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ON BOUNDARY VALUE PROBLEMS OF EXTENSIONS OF SYMMETRIC OPERATORS

BY

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Abstract. Given a symmetric operator in a Hilbert space, then one can consider its selfadjoint extension a Krein space. We show that selfadjoint Krein space extension play a natural role in certain boundary value problems. We will show that boundary value problems with eigenvalue depending boundary conditions are linearized.

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1. Introduction. We shall consider canonical systems of first order differential expressions regular on the compact interval $[a, b]$. For a given symmetric linear relation S in a Hilbert space $\mathcal{H}$ the selfadjoint extensions of S can be characterized as restrictions of the adjoint S^* of S, when S is the minimal relation associated with a formally symmetric ordinary differential expression in L^2-function space, then the restrictions involve linear combinations of the boundary values of the elements in the domain $D(S^*)$ of S^*. When the selfadjoint extensions are canonical within the space $\mathcal{H}$, the coefficients of these combinations can be taken to be constants. In the case of selfadjoint extensions in inner product spaces larger than the given space $\mathcal{H}$, they depend analytically on a parameter, see [3], [9] [11], [18] and [22]. We shall prove that every generalized resolvent $\mathrm{R}(\ell)$ of S can be expressed in terms of a fixed generalized resolvent $G(\ell)$ of S and the Weyl coefficients $\Psi(\ell)$ of $R(\ell)$ relative to $G(\ell)$ as

$$(1.1) \qquad R(\ell)f = G(\ell)f + s(\ell)\Psi(\ell)[f, s(\ell)],$$

where $s(\ell)$ is a holomorphic basis for the null space $v(S^* - \ell)$, see [15], [16], and the spectrum of S can be constructed. Finally, we give examples; some in the classical boundary value problem and the others are the general boundary value problems, see [5], [6], [24], [28] and [32].

In Section 2 some preliminaries will be given; while in Section 3 the characteristic function of unitary colligation will be discussed; finally in Section 4 the resolvent operator which is related to the boundary value problems of symmetric operator, the point in which this paper is interested, will be handled.

We shall use the following notations:

$\mathbb{R}$ = Set of real numbers;

$\mathbb{C}$ = Set of complex numbers;

$\mathbb{C}^{\pm} = \{\ell \in \mathbb{C}, \pm Im(\ell) > 0\}$;

$\mathbb{N} = \{1, 2, ...\}$;

$\overline{D} = \{\ell \in \mathbb{C} | \overline{\ell} \in D\}$ where $D \subset \mathbb{C}$;

$\mathbb{C}^n$ = space of $n \times 1$ vectors with entries from $\mathbb{C}$;

$\mathcal{H}, \mathcal{K}$ = Hilbert space, Krein space respectively;

If S, T are linear (relations) subspaces in $\mathcal{H}^2$ we define:

$S + T = \{\{f, g + k\}\{f, g\} \in S, \{f, k \in T\}$;

$S - \ell I = \{\{f, g - \ell f\} | \{f, g\} \in S\}$, for $\ell \in \mathbb{C}\}$

$T \dot{+} S = \{\{f + h, g + h\} | \{f, g\} \in T, \{h, k\} \in S\}$;

The sum $T \dot{+} S$ is called direct if $S \cap T \ \{\{0, 0\}\}$;

$T \oplus S = T \dot{+} S, T$ and S orthogonal on $\mathcal{H}$;

$\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$;

$I = \{\{f, f\} \in \mathcal{H}^2\}$ identity operator in $\mathcal{H}$;

$D(T) = \{x | \exists y, \{x, y\} \in T\}$ Domain of T;

$R(T) = \{y | \exists x, \{x, y\} \in T\}$ Range of T;

$T(x) = \{y | \{x, y\} \in T\}$ in particular $T(0) = \{y | \{0, y\} \in T\}$ multivalued part;

$v(T) = \{x | \{x, 0\} \in T\}$ null space;

$ST = \{\{x, h\} | \{x, y\} \in T, \{y, h\} \in S\}$ product;

$aT = \{\{x, ay\} | \{x, y\} \in T\} a \in \mathbb{C}$;

$T^{-1} = \{\{y, x\} | \{x, y\} \in T\}$ inverse of T;

$T^* = \{\{u, v\} | [u, x] - [u, y] = 0, \forall \{x, y\} \in T$ Adjoint of T;

when $S^{\perp}$ is orthogonal complement in $\mathcal{H}^2$;
$T \ominus S = T \cap S^{\perp}$.

