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LATTICEAL REPRESENTATIONS OF GROUPS

BY

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Abstract. The aim of this paper is to introduce the concept of latticeal representation of a group and also to give some connections between the linear representations of a group and its latticeal representations.

1. Preliminaries. Let G be a group. The notion of G-lattice (introduced by the author in the paper [6]) appeared by the extension of the situation in which the group G acts on its subgroup lattice via conjugation.

A *G-lattice* is a G-set L (relative to an action ρ of G on L; for $(g, \ell) \in G \times L$, we denote by $g \circ \ell$ the element $\rho(g)(\ell) \in L$) endowed with a lattice structure (we denote by "$\wedge$" and "$\vee$" the binary operations of L) such that there are satisfied the following compatibility conditions:

$$\text{i)} \quad g \circ (\ell_1 \wedge \ell_2) = (g \circ \ell_1) \wedge (g \circ \ell_2),$$
$$\text{ii)} \quad g \circ (\ell_1 \vee \ell_2) = (g \circ \ell_1) \vee (g \circ \ell_2),$$

for any $g \in G$ and $\ell_1, \ell_2 \in L$.

If L and L' are two G-lattices, then a lattice homomorphism $u : L \to L'$ which satisfies the condition $u(g \circ \ell) = g \circ u(\ell)$, for any $(g, \ell) \in G \times L$, is called a *G-lattice homomorphism*. Moreover, if the map u is one-to-one and onto, then it is called a *G-lattice isomorphism*.

In his book [7] the author considered the category G-lat. of G-lattices (i.e. the category whose objects are all G-lattices and whose homomorphisms are G-lattice homomorphisms) and he made some constructions in this category. Among these we remind the following:

A *G-sublattice* of a G-lattice L is a sublattice L' of L satisfying the property:

$$G \circ L' = \{g \circ \ell' \mid g \in G,\ \ell' \in L'\} \subseteq\ L'.$$

If L is a G-lattice having the initial element 0 and $(L_i)_{i\in I}$ is a finite family of G-sublattices of L, then we say that L is the *direct* $\vee$*-sum* of the family $(L_i)_{i\in I}$ (and we denote this fact by $L = \overset{\vee}{\bigoplus\limits_{i\in I}} L_i$), if the following equalities hold:

i) $L = \bigvee\limits_{i\in I} L_i$.

ii) $L_j \wedge \left(\bigvee\limits_{\substack{i\in I\\ i\neq j}} L_i \right) = \{0\}$, for any $j \in I$.

A *G-congruence* on a G-lattice L is a congruence relation "$\sim$" on L which has the property that $\ell \sim \ell'$ $(\ell, \ell' \in L)$ implies $g \circ \ell \sim g \circ \ell'$, for any $g \in G$. If "$\sim$" is a G-congruence on the G-lattice L, then, defining $g \circ [\ell] = [g \circ \ell]$, for all $(g, \ell) \in G \times L$, the factor lattice $L/\!\sim\ = \{[\ell] \mid \ell \in L\}$ of L modulo "$\sim$" becomes a G-lattice, which is called the *factor G-lattice* of L relative to "$\sim$".

Let $(L_i)_{i\in I}$ be a finite family of G-lattices. Then, defining:

$$\begin{aligned}(\ell_i)_{i\in I} \wedge (\ell'_i)_{i\in I} &= (\ell_i \wedge \ell'_i)_{i\in I},\\ (\ell_i)_{i\in I} \vee (\ell'_i)_{i\in I} &= (\ell_i \vee \ell'_i)_{i\in I},\\ g \circ (\ell_i)_{i\in I} &= (g \circ \ell_i)_{i\in I},\end{aligned}$$

the cartesian product $\bigtimes\limits_{i\in I} L_i = \{(\ell_i)_{i\in I} \mid \ell_i \in L_i, \text{ for any } i \in I\}$ becomes a G-lattice, which is called the *direct product* of the G-lattices L_i, $i \in I$.

