(\mathcal{I}, \mathcal{D})$-OUTER KERNELS

BY

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The notion of outer kernel was introduced and studied together with density topology on real number set. Here the outer kernel is called measurable outer kernel and is built relatively to zero measure sets and G_δ-sets. A series of properties of these outer kernels including a theorem of existence, may be found in [1].

Later on, together with the introduction of $\mathcal{I}$-density topology on the real number sets, a new concept of outer kernel was necessary, this time with respect to first category and open sets. In [2], a number of properties of these outer kernels are proved, some of them being dual to those obtained by Abian in [1].

To introduce the notion of $\mathcal{D}$-complete σ-ideal and a density-type topology in [3], the generalization of outer kernel concept was necessary. Thus, in [3] and [4] the notion of $(\mathcal{I}, \mathcal{D})$-outer kernel was defined and analysed. In these articles some sufficient conditions for its existence for each subset of the space are demonstrated.

Existence property of outer kernel is essential in defining a $\mathcal{D}$-complete σ-ideal. It contributes essentially to prove that any family of open sets in the density-type topology generated by a $\mathcal{D}$-complete σ-ideal $\mathcal{I}$ has the union $\left(\mathcal{I} \dot{\Delta} \mathcal{D}\right)$-measurable, assuring in this way the passage from "countable" to "arbitrary" (see [3]).

Then we will study several properties of the $(\mathcal{I}, \mathcal{D})$-outer kernel as a completion for those given in the papers [3] and [4]; we will also demonstrate a structure theorem of these outer kernels.

Let $X \neq \emptyset$ an arbitrary set, $A \subset X$ and $\mathcal{I}, \mathcal{D}$, two nonempty set families of X so that $\mathcal{I} \cap \mathcal{D} = \{\emptyset\}$.

If $\mathcal{D}$ is a topology on X, we denote by $Fr_{\mathcal{D}}A$ the boundary of A in the toplogy $\mathcal{D}$.

We consider the following notations:

$\mathcal{D}(x) = \{D \in \mathcal{D}; x \in D\}$

$\mathcal{I} \dot{\Delta} \mathcal{D} = \{I \Delta D; I \in \mathcal{I} \text{ and } D \in \mathcal{D}\}$

$\mathcal{I}_{\mathcal{D}}(A) = \{x \in X; \forall D \in \mathcal{D}(x), D \cap A \notin \mathcal{I}\}$.

We also consider the follows conditions:

(α) $\forall D_1, D_2 \in \mathcal{D}$ with $D_1 \cap D_2 \neq \emptyset, \exists D_3 \in \mathcal{D} \setminus \{\emptyset\}$ so that $D_3 \subset D_1 \cap D_2$.

(β) $\exists (D_k) \subset \mathcal{D} \setminus \{\emptyset\}$ so that $\forall D \in \mathcal{D} \setminus \{\emptyset\}, \exists k \in \mathbb{N}$ so that $D_k \subset D$.

(β') $\exists (D_k) \subset \mathcal{D} \setminus \{\emptyset\}$ so that $\forall D \in \mathcal{D} \setminus \{\emptyset\}, \exists k \in \mathbb{N}$ so that $D_k \setminus D \in \mathcal{I}$.

(γ) If $\forall D \in \mathcal{D} \setminus \{\emptyset\}, \exists D' \in \mathcal{D} \setminus \{\emptyset\}$ with $D' \subset D$ and $D' \cap A = \emptyset$, then $A \in \mathcal{I}$.

(γ') If $\forall D \in \mathcal{D} \setminus \{\emptyset\}, \exists D' \in \mathcal{D} \setminus \{\emptyset\}$ with $D' \setminus D \in \mathcal{I}$ and $D' \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

(δ) If $A \cap \mathcal{I}_{\mathcal{D}}(A) = \emptyset$ then $A \in \mathcal{I}$.

Remark 1. (1) If $\emptyset \notin \mathcal{D}$, the condition (α) asserts that family $\mathcal{D}$ is a filter base.

(2) If $\mathcal{D}$ is a topology on X so that the topological space $(X, \mathcal{D})$ satisfies second countability axiom, then $\mathcal{D}$ satisfies condition (β). Condition (β') represents a generalization of (β).

(3) If $\mathcal{D}$ is a topology on X and $\mathcal{I}$ is the ideal of nowhere dense subsets of $(X, \mathcal{D})$ then $\mathcal{I}$ and $\mathcal{D}$ satisfy condition (γ).