2. Preliminaries. In this section, we review some of the results, most of which can be found in [13] related to the canonical system of k first order linear differential equations

$$Jf'(t) - H(t)f(t) = \Delta(t)g(t) \tag{2.1}$$

on a closed bounded interval $[a, b]$ satisfying the following conditions:

(a) J is a constant $k \times k$ matrix such that $J^* = J^{-1} = -J$.

(b) H and A are absolutely integrable, $k \times k$ matrix functions on $[a, b]$ such that $H(t) = H(t)^*$ and $\Delta(t) \geq 0$ for almost all $t \in [a, b]$.

(c) The system is definite, which means that if f is an absolutely continuous $k \times 1$ vector function on $[a, b]$ such that $Jf'(t) - H(t)f(t) = 0$ for almost all $t \in [a, b]$ and $\int_0^b f(t)^*\Delta(t)f(t)dt = 0$, then $f(t) = 0$ for almost all $t \in [a, b]$.

The setting is the space $L^2(\Delta dt)$ the Hilbert space of equivalence classes $f, g, \ldots$, consisting of measurable $k \times 1$ vector functions, with inner product

$$[f, g] := \int_a^b \widetilde{g}(t)^*\Delta(t)\widetilde{f}(t)dt, \quad \widetilde{f} \in f, \quad \widetilde{g} \in g. \tag{2.2}$$

If no confusion can arise, we identify equivalence classes and their representatives. Thus, for example, we write $[f, g] = \int_a^b g(t)^*\Delta(t)f(t)dt$. In $L^2(\Delta dt)$, we introduce the linear relation T_{max} as the set of all pairs $\{f, g\} \in (L^2(\Delta dt))^2$ with the property that f contains an absolutely continuous function $\widetilde{f}$ such that for some $\widetilde{g} \in g$

$$J\widetilde{f}(t) - H(t)\widetilde{f} = \Delta(t)\widetilde{g}(t), \quad \text{almost all} \quad t \in [a, b]. \tag{2.3}$$

If $\{f, g\} \in T_{max}$, then, by the assumption that the canonical system is definite, f contains precisely one absolutely continuous $\widetilde{f}$ with the above property and as stated before we identity f with $\widetilde{f}$ and g with $\widetilde{g}$. T_{max} is called the maximal relation associated with the canonical system. For

$\{f,g\},\{h,k\} \in T_{max}$, integration by parts yields

$$\begin{aligned}\langle\{f,g\},\{h,k\}\rangle &= [g,h]-[f,k] = \\ &= \int_a^b h(t)^*\Delta(t)g(t)dt - \int_a^b k(t)^*\Delta(t)^*f(t)dt = \\ &= h(b)^*Jf(b) - h(a)^*Jf(a),\end{aligned} \tag{2.4}$$

which is known as Green's formula (or Lagrange's identity) where $\langle .,.\rangle$ is the inner product defined in $(L^2(\Delta dt))^2$. Next we introduce the minimal relation

$$\begin{aligned}&T_{min} := \{\{f,g\} \in T_{max} | f(\alpha) = O_k^1, f(b) = O_k^1\} \\ &(\text{by } O_k^1 \text{ we mean a zero matrix, column one } (1\times k))\end{aligned} \tag{2.5}$$

It turns out that T_{min} is a closed, symmetric linear relation in $L^2(\Delta dt)$ and $T_{min}^* = T_{max}$. The boundary mapping which takes $\{f,g\} \in T_{max}$ into

$$\begin{pmatrix} f(a) \\ f(b)\end{pmatrix} (2k\times 1) \in \mathbb{C}^{2k}$$

is surjective. Hence there exists an element $\{\sigma_a, T_a\}(I\times k) \in T_{max}$, uniquely determined modulo T_{min}, such that $\sigma_a(a) = -J$ and $\sigma_a(b) = O_k^k$. We use this element as it plays the role of the delta function at the endpoint a. For, by Green's formula, we have that for all $\{f,g\} \in T_{max}$