With respect to these concepts we mention three results (see [7], 2.2, Propositions 4, 6 and 8), which are used in the rest of this paper.

Proposition 1. *Let L be a distributive G-lattice and $(L_i)_{i\in I}$ be a finite family of G-sublattices of L. If L has the initial element* 0 *and the family*

$(L_i)_{i\in I}$ *satisfies the property* $L_j \wedge \left(\bigvee_{\substack{i\in I\\ i\neq j}} L_i\right) = \{0\}$, *for any* $j \in I$, *then the following two conditions are equivalent:*

i) $L = \bigoplus_{i\in I}^{\vee} L_i$.

ii) *Every element* $\ell \in L$ *can be written uniquely as* $\ell = \bigvee_{i\in I} \ell_i$, *where* $\ell_i \in L_i$, *for any* $i \in I$.

Proposition 2. *Let* $(L_i)_{i\in I}$ *be a finite family of* G*-lattices and* L *be its direct product. If the* G*-lattices* L_i, $i \in I$, *have trivial initial elements, then there exists a* G*-sublattice family* $(L'_i)_{i\in I}$ *of* L *which satisfies the following properties:*

a) $L = \bigoplus_{i\in I}^{\vee} L'_i$.

b) $L'_i \cong L_i$ *(*G*-lattice isomorphism), for any* $i \in I$.

Proposition 3. *Let* $(L_i)_{i\in I}$ *be a finite family of* G*-lattices and* $\sim_i$ *be a* G*-congruence on* L_i, *for any* $i \in I$. *Then the binary relation* $\bigtimes_{i\in I} \sim_i$ *on the direct product* $\bigtimes_{i\in I} L_i$, *defined by:*

$$(\ell_i)_{i\in I} \bigtimes_{i\in I} \sim_i (\ell'_i)_{i\in I} \quad \text{iff} \quad \ell_i \sim_i \ell'_i, \quad \text{for all } i \in I,$$

is a G*-congruence.*

Conversely, any G*-congruence on the direct product* $\bigtimes_{i\in I} L_i$ *is of this form.*

2. Main results

2.1. The category of latticeal representations of a group. Similarly to the case of linear representations of groups, where, given a group G

and a field K, there exists an "equivalence" between the concept of $K[G]$-module and the concept of linear K-representation of the group G, for the concept of G-lattice there exists an "equivalent concept". This is the concept of latticeal representation of the group G.

Let $(G, \cdot, e)$ be a group and $L(G)$ be its subgroup lattice.

Definition 1. Let $(L, \wedge, \vee)$ be a lattice and $(\mathrm{Aut}(L), \circ)$ be the automorphism group of L. A group homomorphism $f : G \to \mathrm{Aut}(L)$ is called a *latticeal representation* of G by L (in this case L is called the *lattice of the representation* f). Moreover, if the homomorphism f is one-to-one, then we say that the latticeal representation f is *faithful*.

Remark. Let $(L, \wedge, \vee)$ be a lattice. Then any latticeal representation $f : G \to \mathrm{Aut}(L)$ of the group G by the lattice L defines a monoid homomorphism $\overline{f} : G \to \mathrm{End}(L)$, which is obtained by composition of f with the canonical inclusion $\mathrm{Aut}(L) \to \mathrm{End}(L)$. Conversely, any monoid homomorphism $\varphi : G \to \mathrm{End}(L)$ defines a latticeal representation of the group G by the lattice L. Indeed, since $1_L = \varphi(e) = \varphi(gg^{-1}) = \varphi(g)\circ\varphi(g^{-1}) = \varphi(g^{-1})\circ \varphi(g)$ (so that $\varphi(g) \in \mathrm{Aut}(L)$), for any $g \in G$, we have $\mathrm{Im}\,\varphi \subseteq \mathrm{Aut}(L)$.