(4) In (γ') the hypothesis of (γ) is weakened. It is obvious the fact that if $\emptyset \in \mathcal{I}$ then $(\gamma') \Rightarrow (\gamma)$.

(5) The condition (δ) represents a generalisation of Kuratowski's Theorem: "*a set A of a topological space is of first category iff A is of first category in every point $x \in A$*".

A set $K_A \subset X$ is called $(\mathcal{I}, \mathcal{D})$-*outer kernel of* A if $A \subset K_A$, $K_A \in \mathcal{I} \dot{\Delta} \mathcal{D}$ and $\forall P \in \mathcal{I} \dot{\Delta} \mathcal{D}$ with $A \subset P$ it results that $K_A \setminus P \in \mathcal{I}$.

Remark 2. If $X = \mathbb{R}$, $\mathcal{I}$ is the family of zero measure sets and $\mathcal{D}$ is the family of G_δ-sets, then the notion of $(\mathcal{I}, \mathcal{D})$-outer kernel corresponds to the concept of measurable outer kernel studied by Abian in [1].

In [3] and [4] the following existence theorems of the $(\mathcal{I}, \mathcal{D})$-outer kernel were established.

Theorem 1. [3] *If $\mathcal{D}$ is a topology on X so that $(X,\mathcal{D})$ is a T_1 space and $\mathcal{I}$ is an ideal which satisfies conditions (δ), $\mathcal{I}_{\mathcal{D}}(X)=X$ and $Fr_{\mathcal{D}}D\in\mathcal{I},\forall D\in\mathcal{D}$, then $\forall A\subset X,\exists K_A$, an $(\mathcal{I},\mathcal{D})$-outer kernel of A.*

Corollary. *If (X,d) is a complete metric space, $\mathcal{D}$ is the metric topology and $\mathcal{I}$ is an ideal which satisfies conditions (δ) and $Fr_{\mathcal{D}}D\in\mathcal{I},\forall D\in\mathcal{D}$, then $\forall A\subset X,\exists K_A$, an $(\mathcal{I},\mathcal{D})$-outer kernel of A.*

Theorem 2. [4] *If $\mathcal{I}$ is a σ-ideal which satisfies property (β) and $\mathcal{I}\,\dot{\Delta}\,\mathcal{D}$ is a σ-algebra, then $\forall A\subset X,\exists K_A$, an $(\mathcal{I},\mathcal{D})$-outer kernel of A.*

Corollary. *If (X,d) is a separable metric space, $\mathcal{D}$ is the metric topology and $\mathcal{I}$ is a σ-ideal so that $Fr_{\mathcal{D}}D\in\mathcal{I},\forall D\in\mathcal{D}$, then $\forall A\subset X,\exists K_A$, an $(\mathcal{I},\mathcal{D})$-outer kernel of A.*

This corollary results immediately from the Theorem 2 since the condition of separable metric space implies condition (β) and the property $Fr_{\mathcal{D}}D\in\mathcal{I},\forall D\in\mathcal{D}$, implies that $\mathcal{I}\,\dot{\Delta}\,\mathcal{D}$ is a σ-algebra (see [3, Proposition 2]).

In what follows we suppose that conditions (α) and (β') are satisfied. As a consequence, $\exists\,(D_k)\subset\mathcal{D}\backslash\{\emptyset\}$ so that $\forall D\in\mathcal{D}\backslash\{\emptyset\},\exists k\in\mathbb{N}$ with $D_k\backslash D\in\mathcal{I}$.

Let $H_A=\underset{D_k\subset\mathcal{I}_{\mathcal{D}}(A)}{\cup}D_k$.

Theorem 3. *If $\mathcal{I}$ is a σ-ideal which satisfies condition (γ') then $\forall A\subset X, A\backslash H_A\in\mathcal{I}$.*

Proof. Let us denote:

$K_1=\{k\in\mathbb{N};D_k\subset\mathcal{I}_{\mathcal{D}}(A)\}$ and $K_2=\{k\in\mathbb{N};D_k\cap A\in\mathcal{I}\}$.

Let $M=A\backslash\underset{k\in K_1\cup K_2}{\cup}D_k$ and $D\in\mathcal{D}\backslash\{\emptyset\}$.

From condition (β') it results that there exists $k\in\mathbb{N}$, such that $D_k\backslash D\in\mathcal{I}$.

If $k\in K_1$ then $D_k\cap M=\emptyset\in\mathcal{I}$.