$$\langle\{f,g\},\{\sigma_a,T_a\}\rangle = f(a). \tag{2.6}$$

By $Y(t,\ell)$ we denote the $k\times k$ matrix function which solves the initial value problem

$$\begin{aligned}JY'(t,\ell) - H(t)Y(t,\ell) &= \ell\Delta(t)Y(t,\ell), \text{ almost all } t\in[a,b], \\ Y(a,\ell) &= I_k\end{aligned} \tag{2.7}$$

for all $t \in \mathbb{C}$. For each $t \in [a, b], Y(t, \ell)$ has the following properties:
(2.8)

$$\begin{cases} i. & Y(t,\ell) \text{ is entire in } \ell \in \mathbb{C}, \\ ii. & Y(t,\lambda)^* J Y(t,\ell) - J = (\ell - \overline{\lambda}) \int_a^t Y(s,\lambda)^* \Delta(s) Y(s,\ell) ds, \ell, \lambda \in \mathbb{C}, \\ iii. & Y(t,\overline{\ell})^* J Y(t,\ell) = J = Y(t,\ell) J Y(t,\overline{\ell})^*, \ell \in \mathbb{C}, \\ iv. & \text{The kernel } (Y(t,\lambda)^* - JY(t,\ell) - J)/(\ell - \overline{\lambda}) \text{ and the kernel} \\ & (Y,(t,\ell) J Y(t,\lambda)^* / (\ell - \overline{\lambda}) \text{ are positive.} \end{cases}$$

For $\ell \in \mathbb{C}$ we often denote the function $t \mapsto Y(t, \ell)$ by $Y(\ell)$. Clearly, the k columns of $Y(\ell)$ belong to $L^2(\Delta dt)$, are entire in ℓ, are linearly independent, and span the null space $v(T_{max} \quad - \ell)$. As stated before, we abbreviate this by saying that $Y(\ell)(1 \times k) \in L^2(\Delta dt)$ and forms an entire basis for $v(T_{max} - \ell)$.

By $G(\ell)$ we denote the integral operator

$$(G(\ell)f)(t) := \int_a^b G(t, s, \ell)\Delta(s)f(s)ds, \quad f \in L^2(\Delta dt), \tag{2.9}$$

where the kernel $G(t, s, \ell)$ is given by

$$G(t, s, \ell) = \frac{1}{2} \operatorname{sgn}\,(s - t)Y(t, \ell)JY(s\overline{\ell})^*, \quad s, t \in [a, b] \tag{2.10}$$

For each $\ell \in \mathbb{C}, G(\ell)$ is a bounded operator on $L^2(\Delta dt)$ is a right inverse of T_{max} - ℓ; i.e., for each $f \in L^2(\Delta dt)$ we have

$$\{G(\ell)f, \ell G(\ell)f + f\} \in T_{max},$$

or, equivalently, the function $y = G(\ell)f$ belongs to the domain $D(T_{max})$ and is the solution of the equation

$$Jy'(t) - H(t)y(t) = \ell\Delta(t)y(t) + \Delta(t)f(t). \tag{2.11}$$

It is easily checked that this solution is unique in that it also satisfies the boundary condition

$$Y(b, \ell)y(a) + y(b) = 0, \tag{2.12}$$

as $(G(\ell)f)(a) = \frac{1}{2}J[f, Y(\bar{\ell})]$. $(G(\ell)f)(b) = -\frac{1}{2}Y(b,\ell)J[f, Y(\bar{\ell})]$. This boundary condition is an example of one which depends on the eigenvalue parameter ℓ in the differential equation. Straightforward calculations yield that

$$(I + (\ell - \lambda)G(\ell))Y(\lambda) = Y(\ell)\left(I + \frac{1}{2}(\ell - \lambda)J[Y(\lambda), Y(\bar{\ell})]\right). \tag{2.13}$$

Finally, we note that $G(\ell)$ is entire in ℓ.