Definition 2. Let $f : G \to \mathrm{Aut}(L)$ and $f' : G \to \mathrm{Aut}(L')$ be two latticeal representations of G by the lattices L, respectively L'. We say that f and f' are *isomorphic* or *equivalent* and we denote this fact by $f \cong f'$, if there exists a lattice isomorphism $u : L \to L'$ such that $f'(g) \circ u = u \circ f(g)$, for any $g \in G$.

Proposition 1. *Let $(L, \wedge, \vee)$ be a lattice. Then there exists an one-to-one and onto correspondence between the set of all latticeal representations of G by L and the set of all G-lattice structures on L.*

Proof. If $f : G \to \mathrm{Aut}(L)$ is a latticeal representation of G by L, then L becomes a G-lattice with respect to the action:

$$\begin{aligned} "\circ_f" : G \times L &\to L, \\ (g, \ell) &\mapsto \circ_f(g, \ell) \overset{\text{not.}}{=\!=\!=} g \circ_f \ell = f(g)(\ell). \end{aligned}$$

If L is a G-lattice with respect to the action "$\circ$" of G on L, then the map:

$$f_\circ : G \to \mathrm{Aut}(L),$$
$$f_\circ(g)(\ell) = g \circ \ell,$$

is a latticeal representation of G by L. Moreover, it is easy to see that the maps $f \mapsto$ "$\circ_f$" and "$\circ$" $\mapsto f_\circ$ are inverted to each other.

Remark. From Proposition 1, it results that all examples of G-lattices included in [7] give us examples of latticeal representations of the group G. Therefore we have the following latticeal representations:

1. $f : G \to \mathrm{Aut}(\mathcal{P}(G))$,
   $f(g)(A) = gA = \{ga \mid a \in A\}$, for any $(g, A) \in G \times \mathcal{P}(G)$.

2. $f : G \to \mathrm{Aut}(L(G))$,
   $f(g)(H) = H^g$, for any $(g, H) \in G \times L(G)$.

3. $f : G \to \mathrm{Aut}(G/H)$ (where $H \lhd G$),
   $f(g)(g_1H) = (gg_1)H$, for any $(g, g_1H) \in G \times (G/H)$.

4. $f : G \to \mathrm{Aut}(G)$ (where G is an ordered latticeal group),
   $f(g)(g') = gg'$, for any $(g, g') \in G \times G$.

5. $f : G \to \mathrm{Aut}(L_n)$, $f(g)(d) = g(d)$, for any $(g, d) \in G \times L_n$ (where L_n is the lattice of all natural divisors of $n \in \mathbb{N}^*$, and $G = \{\sigma \in S_n \mid \sigma(L_n) = L_n,\ \sigma((d_1, d_2)) = (\sigma(d_1), \sigma(d_2))$ and $\sigma([d_1, d_2]) = [\sigma(d_1), \sigma(d_2)]$, for all $d_1, d_2 \in L_n\})$.

In the following the latticeal representation which we presented in Example 2 will be called the *regular latticeal representation* of the group G and will be denoted by f_G.

Definition 3. Let $f : G \to \mathrm{Aut}(L)$ and $f' : G \to \mathrm{Aut}(L')$ be two latticeal representations of G by the lattices L, respectively L'. A lattice homomorphism $u : L \to L'$ which satisfies the property $f'(g) \circ u = u \circ f(g)$, for any $g \in G$, is called a *homomorphism of latticeal representations* between f and f'.

Proposition 2. *Let $f : G \to \mathrm{Aut}(L)$, $f' : G \to \mathrm{Aut}(L')$ be two latticeal representations of G by the lattices L, respectively L' and "$\circ$, "$\circ'$" be the actions which give us the structures of G-lattices on L, respectively on L'. Then, for a lattice homomorphism $u : L \to L'$, the following conditions are equivalent:*

(i) *u is a homomorphism of latticeal representations between f and f'.*

(ii) *u is a G-lattice homomorphism between L and L'.*

Proof. (i)$\Longrightarrow$(ii) We have $g \circ' u(\ell) = f'(g)(u(\ell)) = (f'(g) \circ u)(\ell) = (u \circ f(g))(\ell) = u(f(g)(\ell)) = u(g \circ \ell)$, for any $(g, \ell) \in G \times L$.