Now, we suppose that $k\notin K_1$. As a consequence $D_k\not\subset\mathcal{I}_{\mathcal{D}}(A)$, hence there exists $x\in D_k$ so that $x\notin\mathcal{I}_{\mathcal{D}}(A)$.

Since $x\notin\mathcal{I}_{\mathcal{D}}(A)$, there exists $D'\in\mathcal{D}(x)$ so that $D'\cap A\in\mathcal{I}$.

$x\in D_k$ and $x\in D'$ imply that $D_k\cap D'\neq\emptyset$ and, by condition (α) it follows that there is a $D''\in\mathcal{D}\backslash\{\emptyset\}$ so that $D''\subset D_k\cap D'$.

$D''\subset D'\Rightarrow D''\cap A\in\mathcal{I}$.

By $D'' \in \mathcal{D}\backslash\{\emptyset\}$ and condition (β'), there exists $i_k \in \mathbb{N}$ so that $D_{i_k}\backslash D'' \in \mathcal{I}$.

We have:

$D_{i_k}\backslash D \subset (D_{i_k}\backslash D'') \cup (D''\backslash D_k) \cup (D_k\backslash D)$

Since $D_{i_k}\backslash D'' \in \mathcal{I}$, $D''\backslash D_k = \emptyset$ and $D_k\backslash D \in \mathcal{I}$, it results that $D_{i_k}\backslash D \in \mathcal{I}$.

$D_{i_k} \cap A = [(D_{i_k}\backslash D'') \cup (D'' \cap D_{i_k})] \cap A \subset (D_{i_k}\backslash D'') \cup (D'' \cap A)$

Since $D_{i_k}\backslash D'' \in \mathcal{I}$ and $D'' \cap A \in \mathcal{I}$, it results that $D_{i_k} \cap A \in \mathcal{I}$.

It follows that $i_k \in K_2$, hence $D_{i_k} \cap M = \emptyset \in \mathcal{I}$.

As a consequence, if $k \notin K_1$, there exists $i_k \in \mathbb{N}$ so that $D_{i_k}\backslash D \in \mathcal{I}$ and $D_{i_k} \cap M \in \mathcal{I}$.

In conclusion, $\forall D \in \mathcal{D}\backslash\{\emptyset\}, \exists k \in \mathbb{N}$ so that $D_k\backslash D \in \mathcal{I}$ and $D_k \cap M \in \mathcal{I}$.

Then, by condition (γ') it results that $M \in \mathcal{I}$.

We have the inclusion:

$$A\backslash H_A = \left[(A\backslash H_A) \backslash \bigcup_{k\in K_2} D_k\right] \cup \left[(A\backslash H_A) \cap \bigcup_{k\in K_2} D_k\right] \subset M \cup \bigcup_{k\in K_2} (A \cap D_k)$$

Since $\mathcal{I}$ is a σ-ideal and $\forall k \in K_2, A \cap D_k \in \mathcal{I}$, it results that $\bigcup_{k\in K_2} (A \cap D_k) \in \mathcal{I}$ and hence $A\backslash H_A \in \mathcal{I}$.

Theorem 4. *If $\mathcal{I}$ is a σ-ideal which satisfies condition (γ') and $\mathcal{I} \dot{\Delta} \mathcal{D}$ is a σ-algebra, then $\forall A \subset X$, the set $A \cup H_A$ is an $(\mathcal{I}, \mathcal{D})$-outer kernel of A.*

Proof. Let $K_A = A \cup H_A$.

Since $\mathcal{I} \dot{\Delta} \mathcal{D}$ is a σ-algebra and $\mathcal{D} \subset \mathcal{I} \dot{\Delta} \mathcal{D}$, it results that $H_A \in \mathcal{I} \dot{\Delta} \mathcal{D}$.

As $K_A = H_A \cup (A\backslash H_A)$ and because from Theorem 3, $A\backslash H_A \in \mathcal{I}$, it follows that $K_A \in \mathcal{I} \dot{\Delta} \mathcal{D}$.

Since $A \subset K_A$, it is sufficient to prove that $\forall P \in \mathcal{I} \dot{\Delta} \mathcal{D}$ with $A \subset P$ it results that $K_A\backslash P \in \mathcal{I}$.

We suppose that there exists $P \in \mathcal{I} \dot{\Delta} \mathcal{D}$ so that $A \subset P$ and $K_A\backslash P \notin \mathcal{I}$.