Now, we shall present the characterization of self adjoint extensions in (possibly indefinite) inner product spaces of a given symmetric relation; see [2], [4]. In what follows S stands for a closed symmetric linear relation in a Hilbert space $\mathcal{H}$. The adjoint S^* of S can be written as

$$S^* = S\dot{+}M\mu(S)\dot{+}M_{\overline{\mu}}(s), \text{ direct sum in } \mathcal{H}^2. \tag{2.14}$$

In this formula, known as von Neumann's formula, $\mu \in \mathbb{C}_0$ and for $\ell \in \mathbb{C}_0$

$$M_\ell(S) = \{\{f, g\} \in S^* | g = \ell f\}. \tag{2.15}$$

Note that $D(M_\ell(S)) = \varphi(S^* - \ell)$. The subspace $M_\ell(S)$ is called the defect space of S. Its dimension is constant for $\ell \in \mathbb{C}^+$ and constant for $\ell \in \mathbb{C}^-$. The two constants are the defect numbers of S, in this case there exists a self adjoint for this S; let it be A in Krein space $\mathcal{K}$. So extend S to selfadjoint relation A in Krein space $\mathcal{K}$, i.e.,

$$\mathcal{H} \subset \mathcal{K}, \ \ \mathcal{K} \text{ Krein space}$$
$$S \subset A, A \text{ selfadjoint in } \mathcal{K}, \rho(A) \neq \varphi.$$

Describe extension of S in terms of $\mathcal{H}$.

$$T(\ell) = \{\{P_{\mathcal{H}}\widetilde{h}, P_{\mathcal{H}}\widetilde{k}\} | \{\widetilde{h}, \widetilde{k}\} \in A, \widetilde{k} - \ell\widetilde{h} \in \mathcal{H}\}.$$

Theorem 2.1. *There exists* $\ominus(z) \in L(v(S^* - \overline{\mu}), v(S^*, \mu))$, *holomorphic in a neighborhood of* 0, *such that*

$$T(\ell) = S\dot{+}\{\{I - \ominus(z(\ell)))f, (\overline{\mu} - \mu \ominus (z(\ell)))f\} | f \in v(S^* - \overline{\mu})\}$$

where

$$z(\ell) = \frac{\ell - \mu}{\ell - \overline{\mu}}$$

$$v(S^* - \ell) = D(M_\ell(S^*)) = \{f | \{f, \ell f\} \in S^*\}.$$

Conversely, to any $\ominus(z) \in L(v(S^* - \overline{\mu}), v(S^* - \mu))$, *holomorphic in some neighborhood of* 0*; there exists self adjoint extension* A *of* S *in a Krein space* $\mathcal{K}$ *with* $\mu \in \rho(A)$ *and such that*

$$T(\ell) = S \dot{+} \{(I - \ominus(z(\ell))), f, (\overline{\mu} - \mu \ominus (z(\ell)))f | f \in v(S^* - \overline{\mu})\}$$

Krein space $\mathcal{K}$ *can be chosen minimal, i.e.,*

$$\mathcal{K} = c.l.s.\{I + (\ell - \mu)(A - \ell)^{-1} f | f \in \mathcal{H},\ \ell \in \rho(A)\}.$$

Also

$$\#\ \textit{positive/negative squares of}\ \mathcal{K} \ominus \mathcal{H} =$$

$$\#\ \textit{positive/negative squares of}\ S_\ominus(z(\ell), z(\lambda))$$

where

$$S_\theta(z, w) = \frac{I - \Theta(w)^* \Theta(z)}{1 - z\overline{w}}.$$

Proof. We refer to [8] and [10].

3. Characteristic functions. Previous results are based on characteristic functions of unitary colligations.

Theorem 3.1. *Let* F, g *be Hilbert spaces and* $\hat{\mathcal{K}}$ *be a Krein space, see [13], [14] and [25]*

If:

$$U = \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \widehat{K} \\ \mathcal{F} \end{pmatrix} \to \begin{pmatrix} \widehat{\mathcal{K}} \\ g \end{pmatrix}^* \ \textit{unitary operator} \tag{3.1}$$

then

$$\ominus(z) = H + zG(I - zT)^{-1} F \in L(\mathcal{F}, g)$$

and holornorphic on a neighborhood of $z = 0$.