(ii)$\Longrightarrow$(i) We have $(f'(g) \circ u)(\ell) = f'(g)(u(\ell)) = g \circ' u(\ell) = u(g \circ \ell) = u(f(g)(\ell)) = (u \circ f(g))(\ell)$, for any $(g, \ell) \in G \times L$, therefore $f'(g) \circ u = u \circ f(g)$, for any $g \in G$.

We denote by $\operatorname{Repr} G$ the category of latticeal representations of the group G, i.e. the category in which the objects are all latticeal representations of G and the homomorphisms are homomorphisms of latticeal representations of G.

The following result gives us the main connection between the categories $\operatorname{Repr} G$ and G-lat.

Proposition 3. *The categories* $\operatorname{Repr} G$ *and G-lat. are isomorphic.*

Proof. We have a functor $T : \operatorname{Repr} G \to G$-lat. given by:

(i) for a latticeal representation $f : G \to \operatorname{Aut}(L)$ of G by the lattice L, $T(f)$ is the G-lattice L,
(ii) for a homomorphism of latticeal representations $u \in \operatorname{Repr} G(f, f')$, $T(u)$ is the G-lattice homomorphism u.

We also have a functor $S : G$-lat.$\to \operatorname{Repr} G$, given by:

(i) for a G-lattice L (with respect to the action "$\circ$" of G on L) $S(L)$ is the latticeal representation $f : G \to \operatorname{Aut}(L)$ of G by the lattice L, defined by $f(g)(\ell) = g \circ \ell$, for any $(g, \ell) \in G \times L$,
(ii) for a G-lattice homomorphism $u : L \to L'$, $S(u)$ is the homomorphism of latticeal representations u.

Moreover, it is easy to see that $T \circ S = 1_{G-\text{lat.}}$ and $S \circ T = 1_{\operatorname{Repr} G}$, i.e. the functors T and S are isomorphisms of categories.

In the following we introduce some concepts in the category $\operatorname{Repr} G$, by duality to the concepts which we constructed in the category G-lat.

Definition 4. Let $f : G \to \operatorname{Aut}(L)$ be a latticeal representation of G by the lattice L and let L' be a G-sublattice of L. Then the map

$f|L' : G \to \mathrm{Aut}(L')$, defined by $(f|L')(g)(\ell') = f(g)(\ell')$, for any $(g, \ell') \in G \times L'$, is a latticeal representations of G by the lattice L', which is called the *latticeal subrepresentation* of f determined by the G-sublattice L' of L.

Remark. The latticeal subrepresentations of latticeal representations of the group G are just the subobjects of the category $\mathrm{Repr}\, G$.

Definition 5. Let $f : G \to \mathrm{Aut}(L)$ be a latticeal representation of G by the lattice L and $f_i : G \to \mathrm{Aut}(L_i)$, $i \in I$, be a finite family of latticeal subrepresentations of f. If the G-lattice L has an initial element, then we say that f is the *direct* $\vee$*-sum* of its latticeal subrepresentations f_i, $i \in I$ (and we denote this fact by $f = \overset{\vee}{\bigoplus\limits_{i\in I}} f_i$), if $L = \overset{\vee}{\bigoplus\limits_{i\in I}} L_i$.

Definition 6. Let $f_i : G \to \mathrm{Aut}(L_i)$, $i \in I$, be a finite family of latticeal representations of G by the lattices L_i, $i \in I$. Then the latticeal representation of G:

$$\bigtimes_{i\in I} f_i : G \to \mathrm{Aut}\left(\bigtimes_{i\in I} L_i\right),$$
$$\left(\bigtimes_{i\in I} f_i\right)(g)((\ell_i)_{i\in I}){=}(f_i(g)(\ell_i))_{i\in I}, \text{ for any } (g, (\ell_i)_{i\in I}) \in G\times \left(\bigtimes_{i\in I} L_i\right),$$

is called the *direct product* of the latticeal representations f_i, $i \in I$.

