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DIRECT AND INVERSE THEOREMS FOR GENERALIZED BASKAKOV OPERATORS IN POLYNOMIAL WEIGHT SPACES

BY

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Abstract. The object of this paper is to establish direct and inverse theorem in polynomial weight spaces for Linear Positive Operator $B_n^a(f;x)$ introduced by Mihesan and defined as:

$$B_n^a(f;x) = e^{\frac{-ax}{x+1}} \sum_{k=0}^{\infty} \frac{p_k(n,a)}{k!} \cdot \frac{x^k}{(1+x)^{n+k}} f(k/n),$$
$$a \geq 0, x \geq 0, k = 0,1,2,\ldots, n = 1,2,\ldots,$$

where,

$$p_k(n,a) = \sum_{i=0}^{k} {}^kC_i (n)_i a^{k-i},$$

with

$$(n)_0 = 1, (n)_i = n(n+1)\ldots(n+i-1), \text{ for } i \geq 1.$$

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1. Introduction. The inverse problems for Bernstein polynomials, Szász-Mirakjan operators and Baskakov operators have been intensively studied in papers [1], [3]. For Bernstein polynomials, direct and inverse theorem was first established by Berens and Lorentz [3] in 1972 and given by the following theorem:

Theorem 1.1. *For $f \in C[0,1), \alpha \in (0,2]$, the following statements are equivalent:*

$$|B_n(f;x) - f(x)| \leq M \left[\frac{x(1-x)}{n}\right]^{\alpha/2}, \quad (n \in N, x \in [0,1]) \tag{1.1}$$

(1.2)
$f \in Lip^*_\alpha := \{f \in C[0,1]; \sup |f(x+h) - 2f(x) + f(x-h)| = O(h^\alpha), h \to 0+\}$.

M. Becker [2] derived the similar results for Szász-Mirakjan and Baskakov operators in polynomial weight spaces. He proved the following theorem with weights $w_p(x), p \in W = N \cup \{0\}$ as

$$w_0(x) = 1, \quad w_p(x) = (1+x^p)^{-1}, \quad x \geq 0, p > 0, \tag{1.3}$$

spaces
(1.4)
$C_p = \{f \in C[0,\infty) : w_p f \text{ uniformly continuous and bounded on } [0,\infty)\}$

and

$$\|f\|_p = \sup_{x \geq 0} w_p(x)|f(x)| \Rightarrow w_p(x)|f(x)| \leq \|f\|_p. \tag{1.5}$$

Theorem 1.2. *Let $p \in W, f \in C_p, \alpha \in (0,2]$. Then for Szász-Mirakjan or Baskakov operators, the approximation rate*

$$w_p(x)|L_n(f;x) - f(x)| \leq M_p \left[\frac{\varphi(x)}{n}\right]^{\alpha/2}, \quad n \in N, x \geq 0 \tag{1.6}$$

is equivalent to $f \in \mathrm{Lip}^2_p\,\alpha$, where $\varphi(x) = x$ or $x(1+x)$.

The aim of this paper is to derive similar results for the following generalized Baskakov operators introduced by Mihesan [5] in 1998.

$$B^a_n(f;x) = \sum_{k=0}^{\infty} p_{n,k}(x,a) f(k/n), \tag{1.7}$$
$$a \geq 0, x \geq 0, \ k = 0,1,2,\ldots, \ n = 1,2,\ldots,$$

where,

(1.8) $$p_{n,k}(x,a) = e^{\frac{-ax}{x+1}} \cdot \frac{p_k(n,a)}{k!} \cdot \frac{x^k}{(1+x)^{n+k}}$$

and

(1.9) $$p_k(n,a) = \sum_{i=0}^{k} {}^kC_i(n)_i a^{k-i},$$
$$\text{with } (n)_0 = 1, (n)_i = n(n+1)\dots(n+i-1), \text{ for } i \geq 1$$

defined for $f \in C[0,\infty)$, the space of functions continuous on $[0,\infty)$.