$\ominus(z)$ *is called characteristic function of unitary colligation. Conversely, every* $\ominus(z) \in L(\mathcal{F}, g)$ *holornorphic in a neighborhood of* 0 *is obtained in such a way, i.e.,* $\exists$ Krein space K, and unitary

$$U = \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \widehat{K} \\ \mathcal{F} \end{pmatrix} \to \begin{pmatrix} \widehat{\mathcal{K}} \\ g \end{pmatrix} \text{ with } \ominus(z) = H + G(I - zT)^{-1}F. \tag{3.2}$$

The Krein space is defined only up to weak isomorphisms. Note
$\#$ positive/negative squares of $\widehat{\mathcal{K}} = \dfrac{I - \Theta(w)^*\Theta(z)}{1 - z\overline{w}}$

Lemma 3.1 Special case. *V is isometry in $\mathcal{H}$, Hilbert space, $\overline{V}$ trivial extension*

$$\begin{pmatrix} \widetilde{V} & I|_{R(V)^\perp} \\ P_{D(V)^\perp} & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ R(V)^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ D(V)^\perp \end{pmatrix} \tag{3.3}$$

is unitary with characteristic function

$$X(z) = zP_{D(V)^\perp}(I - z\widetilde{V})^{-1}|_{R(V)^\perp} \tag{3.4}$$

$X(z)$ determines simple part of V. See [1], [12] and [18].

Problem 3.1. Fix self adjoint extension A_o of S (in possibly Krein space) and define

$$\Gamma_T^0(\ell) = P_{\mathcal{H}}(I + (\ell - T)(A_0 - \ell)^{-1})P_v(S^* - T) \tag{3.5}$$

and $\ominus_o$ is corresponding mapping $L(\mu(S^* - \overline{\mu}), v(S^* - \mu))$. Describe all self adjoint extensions in terms of fixed Ao:

$$R(\ell) = P_H(A - \ell)^{-1}|_{\mathcal{H}}, \;\; R_0(\ell) = P_{\mathcal{H}}(A_0 - \ell)^{-1}|_{\mathcal{H}}$$

generalized (or compressed) resolvents

$$R(\ell) = R_0(\ell) + \frac{1}{\mu - \overline{\mu}}\Gamma_\eta^0(\ell)\varepsilon(\ell)\Gamma_{\overline{\mu}}^0(\bar{\ell})^* \tag{3.6}$$

where

$$\varepsilon(\ell) = (I - \Theta z(\ell))Xz(\ell)((I - \Theta z(\ell))Xz(\ell))^{-1}(\Theta(z(\ell)) - \Theta_0(z(\ell))). \tag{3.7}$$

See [7], [17], [19], [20] and [23].

Lemma 3.2. *In particular if S has equal defect numbers and A_0 is canonical self adjoint extension, then*

$$R(\ell) = R_0(\ell) - \Gamma^0_{\overline{\mu}}(\ell)(Q(\ell) + \mathcal{F}(\ell))^{-1}\Gamma^0_{\overline{\mu}}(\overline{\ell})^* \tag{3.8}$$

with relation $\mathcal{F}(f)$ given by

$$\mathcal{F}(\ell) = \{\{(\Theta(z(\ell)) - I)f, (\overline{\mu}\Theta(\ell)) - \mu)f \in v(S^* - \overline{\mu})\}$$

and $Q[f)$ is so called Q-function

$$Q(\ell) = -Re\,\mu + P_{v(S^*-\overline{\mu})} + ((\ell - Re\,\mu) + (\ell - \mu)(\ell - \mu))(A_0 - \ell)^{-1})P_{v(S^*-\overline{\mu})}$$

with

$$\frac{Q(\ell) - Q(\lambda)^*}{\ell - \overline{\lambda}} = \Gamma_{\overline{\mu}}(\lambda)^*\Gamma^0_{\overline{\mu}}(\ell). \tag{3.9}$$

This leads to so-called Resolvent matrix, see [29] and [31].

The Krein space is defined only up to weak isomorphisms.

Example 3.1. is of interest for boundary value problems (spectral matrix).

Riesz-Herglotz

$$\begin{gathered} N(\ell) = A + B\ell + \int_R \left(\frac{1}{t-\ell} - \frac{t}{t^2+1}\right) d\Sigma(t), \\ \Rightarrow K_N(\ell,\lambda) = B + \int_R \frac{d\Sigma(t)}{(t-\ell)(t-\overline{\lambda})} \end{gathered} \tag{3.10}$$

$\mathcal{L}(N)$ is isornorphic to

$$\begin{gathered} F(\ell) = Bc + \int_R \frac{d\Sigma(t)f(t)}{t-\ell} \quad f \in L^2(d\Sigma)c \in \mathbb{C}^n \\ \|F\|^2 = c^*Bc + \|f\|^2_\Sigma \end{gathered} \tag{3.11}$$

Recall $A_o(0) = \mathcal{L}(N) \cap \mathbb{C}^n = R(B)$.