Remark. The direct products of latticeal representations of the group G are just the direct products of the category $\mathrm{Repr}\, G$.

Definition 7. Let $f : G \to \mathrm{Aut}(L)$ be a latticeal representation of G by the lattice L and "$\sim$" be a G-congruence on L. The the latticeal representation of G:

$$f_\sim : G \to \mathrm{Aut}(L/\sim),$$
$$f_\sim(g)([\ell]) = [f(g)(\ell)], \text{ for any } (g, [\ell]) \times G \times (L/\sim),$$

is called the *factor latticeal representation of* f with respect to "$\sim$".

Next result can be easily proved:

Proposition 4. *Let $f : G \to \mathrm{Aut}(L)$ be a latticeal representation of G by the lattice L. Then the map:*

$$\overline{f} : G \to \mathrm{Aut}(\mathcal{P}(L)),$$
$$\overline{f}(g)(A) = f(g)(A) = \{f(g)(a) \mid a \in A\}, \text{ for any } (g, A) \in G \times \mathcal{P}(L),$$

is a latticeal representation of G by the lattice $\mathcal{P}(L)$.

Moreover, if L is fully ordered, then the map $u : L \to \mathcal{P}(L)$, defined by $u(\ell) = \{\ell' \in L \mid \ell' \leq \ell\}$, for any $\ell \in L$, is a homomorphism of latticeal representations between f and $\overline{f}$.

In the following we introduce some variants of the decomposability concept of the latticeal representations. In the case of the regular latticeal representations, these give us particularizations of the L-decomposability concept (see [5], page 346).

Let $f : G \to \mathrm{Aut}(L)$ be a latticeal representation of G with finite representation lattice, $\mathrm{Fix}_G(L) = \{\ell \in L \mid f(g)(\ell) = \ell$, for any $g \in G\}$ and 0 (respectively 1) be the initial (respectively the final) element of the G–latice L.

Definition 8. We say that f (or that the G-lattice L) is:

(i) *irreducible*, if $\{0, 1\} \subseteq \mathrm{Fix}_G(L)$ and the only G–sublattices of L are $\{0\}$, $\{1\}$ and L; otherwise, f (or the G-lattice L) is called *reducible.*

(ii) *completely reducible*, if there exists a family $(L_i)_{i \in I}$ of irreducible G-sublattices of L such that $L = \overset{\vee}{\bigoplus\limits_{i \in I}} L_i$.

(iii) *fully decomposable*, if there exists a family $(L_i)_{i \in I}$ of fully ordered G-sublattices of L such that $L = \overset{\vee}{\bigoplus\limits_{i \in I}} L_i$.

Next result shows that the class of completely reducible latticeal representation with finite distributive representation lattices is closed under subobjects and factor objects (for simplicity, we formulate this into a "G-lattice language").

Proposition 5. *Any G-sublattice and any factor G-lattice of a completely reducible distributive finite G-lattice are completely reducible.*

Proof. Let L be a completely reducible distributive finite G-lattice and $(L_i)_{i\in I}$ be a family of irreducible G-sublattices of L such that $L = \overset{\vee}{\bigoplus_{i\in I}} L_i$. Then L is isomorphic with the direct product $\bigtimes_{i\in I} L_i$ of the G-lattices L_i, $i \in I$. For each $i \in I$, let $L_i = \{0, \ell_i\}$, where 0 is the initial element of L.

If L' is a G-sublattice of L, then $L' = \overset{\vee}{\bigoplus_{i\in J}} L_i \cong \bigtimes_{i\in J} L_i$, where $J = \{i \in I \mid \ell_i \in L'\}$. Therefore L' is completely reducible.