Corresponding to the unbounded intervals the functions are indeed allowed to be unbounded, with some restriction over the growth of f at infinity. For functions of polynomial growth, we consider the space C_p with weights w_p as defined in (1.3) and (1.4). The corresponding Lipschitz classes are given for $0 < \alpha \leq 2$ by $(h > 0)$

$$\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x),$$

(1.10) $$\omega_p^2(f;\delta) = \sup_{o<h\leq\delta} \|\Delta_h^2 f\|_p,$$

(1.11) $$\text{Lip}_p^2\,\alpha = \{f \in C_p : \omega_p^2(f;\delta) = O(\delta^\alpha), \delta \to 0+\}.$$

2. Some properties of the operator. Some properties of the operator (1.7) are given below.

With $\varphi(x) = x(1+x)$, we have,

(2.1) $$\varphi(x)p'_{n,k}(x,a) = \left[(k-nx) - \frac{ax}{1+x}\right] p_{n,k}(x,a).$$

Introducing r-th moment $(r \in W)$ by

(2.2) $$T_{n,r}(x,a) = B_n^a((t-x)^r;x) = \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^r p_{n,k}(x,a).$$

one can easily derive the following recurrence formula $(T_{n,-1}(x,a) = 1)$

(2.3)
$$T_{n,r+1}(x,a) = \frac{\varphi(x)}{n}\left[T'_{n.r}(x,a) + rT_{n,r-1}(x,a) + \frac{a}{(1+x)^2}T_{n,r}(x,a)\right].$$

As a first consequence, we have,

$$T_{n,0}(x,a) = B_n^a(1;x) = 1; \;\; T_{n,1}(x,a) = B_n^a(t-x;x) = \frac{ax}{n(1+x)}, \tag{2.4}$$

$$T_{n,2}(x,a) = B_n^a((t-x)^2;x) = \frac{x(1+x)}{n} + \frac{1}{n^2} \cdot \frac{ax}{1+x} \cdot \frac{(a+1)x+1}{1+x}, \tag{2.5}$$

which have been proved by Mihesan [5].

Next, we characterize higher order moments and give a representation for the r-th moment.

Lemma 2.1. *Let $r \geq 2, \delta_{r,j} = 1$ if r is odd, $\delta_{r,j} = 0$ if r is even. Then r-th moment for operator (1.7) is given by*

$$T_{n,r}(x,a) = \sum_{j=1}^{[r/2]} \beta_{nrj} \left[\frac{x(1+x)}{n}\right]^j \left[\frac{1+2x}{n}\right]^{\delta_r} + P(x,n,a) \tag{2.6}$$

with positive coefficients β_{nrj} bounded with respect to n and $P(x,n,a)$ is a polynomial in x and a. In particular, first term on the right hand side is a polynomial of degree r without a constant term.

Proof. From (2.3) it follows that for $r = 2$

$$T_{n,3}(x,a) = \frac{x(1+x)}{n} \cdot \frac{1+2x}{n} + \frac{3ax^2}{n^2} + \frac{1}{n^3}\left[\frac{ax}{1+x} + \frac{3a^2x^2}{(1+x)^2} + \frac{a^3x^3}{(1+x)^3}\right].$$

This implies (2.6) is valid for $r = 2, 3$. Let (2.6) be valid for $r \leq 2k+1$. Using induction, (2.3) and $(1+2x)^2 = 1 + 4[x(1+x)]$, we obtain,

$$T_{n,2k+2}(x,a) =$$

$$= \frac{x(x+1)}{n}\left[T'_{n.,2k+1}(x,a) + (2k+1)T_{n,2k}(x,a) + \frac{a}{(1+x)^2}T_{n,2k+1}(x,a)\right] =$$

$$= \frac{x(1+x)}{n}\left[\sum_{j=1}^{k} \beta_{n,2k+1,j} \cdot \frac{j}{n^2}\left[\frac{x(1+x)}{n}\right]^{j-1} + \right.$$

$$+ \sum_{j=1}^{k} \beta_{n,2k+1,j} \cdot \frac{4j}{n}\left[\frac{x(1+x)}{n}\right]^{j} +$$

$$+\sum_{j=1}^{k}\beta_{n,2k+1,j}\cdot\frac{2}{n}\left[\frac{x(1+x)}{n}\right]^{j}+{P_1}'(x,n,a)+$$

$$+(2k+1)\left\{\sum_{j=1}^{k}\left[\frac{x(1+x)}{n}\right]^{j}\left[\frac{1+2x}{n}\right]+P_2(x,n,a)\right\}+$$

$$+\frac{a}{(1+x)^2}\left\{\sum_{j=1}^{k}\beta_{n,2k+1,j}\left[\frac{x(1+x)}{n}\right]^{j}\left[\frac{1+2x}{n}\right]+P_1(x,n,a)\right\}\Bigg],$$

where ${P_1}'(x,n,a)$ is first derivative of $P_1(x,n,a)$.