Example 3.2. Let $N(\ell), \ell \in \mathbb{C}\backslash\mathbb{R}$ be a $n \times n$ matrix function with

(3.12) $$K_N(\ell,\lambda)=\frac{N(\ell)-N(\ell)^*}{\ell-\bar{\lambda}} \text{ non-negative kernel.}$$

In fact, we assume

$$\frac{N(\ell)-N(\ell)^*}{\ell-\bar{\ell}} \text{ positive.}$$

The kernel $K_N(\ell,\lambda)$ gives rise to a reproducing kernel Hilbert space $\mathcal{L}(N)$ of functions holomorphic on $\mathbb{C}\backslash\mathbb{R}$ such that

(3.13) $$[F(.),K_N(.,\lambda)d]=d^*F(\lambda), d\in\mathbb{C}^n$$

in the space $\mathcal{L}(N)$ "multiplication by ℓ" is a symmetric operator S, with adjoint

$$S^*=\{\{F,G\}\in\mathcal{L}(N)^2|G(\ell)-\ell F(\ell)-\alpha+N(\ell)\beta, \text{ some } \alpha,\beta\in\mathbb{C}^n\}.$$

The operator S has equal defect numbers $n\times n$, and

(3.14) $$A_0=\{\{F,G\}\in\mathcal{L}(N)^2|G(\ell)-\ell F(\ell)=\alpha, \text{ for some } \alpha\in\mathbb{C}^n\}$$

is a canonical selfadjoint of S. In fact,

(3.15) $$(A_0-\lambda)^{-1}F(\ell)=\frac{F(\ell)-F(\lambda)}{\ell-\lambda}$$

and

$$N(\ell) \text{ is the } Q- \text{ function of } S.$$

See [3] and [27]. We define the Nevanlinna class $\mathbb{N}^n$ as the class of all $n\times n$ matrix functions $N(\ell)$ which are holomorphic in $\mathbb{C}\backslash\mathbb{R}$.

4. Resolvent Operator. We define the resolvent operator associated with a closed symmetric relation and a space yield an associated nonnegative kernel and associated to it the reproducing kernel Hilbert space which we are interested in.

Theorem 4.1. *Let $N(\ell) \in \mathbb{N}^n$ have the integral representation (3.10). The Hilbert space $\mathcal{L}(N)$ is isornorphic to the space of all $n \times 1$ vector functions $F(\ell)$ of the form*

$$F(\ell) = Bc + \int_R \frac{d\Sigma(t) f(t)}{t-\ell}, \ell \in \mathbb{C}\backslash\mathbb{R} \tag{4.1}$$

where $c \in \mathbb{C}^n$, $f \in L^2(d\Sigma)$ are uniquely determined $F(\ell)$, with norm given by

$$\|F\|^2 = c^* Bc + \|f\|^2_\Sigma. \tag{4.2}$$

Proof. Let $F(\ell)$ be an element of the form (4.1). We first check that Bc and f are uniquely determined by $F(\ell)$. Indeed, if $F(\ell)$ admits two such representations with $f \in L^2(d\Sigma), c \in \mathbb{C}^n$ and $\widetilde{f} \in L^2(d\Sigma), c \in \mathbb{C}^n$, we have

$$B(\widetilde{c} - c) = \int_R \frac{d\Sigma(t)(f(t) - \widetilde{f}(t))}{t-\ell} = [f(t) - \widetilde{f}(t), (t-\widetilde{\ell})^{-1}]_\Sigma$$

Letting $\ell \to \infty$ we obtain $Bc = B\widetilde{c}$ and $f = \widetilde{f}$. The set of functions of the form (4.2) with norm (4.2) is easily seen to be a Hilbert space, which we denote by $\mathcal{K}$.