If $\sim$ is a G-congruence on L, then, for any $i \in I$, there exists a G-congruence $\sim_i$ on L_i such that $\sim = \bigtimes_{i\in I} \sim_i$. Since any factor G-lattice $L_i/\sim_i$, $i \in I$, is irreducible, from the G-lattice isomorphism:

$$L/\sim \cong \bigtimes_{i\in I} (L_i/\sim_i),$$

we obtain that the factor G-lattice $L/\sim$ is completely reducible.

Concerning the subgroup lattice of a finite group, we have the following result, whose proof is a simple exercise:

Proposition 6. *Let G be a finite group of order n and $L(G)$ be its subgroup lattice. Then the following statements hold:*

i) *$L(G)$ is a irreducible G-lattice, if and only if the group G is cyclic and n is a prime number.*
ii) *If the group G is cyclic, then:*
    a) *$L(G)$ is a fully decomposable G-lattice;*
    b) *$L(G)$ is a completely reducible G-lattice, if and only if n is a square–free number.*

2.2. Connections between the linear representations of a group and its latticeal representations. Let K be a field and $\mathrm{Repr}_K(G)$ be the category of linear K-representations of the group G. We denote by $\mathrm{Aut}_K(V)$ the group of all invertible linear transformations of a vector space V (over the field K) onto itself.

Remarks. 1) If $\rho : G \to \mathrm{Aut}_K(V)$ is a K-representation of the group G, then the set $L_G^{\rho}(V)$ consisting of all G-subspaces of V is a G-lattice, where

$g*V' = \rho(g)(V')$, for any $(g, V') \in G\times L_G^{\rho}(V)$. Therefore we have a latticeal representation $f_\rho : G \to \mathrm{Aut}(L_G^{\rho}(V))$ of the group G with representation lattice $L_G^{\rho}(V)$, which is called *the latticeal representation induced by the K-representation ρ.*

2) If $\rho_1 : G \to \mathrm{Aut}_K(V_i)$, $i = 1, 2$, are two K-representations of the group G and $u : V_1 \to V_2$ is a homomorphism of K-representations (in the category $\mathrm{Repr}_K(G)$) between ρ_1 and ρ_2, then $\bar{u} : L_G^{\rho_1}(V_1) \to L_G^{\rho_2}(V_2)$, $\bar{u}(V_1') = u(V_1')$, for any $V_1' \in L_G^{\rho_1}(V_1)$, is a homomorphism of latticeal representations (in the category $\mathrm{Repr}\, G$) between f_{ρ_1} and f_{ρ_2}.

3) From the remarks 1), 2) we have a functor $F : \mathrm{Repr}_K(G) \to \mathrm{Repr}\, G$.

We consider the following problem:

"If $f : G \to \mathrm{Aut}(L)$ is a latticeal representation of G with finite representation lattice, then there exists a K-representation of finite degree $\rho : G \to \mathrm{Aut}_K(V)$ of G such that f is isomorphic with a subrepresentation of the latticeal representation f_ρ induced by the K-representation ρ?"

We shall solve this problem in the particularly case when G is a finite cyclic group, f is a fully decomposable latticeal representation and L is a distributive lattice.

Proposition 1. *If G is a finite cyclic group and K is a field, then any fully decomposable latticeal representation $f : G \to \mathrm{Aut}(L)$ of G with distributive finite representation lattice is isomorphic with a subrepresentation of the latticeal representation f_ρ induced by a K-representation of finite degree ρ of G.*