Taking

$$P(x,n,a)=\frac{x(1+x)}{n}\Bigg[{P_1}'(x,n,a)+(2k+1)P_2(x,n,a)+ \\ +\frac{a}{(1+x)^2}\left\{\sum_{j=1}^{k}\beta_{n,2k+1,j}\left[\frac{x(1+x)}{n}\right]^{j}\left[\frac{1+2x}{n}\right]+P_1(x,n,a)\right\}\Bigg]$$

we have

$$T_{n,2k+2}(x,a)=\sum_{j=1}^{k}\beta_{n,2k+1,j}\cdot\frac{j}{n^2}\left[\frac{x(1+x)}{n}\right]^{j}+$$

$$+\sum_{j=1}^{k}\left[\frac{4j+2}{n}\beta_{n,2k+1,j}+(2k+1)\beta_{n,2k,j}\right]\left[\frac{x(1+x)}{n}\right]^{j+1}+P(x,n,a)=$$

$$=\sum_{j=1}^{k}\beta_{n,2k+1,j}\cdot\frac{j}{n^2}\left[\frac{x(1+x)}{n}\right]^{j}+$$

$$+\sum_{j=2}^{k+1}\left[\frac{4(j-1)+2}{n}\beta_{n,2k+1,j-1}+(2k+1)\beta_{n,2k,j-1}\right]\left[\frac{x(1+x)}{n}\right]^{j}+P(x,n,a).$$

Thus (2.6) is valid for $r=2k+2$.

Further,

$$T_{n,2k+3}(x,a)=\frac{x(1+x)}{n}\left[T'_{n.,2k+2}(x,a)+(2k+2)T_{n,2k+1}(x,a)+\right.$$

$$+\frac{a}{(1+x)^2}T_{n,2k+2}(x,a)\Bigg] =$$

$$=\frac{x(1+x)}{n}\left[\sum_{j=1}^{k+1}\beta_{n,2k+2,j}\cdot j\left[\frac{x(1+x)}{n}\right]^{j-1}\left[\frac{1+2x}{n}\right]+{P_3}'(x,n,a)+\right.$$

$$+(2k+2)\left\{\sum_{j=1}^{k}\beta_{n,2k+1,j}\left[\frac{x(1+x)}{n}\right]^{j}\left[\frac{1+2x}{n}\right]+P_4(x,n,a)\right\}+$$

$$\left.+\frac{a}{(1+x)^2}\left\{\sum_{j=1}^{k}\beta_{n,2k+2,j}\left[\frac{x(1+x)}{n}\right]^{j}+P_3(x,n,a)\right\}\right] =$$

$$=\sum_{j=1}^{k+1}\beta_{n,2k+2,j}\cdot j\left[\frac{x(1+x)}{n}\right]^{j}\left[\frac{1+2x}{n}\right]+$$

$$+(2k+2)\sum_{j=1}^{k}\beta_{n,2k+1,j}\left[\frac{x(1+x)}{n}\right]^{j+1}\left[\frac{1+2x}{n}\right]+P_5(x,n,a).$$

This implies that (2.6) is valid for $r = 2k+3$ and thus for all $r \geq 2$. The boundedness of the coefficients β_{nrj} with respect to n follows from the above representation of $T_{n,2k+2}(x,a)$ and $T_{n,2k+3}(x,a)$.

Now with the above representation of Lemma 2.1, we can prove some of the fundamental properties of $B_n^a(f;x)$ necessary for characterizing its approximate properties.