The representation (3.10) shows that the function $\ell \to K_N(\ell, \lambda)d$ belongs to $\mathcal{K}$ for any $d \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Moreover, for $F(\ell)$ of the form (4.2) we have the reproducing property of the kernel $K_N(\ell, \lambda)$:

$$[F, K_N(.,\lambda)d] = d^* F(\lambda), \quad d \in \mathbb{C}^n, \tag{4.3}$$

where $[.,.]$ denotes the inner product associated with the norm (4.2). The uniqueness of the reproducing kernel Hilbert space with reproducing kernel $K_N(\ell, \lambda)$ implies that $\mathcal{K}$ and $\mathcal{L}(N)$ are isomorphic.

Note that we may write $\mathcal{L}(N) = \mathcal{L}(B\ell) \oplus \mathcal{L}(N - B\ell) = \mathbb{C}^n_B \oplus \mathcal{L}(N - B\ell)$. The mapping

$$f \to \int_R \frac{d\Sigma(t) f(t)}{t-\ell}, \quad f \in L^2(d\Sigma), \tag{4.4}$$

is an isometry from $L^2(d\Sigma)$ onto $\mathcal{L}(N - B\ell)$ and the mapping denned by (4.1) is an isometry from $\mathbb{C}^n_B \oplus L^2(d\Sigma)$ onto $\mathcal{L}(N)$. The elements $\mathcal{L}(N)$ are

$n \times 1$ vector functions, which are defined and holomorphic on $\mathbb{C}\backslash\mathbb{R}$. This follows from the fact that $\mathcal{L}(N)$ is a reproducing kernel Hilbert space, but also from (4.1). We shall identify $\mathcal{L}(N)$ with space $\mathbb{C}_B^n \oplus L^2(d\Sigma)$. For any $n \times 1$ vector function $F(\ell)$, holomorphic on $\mathbb{C}\backslash\mathbb{R}$, and any $\lambda \in \mathbb{C}\backslash\mathbb{R}$, we define the operator R_λ by

$$(R_\lambda F)(\ell) = \frac{F(\ell) - F(\lambda)}{\ell - \lambda}, \quad \ell \in \mathbb{C}\backslash\mathbb{R},\ \ell \neq \lambda, (R_\lambda F)(\lambda) = F'(\ell) \tag{4.5}$$

For Hilbert spaces $\mathcal{K}, \mathcal{R}$ we denote by $L(\mathcal{K}, \mathcal{R})$ the space of all bounded linear operators from $\mathcal{K}$. to $\mathcal{R}$; we write $L(\mathcal{K}) = L(\mathcal{K}, \mathcal{K})$. See [21], [26] and [30].

Theorem 4.2. *For all $\lambda \in \mathbb{C}\backslash\mathbb{R}, R_\lambda \in L(\mathcal{L}(N))$. In fact, we have $\|R_\lambda F\| \leq |Im\lambda|^{-1}\ \|F\|, F \in \mathcal{L}(N)$.*

(ii) For all $\lambda \in \mathbb{C}\backslash\mathbb{R}$, $(R_\lambda)^* = R_{\overline{\lambda}}$.

(iii) The resolvent identity $R_\lambda - R_\mu, = (\lambda - \mu)R_\mu R_\lambda$ holds for all $\lambda, \mu, \in \mathbb{C}\backslash\mathbb{R}$.

Proof. Let $F(\ell) \in \mathcal{L}(N)$ have representation (4.1), then for $\lambda \in \mathbb{C}\backslash\mathbb{R}$

$$(R_\lambda F)(\ell) = \int_R \frac{d\Sigma(t) f(t)}{(t - \ell)(t - \lambda)}, \tag{4.6}$$

which on account of Theorem 3.1 implies that $R_\lambda F \in \mathcal{L}(N), \lambda \in \mathbb{C}\backslash\mathbb{R}$. By (4.5)

$$\|F_\lambda F\|^2 = \|f(t)/(t-\lambda)\|_\Sigma^2 = \int_R 1 \left(\frac{f(t)}{t-\ell}\right)^* d\Sigma(t) \left(\frac{f(t)}{t-\lambda}\right) \leq$$

$$\leq |Im\lambda|^{-2} \int_R f(t)^* d\Sigma(t) f(t) = |Im\lambda|^{-2} \|f\|_\Sigma^2 \leq |Im\lambda|^{-2} \|F\|^2,$$

for $\lambda \in \mathbb{C}\backslash\mathbb{R}$, which shows that R_λ is a bounded operator in $\mathcal{L}(N)$. This proves (i). Items (ii) and (iii) also follow from (4.6).