Proof. Let g be a generator of the finite cyclic group G and $n = |G|$. Since f is a fully decomposable latticeal representation and the lattice L is distributive, there exists a family $(L_i)_{i=\overline{1,s}}$ of fully ordered G-sublattices of L such that:

$$L = \bigoplus_{i=\overline{1,s}}^{\vee} L_i \cong \bigtimes_{i=1}^{s} L_i.$$

For each $i \in \{1, 2, ..., s\}$, let $L_i = \{0, \ell_{i1}, \ell_{i2}, ..., \ell_{i\alpha_i}\}$, where 0 is the initial element of L and $0 < \ell_{i1} < \ell_{i2} < \cdots < \ell_{i\alpha_i}$. We consider the vector space $V = K^\alpha$, where $\alpha = \sum_{i=1}^{s} \alpha_i$ and the linear transformation $u : V \to V$ defined by $u(x_{11}, x_{12}, ..., x_{1\alpha_1}, x_{21}, x_{22}, ..., x_{2\alpha_2}, ..., x_{s1}, x_{s2}, ..., x_{s\alpha_s}) = (x_{11} + x_{12} +$

$\cdots + x_{1\alpha_1}, x_{12} + x_{13} + \cdots + x_{1\alpha_1}, ..., x_{1\alpha_1}, x_{21} + x_{22} + \cdots + x_{2\alpha_2}, x_{22} + x_{23} + \cdots + x_{2\alpha_2}, ..., x_{2\alpha_2}, ..., x_{s1} + x_{s2} + \cdots + x_{s\alpha_s}, x_{s2} + x_{s3} + \cdots + x_{s\alpha_s}, ..., x_{s\alpha_s})$, for any $(x_{11}, x_{12}, ..., x_{1\alpha_1}, x_{21}, x_{22}, ..., x_{2\alpha_2}, ..., x_{s1}, x_{s2}, ..., x_{s\alpha_s}) \in V$. Since the matrix of u in the canonical basis of K^α is invertible, we have $u \in \mathrm{Aut}_K(V)$. Then the map $\rho : G \to \mathrm{Aut}_K(V)$, $\rho(g^k)(v) = u^k(v)$, for any $k \in \{0, 1, ..., n-1\}$ and any $(g, v) \in G \times V$, is a K-representation of degree α of the group G. Using the fact that G is cyclic, we obtain that the G-subspaces of V are the subspaces of the vector space V which are invariant by u. We consider the following G-subspaces of V:

$$
\begin{aligned}
&W_0 = 0,\\
&W_{11} = \{(x_{11}, 0, ..., 0) \mid x_{11} \in K\},\\
&W_{12} = \{(x_{11}, x_{12}, 0, ..., 0) \mid x_{11}, x_{12} \in K\},\\
&\vdots\\
&W_{1\alpha_1} = \{(x_{11}, x_{12}, ..., x_{1\alpha_1}, 0, ..., 0) \mid x_{11}, x_{12}, ..., X_{1\alpha_1} \in K\},\\
&W_{21} = \{(\underbrace{0, 0, ..., 0}_{\alpha_1 \text{ positions}}, x_{21}, 0, ..., 0) \mid x_{21} \in K\},\\
&W_{22} = \{(\underbrace{0, 0, ..., 0}_{\alpha_1 \text{ positions}}, x_{21}, x_{22}, 0, ..., 0) \mid x_{21}, x_{22} \in K\},\\
&\vdots\\
&W_{2\alpha_2} = \{(\underbrace{0, 0, ..., 0}_{\alpha_1 \text{ positions}}, x_{21}, x_{22}, ..., x_{2\alpha_2}, 0, ..., 0) \mid x_{21}, x_{22}, ..., x_{2\alpha_2} \in K\},\\
&\vdots\\
&W_{s1} = \{(\underbrace{0, 0, ..., 0}_{\alpha-\alpha_s \text{ positions}}, x_{s1}, 0, ..., 0) \mid x_{s1} \in K\},\\
&W_{s2} = \{(\underbrace{0, 0, ..., 0}_{\alpha-\alpha_s \text{ positions}}, x_{s1}, x_{s2}, 0, ..., 0) \mid x_{s1}, x_{s2} \in K\},\\
&\vdots\\
&W_{s\alpha_s} = \{(\underbrace{0, 0, ..., 0}_{\alpha-\alpha_s \text{ positions}}, x_{s1}, x_{s2}, ..., x_{s\alpha_s}) \mid x_{s1}, x_{s2}, ..., x_{s\alpha_s} \in K\},
\end{aligned}
$$