Since $K_A\backslash P \in \mathcal{I} \dot{\Delta} \mathcal{D}$, there are $D \in \mathcal{D}$ and $I_1, I_2 \in \mathcal{I}$ so that $K_A\backslash P = (D\backslash I_1) \cup I_2$.

From $K_A\backslash P \notin \mathcal{I}$ it follows that $D \neq \emptyset$. Then, from condition (β') it results that there exists $k \in \mathbb{N}$ so that $D_k\backslash D \in \mathcal{I}$.

We have:

$D_k = (D_k\backslash D) \cup (D_k \cap D) \Rightarrow D_k \cap A \subset (D_k\backslash D) \cup (D \cap A) = (D_k\backslash D) \cup ((D\backslash I_1) \cap A) \cup (D \cap I_1 \cap A)$.

By $D_k \backslash D \in \mathcal{I}$, $(D\backslash I_1) \cap A = \emptyset$ and $D \cap I_1 \cap A \in \mathcal{I}$, it follows that $D_k \cap A \in \mathcal{I}$.

Now, we prove that $\forall j \in \mathbb{N}$ with $D_j \subset \mathcal{I}_{\mathcal{D}}(A)$, $D_j \cap D_k \in \mathcal{I}$.

If there exists $j \in \mathbb{N}$ with $D_j \subset \mathcal{I}_{\mathcal{D}}(A)$ and $D_j \cap D_k \notin \mathcal{I}$, since $D_j \cap D_k \neq \emptyset$, from condition (α), $\exists D' \in \mathcal{D}\backslash \{\emptyset\}$ so that $D' \subset D_j \cap D_k$.

As $D' \subset D_k$ and $D_k \cap A \in \mathcal{I}$, it follows that $D' \cap A \in \mathcal{I}$.

On the other hand, $\forall x \in D'$, $x \in D_j$, so that $x \in \mathcal{I}_{\mathcal{D}}(A)$. Hence $D' \cap A \notin \mathcal{I}$; contradiction!

Since $\forall j \in \mathbb{N}$ with $D_j \subset \mathcal{I}_{\mathcal{D}}(A)$, $D_j \cap D_k \in \mathcal{I}$ and $\mathcal{I}$ is a σ-ideal, it follows that $D_k \cap H_A \in \mathcal{I}$.

We have the following relations:

$D\backslash I_1 \subset K_A \backslash P \subset K_A = H_A \cup (A\backslash H_A) \Rightarrow$

$D \subset D \cup I_1 = (D\backslash I_1) \cup I_1 \subset H_A \cup ((A\backslash H_A) \cup I_1)$ and hence

$D \cap D_k \subset (H_A \cap D_k) \cup ((A\backslash H_A) \cap D_k) \cup (I_1 \cap D_k)$.

Since $H_A \cap D_k \in \mathcal{I}$, $A\backslash H_A \in \mathcal{I}$ and $I_1 \cap D_k \in \mathcal{I}$, it follows that $D \cap D_k \in \mathcal{I}$.

As a consequence, $D_k = (D_k\backslash D) \cup (D \cap D_k) \in \mathcal{I}$. This is a contradiction because $D_k \in \mathcal{D}\backslash \{\emptyset\}$ and $\mathcal{D} \cap \mathcal{I} = \{\emptyset\}$.

Theorem 5. *If $\mathcal{I}$ is a σ-ideal which satisfies condition (γ') and $\mathcal{I} \dot{\Delta} \mathcal{D}$ is a σ-algebra, then $\forall K_A$, an $(\mathcal{I},\mathcal{D})$-outer kernel of A, there are $I_1, I_2 \in \mathcal{I}$ so that $K_A = (H_A\backslash I_1) \cup I_2$.*

Proof. Let K_A be a $(\mathcal{I},\mathcal{D})$-outer kernel of A. It results that $A \subset K_A$, $K_A \in \mathcal{I} \dot{\Delta} \mathcal{D}$ and $\forall P \in \mathcal{I} \dot{\Delta} \mathcal{D}$ with $A \subset P$ it follows that $K_A \backslash P \in \mathcal{I}$.

We have:

$K_A = [K_A \backslash (A \cup H_A)] \cup [(A \cup H_A) \cap K_A] =$

$[(H_A \cup (A\backslash H_A)) \backslash ((A \cup H_A) \backslash K_A)] \cup [K_A \backslash (A \cup H_A)] =$

$[H_A \backslash (H_A \backslash K_A)] \cup (A\backslash H_A) \cup [K_A \backslash (A \cup H_A)]$

Let $I_1 = H_A \backslash K_A$ and $I_2 = (A\backslash H_A) \cup [K_A \backslash (A \cup H_A)]$.