Combining (ii) and (iii), we obtain the identity

$$R_\lambda - R_\mu^* = (\lambda - \overline{\mu}) R_\mu^* R_\lambda, \quad \lambda, \mu \in \mathbb{C}\backslash\mathbb{R} \tag{4.7}$$

Conversely, if $R_\lambda \in L(\mathcal{L}(N))$ satisfies (4.7), then the inequality in (i), (ii) and (iii) follow. From a selfadjoint relation (i.e., multivalued operator) A

in a Hilbert space $\mathcal{K}, A(0) = \{\varphi \in \mathcal{K} | \{0, \varphi\} \in A\}$ stands for the multivalued part of A and:

$A_s = A \cap (\mathcal{K}(\ominus A(0))^2$ is the operator part of A, which is a (densely defined) selfadjoint operator in $H \ominus A(O)$ with $D(A_s) = D(A)$.

Theorem 4.3. *Let $N(\alpha) \in \mathbb{N}^n$ with $Im N(\alpha)$ invertible for some $\alpha \in \mathbb{C} \backslash \mathbb{R}$. Then S is simple with defect numbers (n, n) and $N(\ell)$ is the Q function of S and its canonical selfadjoint extension A.*

Proof. On account of (4.3) the spaces $R(S-\alpha)$ and $K_N(., \alpha)\mathbb{C}^n, \alpha \in \mathbb{C} \backslash \mathbb{R}$, are orthogonal. Their sum is all of $\mathcal{L}(N)$, since each element $F(\ell)$ in $\mathcal{L}(N)$ can be written as

$$F(\ell) = (\ell - \alpha)G(\alpha) + K_N(\ell, \alpha)K_N(\alpha, \alpha)^{-1}F(\alpha), \tag{4.8}$$

for some $G_\alpha(\ell) \in \mathcal{L}(N)$. To see this, define $G_\alpha(\ell)$ by

$$G_\alpha(\alpha) = \frac{F(\ell) - K_N(\ell, \alpha)K_N(\alpha, \alpha)^{-1}F(\alpha)}{\ell - \alpha} =$$
$$= \frac{F(\ell) - F(\alpha)}{\ell - \alpha} + \frac{K_N(\alpha, \alpha) - K_N(\ell, \alpha)}{\ell - \alpha} K_N(\alpha, \alpha)^{-1}F(\alpha).$$

Then, since $\mathcal{L}(N)$ is resolvent invariant, $G_\alpha(\ell) \in \mathcal{L}(N)$ and by (4.8) $\ell G_\alpha(\ell) \in \mathcal{L}(N)$. Hence, we obtain

$$\mathcal{L}(N) = R(S - \alpha) \oplus K_N(., \alpha)\mathbb{C}^n, \alpha \in \mathbb{C} \backslash \mathbb{R}. \tag{4.9}$$

It follows that the null space $v(S^* - \overline{\alpha}), \alpha \in \mathbb{C} \backslash \mathbb{R}$, is spanned by the column vectors of $K_N(., \alpha)$ and therefore n dimensional. This implies that the defect numbers of S are (n, n). By (4.9), to prove the simplicity of S, it suffices to show that $\mathcal{L}(N)$ is equal to the closed linear span of $K_N(., \alpha)\mathbb{C}^n$, where α runs through $\mathbb{C} \backslash \mathbb{R}$. But this is a direct consequence of the reproducing kernel property. In order to show that $N(\ell)$ is a Q-function for S and A we define the mapping Γ_ℓ from $\mathbb{C}^n$ onto $v(S^* - \ell)$ by $c \to K_N(., \overline{\ell})c$.

It is clear that

(4.10)
$$[\Gamma_\ell c, \Gamma_\lambda d]_{\mathcal{L}(N)} = d^*[L_N(., \overline{\ell}), K_N(., \overline{\lambda})]_{\mathcal{L}(N)} = d^* \frac{N(\ell) - N(\lambda)^*}{\ell - \overline{\lambda} c} c,\ c, d \in \mathbb{C}^n,$$

which shows that $N(\ell)$ is a Q function for S and A. This concludes the proof.

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