and let L' be the G-sublattice of $L_G^\rho(V)$ generated by them. Therefore, we have a subrepresentation $f' : G \to \mathrm{Aut}(L')$ of the latticeal representation f_ρ induced by ρ. Moreover, the map $\varphi : L \to L'$ defined by $\varphi(0) = 0$ and $\varphi(\ell_{ij}) = W_{ij}$, for any $i = \overline{1, s}$ and any $j = \overline{1, \alpha_i}$, is a G-lattice isomorphism, thus the latticeal representations f and f' are isomorphic.

Corollary. *If G is a finite cyclic group and K is a field, then the regular latticeal representation f_G of G is isomorphic with a subrepresentation of the latticeal representation f_ρ induced by a K-representation of finite degree ρ of G.*

Proof. Since the regular latticeal representation f_G of the finite cyclic group G is fully decomposable and the lattice $L(G)$ is distributive (see [2], Theorem 4, page 267), the statement results from Proposition 1.

In the following, let G be a finite cyclic group, K be a field and $f : G \to \mathrm{Aut}(L)$ be a fully decomposable latticeal representation of G with distributive finite representation lattice. Under these assumptions, we study the existence of a K-representation of finite degree ρ of G such that the latticeal representations f and f_ρ are isomorphic.

We shall solve this problem under the supplementary condition that $\mathrm{char}(K) \not| |G|$. The case when $\mathrm{char}(K)|\ |G|$ remains open.

Proposition 2. *Let G be a finite cyclic group and K be a field with the property* $\mathrm{char}(K) \not| |G|$. *Then, for a fully decomposable latticeal representation $f : G \to \mathrm{Aut}(L)$ of G with distributive finite representation lattice, the following conditions are equivalent:*

i) *There exists a K-representation of finite degree ρ of G such that $f \cong f_\rho$.*

ii) *f is a completely reducible latticeal representation.*

Proof. i) $\Longrightarrow$ ii) From the Maschke's theorem, the K-representation ρ is completely reducible. Then $L_G^\rho(V)$ (where V is the representation space of ρ) is a complemented G-lattice. Therefore the G-lattice L is complemented.

Let $(L_i)_{i\in I}$ be a family of fully ordered G-sublattices of L such that $L = \overset{\vee}{\bigoplus_{i\in I}} L_i \cong \bigtimes_{i\in I} L_i$. We obtain that the G-lattice L_i is complemented, for any $i \in I$. Since L_i is fully ordered, it results that L_i is irreducible, $i \in I$. Therefore L is completely reducible.

ii) $\Longrightarrow$ i) Let $(L_i)_{i\in I}$ be a family of irreducible G-sublattices of L such that $L = \overset{\vee}{\bigoplus_{i\in I}} L_i \cong \bigtimes_{i\in I} L_i$. If $\rho : G \to \mathrm{Aut}_K(V)$ is a K-representation of finite degree of G, then the G-lattice $L_G^\rho(V)$ is isomorphic with the G-lattice $L_{K[G]}(V)$ consisted of all submodules of the $K[G]$-module V. Let V_0 be a simple $K[G]$-module, $s = |I|$ and $V = V_0^s$. Since $K[G]$ is a semisimple ring, we obtain that V is semisimple $K[G]$-module. It results that $L_{K[G]}(V) \cong \bigtimes_{i\in I} L_i$. Thus the latticeal representations f and f_ρ are isomorphic.

Corollary. *Let G be a finite cyclic group and K be a field with the property* $\mathrm{char}(K) \not| |G|$. *Then the regular latticeal representation f_G of G is isomorphic with the latticeal representation induced by a K-representation of finite degree of G, if and only if $|G|$ is a square–free number.*

Proof. The statement results from Proposition 6 (2.1) and Proposition 2 (2.2).

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