From Theorem 4 it results that $A \cup H_A$ is a $(\mathcal{I},\mathcal{D})$-outer kernel of A.

Since $K_A \in \mathcal{I} \dot{\Delta} \mathcal{D}$ and $A \subset K_A$, it follows that $(A \cup H_A) \backslash K_A \in \mathcal{I}$, hence $I_1 = (A \cup H_A) \backslash K_A \in \mathcal{I}$.

By Theorem 3 we have that $A\backslash H_A \in \mathcal{I}$.

As $A \cup H_A \in \mathcal{I} \dot{\Delta} \mathcal{D}$, $A \subset A \cup H_A$ and K_A is an $(\mathcal{I},\mathcal{D})$-outer kernel of A, it follows that $K_A \backslash (A \cup H_A) \in \mathcal{I}$. Hence $I_2 \in \mathcal{I}$.

As a conclusion, there are $I_1, I_2 \in \mathcal{I}$ so that $K_A = (H_A\backslash I_1) \cup I_2$.

Corollary. *If $\mathcal{I}$ is a σ-ideal which satisfies condition (γ') and $\mathcal{I} \dot{\Delta} \mathcal{D}$ is a σ-algebra, then $\forall A \in \mathcal{I} \dot{\Delta} \mathcal{D}$, $A \Delta H_A \in \mathcal{I}$.*

Proof. Let $A \in \mathcal{I} \dot{\Delta} \mathcal{D}$. From Theorem 3, $A \backslash H_A \in \mathcal{I}$.

Since $A \in \mathcal{I} \dot{\Delta} \mathcal{D}$ it results that A is an $(\mathcal{I}, \mathcal{D})$-outer kernel of A. Then, from Theorem 5 it follows that there are $I_1, I_2 \in \mathcal{I}$ so that $A = (H_A \backslash I_1) \cup I_2$.

As a consequence $H_A \backslash A = H_A \backslash [(H_A \backslash I_1) \cup I_2] = H_A \cap I_1 \cap C I_2 \subset I_1 \in \mathcal{I}$, hence $H_A \backslash A \in \mathcal{I}$.

In conclusion $A \Delta H_A \in \mathcal{I}$.

In what follows we suppose that conditions (α) and (β) are satisfied. The set H_A is defined as above.

The following theorems are demonstrated analogously as Theorems 3, 4, 5.

Theorem 6. *If $\mathcal{I}$ is a σ-ideal which satisfies condition (γ) then $\forall A \subset X, A \backslash H_A \in \mathcal{I}$.*

Theorem 7. *If $\mathcal{I}$ is a σ-ideal which satisfies condition (γ) and $\mathcal{I} \dot{\Delta} \mathcal{D}$ is a σ-algebra, then $\forall A \subset X$, the set $A \cup H_A$ is an $(\mathcal{I}, \mathcal{D})$-outer kernel of A.*

Corollary. *If $\mathcal{D}$ is a topology on X so that the topological space $(X, \mathcal{D})$ satisfies the second countability axiom and $\mathcal{I}$ is the σ-ideal of first category sets, then $\forall A \subset X$, the set $A \cup H_A$ is an $(\mathcal{I}, \mathcal{D})$-outer kernel of A.*

Theorem 8. *If $\mathcal{I}$ is a σ-ideal which satisfies condition (γ) and $\mathcal{I} \dot{\Delta} \mathcal{D}$ is a σ-algebra, then $\forall K_A$, an $(\mathcal{I}, \mathcal{D})$-outer kernel of A, there are $I_1, I_2 \in \mathcal{I}$ so that $K_A = (H_A \backslash I_1) \cup I_2$.*

Corollary. *If $\mathcal{D}$ is a topology on X so that the topological space $(X, \mathcal{D})$ satisfies the second countability axiom and $\mathcal{I}$ is the σ-ideal of first category sets, then $\forall K_A$, an $(\mathcal{I}, \mathcal{D})$-outer kernel of A, there are $I_1, I_2 \in \mathcal{I}$ so that $K_A = (H_A \backslash I_1) \cup I_2$.*

This corollary results from Theorem 8 and Remark 1 (2),(3). In this case the family $\mathcal{I} \dot{\Delta} \mathcal{D}$ represents the σ-algebra of the sets of X, which satisfy Baire property.

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