Lemma 2.2. *For each $p \in W$ there is a constant $M_p(a)$ such that for $n \in N, x \geq 0$*

$$w_p(x)B_n^a\left(\frac{1}{w_p(t)};x\right) \leq M_p(a). \tag{2.7}$$

In particular, for any $f \in C_p$

$$\|B_n^a(f;x)\|_p \leq M_p(a)\|f\|_p. \tag{2.8}$$

Proof. Obviously, (2.7) is valid for $p = 0$.

For $p \geq 1$, we have,

$$
\begin{aligned}
w_p(x) B_n^a\left(\frac{1}{w_p(t)}; x\right) &= w_p(x) B_n^a(1+t^p; x) = \\
&= w_p(x)[B_n^a(1;x) + B_n^a((t-x+x)^p; x)] = \\
&= w_p(x)\left[1 + B_n^a\left(\sum_{j=0}^{p} {}^pC_j (t-x)^j x^{p-j}; x\right)\right] = \\
&= 1 + w_p(x) \sum_{j=0}^{p} {}^pC_j T_{n,j}(x,a) x^{p-j} \leq M_p(a).
\end{aligned}
$$

Since $T_{n,j}(x,a)$ is a polynomial of degree j, the sum is bounded in x. The boundedness with respect to n follows from the bounded coefficients of $T_{n,j}(x,a)$ in n.

Since, for any $f \in C_p$, we get

$$
\begin{aligned}
w_p(x)|B_n^a(f;x)| &\leq w_p(x) \sum_{k=0}^{\infty} w_p\left(\frac{k}{n}\right) \left|f\left(\frac{k}{n}\right)\right| \left(w_p\left(\frac{k}{n}\right)\right)^{-1} p_{n,k}(x,a) \leq \\
&\leq \|f\|_p w_p(x) B_n^a\left(\frac{1}{w_p(t)}; x\right) \leq M_p(a)\|f\|_p.
\end{aligned}
$$

Lemma 2.3. *For each $p \in W$ there is a constant $M_p(a)$ s.t. for all $n \in N$*

$$
w_p(x) B_n^a\left(\frac{(t-x)^2}{w_p(t)}; x\right) \leq M_p(a) \frac{\varphi(x)}{n}. \tag{2.9}
$$

Furthermore,

$$
(1+2x) w_p(x) B_n^a\left(\frac{(t-x)}{w_p(t)}; x\right) \leq M_p(a) \frac{\varphi(x)}{n}. \tag{2.10}
$$

Proof. For $p = 0$, (2.9) reduces to (2.5). For $Pp \geq 1$, we have,

$$
\begin{aligned}
w_p(x) B_n^a\left(\frac{(t-x)^2}{w_p(t)}; x\right) &= w_p(x) B_n^a((t-x)^2(1+t^p); x) = \\
= w_p(x)&\left[T_{n,2}(x,a) + \sum_{r=0}^{p} {}^pC_r B_n^a((t-x)^{r+2}; x) x^{p-r}\right] =
\end{aligned}
$$

$$= w_p(x)\left[T_{n,2}(x,a)+\sum_{r=0}^{p}{}^{p}C_r T_{n,r+2}(x,a)x^{p-r}\right]=$$

$$= w_p(x)\left[\frac{x(1+x)}{n}+\frac{1}{n^2}\cdot\frac{ax}{1+x}\cdot\frac{(a+1)x+1}{1+x}+\right.$$

$$\left.+\sum_{r=0}^{p}{}^{p}C_r\left\{\sum_{j=1}^{[r/2]+1}\beta_{n,r+2,j}\left[\frac{x(1+x)}{n}\right]^j\left[\frac{1+2x}{n}\right]^{\delta_{r+2}}+P_1(x,n,a)\right\}x^{p-r}\right]=$$

$$=\frac{x(1+x)}{n}w_p(x)\left[1+\sum_{r=0}^{p}{}^{p}C_r T_{n,r+2}(x,a)x^{p-r}\right]=$$

$$=\frac{\varphi(x)}{n}\frac{\Pi_{p,n}(x,a)}{1+x^p}\le M_p(a)\frac{\varphi(x)}{n},$$

as $\Pi_{p,n}(x,a)$ is a polynomial of degree p and coefficients are bounded in n.

Now,

$$w_p(x)B_n^a\left(\frac{(t-x)}{w_p(t)};x\right)=w_p(x)B_n^a((t-x)(1+t^p);x)=$$

$$=w_p(x)\left[T_{n,1}(x,a)+\sum_{r=0}^{p}{}^{p}C_r B_n^a((t-x)^{r+1};x)x^{p-r}\right]=$$

$$=w_p(x)\left[T_{n,1}(x,a)+\sum_{r=0}^{p}{}^{p}C_r T_{n,r+1}(x,a)x^{p-r}\right]=$$

$$=w_p(x)\left[T_{n,1}(x,a)+T_{n,1}(x,a)x^p+\sum_{r=1}^{p}{}^{p}C_r T_{n,r+1}(x,a)x^{p-r}\right]=$$

$$=T_{n,1}(x,a)+w_p(x)\sum_{r=0}^{p-1}{}^{p}C_{r+1}T_{n,r+2}(x,a)x^{p-r}=$$

$$=\frac{x(1+x)}{n}\left[w_p(x)\sum_{r=0}^{p-1}{}^{p}C_{r+1}T_{n,r+2}(x,a)x^{p-r-1}\right]=$$

$$=\frac{\varphi(x)}{n}\frac{\Pi_{p,n}(x,a)}{1+x^p}.$$

This implies,

$$(1+2x)w_p(x)B_n^a\left(\frac{(t-x)}{w_p(t)};x\right)=\frac{\varphi(x)}{n}\frac{(1+2x)\Pi_{p,n}(x,a)}{1+x^p}\le M_p(a)\frac{\varphi(x)}{n}.$$

3. Direct theorem. We will prove the direct theorem using a Jackson-type inequality, the Steklov means and appropriate results for the moments of the operator. Jackson-type inequality is proved in Lemma (3.1) below.

Lemma 3.1. *Let $p \in W$, $g \in C_p^2 = \{f \in C_p; f'' \in C_p\}$. Then there exist a constant $M_p(a)$ s.t. for all $n \in N, x \geq 0$, we obtain,*

(3.1)
$$w_p(x)|B_n^a(g(t);x) - g(x)| \leq \|g''\|_p \frac{x(1+x)}{n} M_p(a) + w_p(x)g'(x)\frac{ax}{n(1+x)}.$$

Proof. From [2], we have,

$$\begin{aligned}|g(t) - g(x)| &= |(t-x)g'(x)| + \int_x^t \int_x^s \left|g''(u)\right| duds \leq \\ &\leq |(t-x)|g'(x) + \frac{1}{2}\|g''\|_p (t-x)^2 \left[\frac{1}{w_p(x)} + \frac{1}{w_p(t)}\right].\end{aligned}$$

Using (2.4), (2.5), (2.9) and (3.2), we get

$$\begin{aligned}w_p(x)\,|B_n^a(g(t);x) - g(x)| &\leq w_p(x)\left|\sum_{k=0}^{\infty} p_{n,k}(x,a)[g(t)-g(x)]\right| \leq \\ &\leq w_p(x)\sum_{k=0}^{\infty} p_{n,k}(x,a) \times \\ &\times \left[|(t-x)|g'(x) + \frac{1}{2}\|g''\|_p (t-x)^2 \left[\frac{1}{w_p(x)} + \frac{1}{w_p(t)}\right]\right] \leq \\ &\leq w_p(x)g'(x)|B_n^a(t-x;x) + \\ &+ \frac{1}{2}\|g''\|_p \left[B_n^a((t-x)^2;x) + w_p(x)B_n^a\left(\frac{(t-x)^2}{w_p(t)};x\right)\right] \leq \\ &\leq w_p(x)g'(x)\frac{ax}{n(1+x)} + \\ &+ \frac{1}{2}\|g''\|_p \left[\frac{x(1+x)}{n} + \frac{1}{n^2}\cdot\frac{ax}{1+x}\cdot\frac{(a+1)x+1}{1+x} + M_p(a)\frac{\varphi(x)}{n}\right].\end{aligned}$$

This implies,

$$w_p(x)|B_n^a(g(t);x) - g(x)| \leq \|g''\|_p \frac{x(1+x)}{n} M_p(a) + w_p(x)g'(x)\frac{ax}{n(1+x)},$$

for large n.
Modified Steklov's mean [4] for $h > 0$ is given by

$$f_h(x) = \left(\frac{2}{h}\right)^2 \int_0^{h/2} \int_0^{h/2} [2f(x+s+t) - f(x+2(s+t)]dsdt,$$

this implies

$$f(x) - f_h(x) = \left(\frac{2}{h}\right)^2 \int_0^{h/2} \int_0^{h/2} \Delta^2_{s+t} f(x)dsdt$$

and

$$f_h{}''(x) = h^{-2}[8\Delta^2_{h/2} f(x) - \Delta^2_h f(x)], \tag{3.3.}$$

hence,

$$\|f - f_h\| \le \omega_p^2(f;h), \quad \|f_h{}''\| \le \frac{9}{h^2} w_p^2(f;h). \tag{3.4}$$

Now we will prove the direct theorem.

Theorem 3.1. *Let $B_n^a(f;x)$ be the operator defined by (1.7) and $\phi(x) = x(1+x)$. For any $p \in W, f \in C_p$, there holds for all $n \in N, x \ge 0$*

$$w_p(x)|B_n^a(f;x) - f(x)| \le M_p(a)\omega_p^2\left(f; \sqrt{\frac{\varphi(x)}{n}}\right) + w_p(x) f_h'(x) \frac{ax}{n(1+x)}.$$

In particular, if $f \in \mathrm{Lip}_p^2\,\alpha$ for some $\alpha \in (0,2]$, then

$$w_p(x)|B_n^a(f;x) - f(x)| \le M_p(a) \left[\frac{\varphi(x)}{n}\right]^{\alpha/2} + w_p(x) f_h'(x) \frac{ax}{n(1+x)}.$$

Proof. For $x = 0$, the assertion is trivial. For $f \in C_p, h > 0$, one has by Lemmas (2.2), (3.1) and by (3.4) that

$$\begin{aligned} w_p(x)\ &|B_n^a(f;x) - f(x)| \le \\ &\le w_p(x)|B_n^a(f - f_h;x)| + w_p(x)|B_n^a(f_h;x) - f_h(x)| + \\ &+ w_p(x)|f_h(x) - f(x)| \le \|f - f_h\| w_p(x) B_n^a\left(\frac{1}{w_p(t)};x\right) + \\ &+ \|f_h{}''\|_p \frac{x(1+x)}{n} M_p(a) + w_p(x) f_h{}'(x) \frac{ax}{n(1+x)} + \|f - f_h\| \end{aligned}$$

and using (2.7) and (3.4), we have,

$$\leq M_p(a)\omega_p^2(f;h) + M_p(a)\frac{x(1+x)}{n}\frac{9}{h^2}\omega_p^2(f;h) +$$
$$+ w_p(x){f_h}'(x)\frac{ax}{n(1+x)} + \omega_p^2(f;h) \leq$$
$$\leq M_p(a)\omega_p^2(f;h)\left[1 + \frac{\varphi(x)}{nh^2}\right] + w_p(x){f_h}'(x)\frac{ax}{n(1+x)},$$

where $\varphi(x) = x(1+x)$.

Taking $h = \sqrt{\frac{\varphi(x)}{n}}$, we get,

$$w_p(x)|B_n^a(f;x) - f(x)| \leq M_p(a)\omega_p^2\left(f;\sqrt{\frac{\varphi(x)}{n}}\right) + w_p(x){f_h}'(x)\frac{ax}{n(1+x)}.$$

4. Inverse theorem. In order to prove the inverse theorem in the non-optimal case $0 < \alpha < 2$, we will make use of an appropriate type inequality as the main tool.

Lemma 4.1. *For $f \in C_p, x, \delta > 0$ there holds*

$$\begin{aligned} &w_p(x)|{B_n^a}''(f;x)| \leq \\ &\leq M_p(a)\omega_p^2(f;\delta)\left[\frac{n}{\varphi(x)} + \frac{1}{\delta^2} + \frac{n}{\varphi(x)}\left\{\frac{2ax}{(1+x)^2} + \frac{a^2x}{n(1+x)^3}\right\}\right]. \end{aligned} \tag{4.1}$$

Proof. We consider the two representations of ${B_n^a}''(f;x)$.

First one is

$${B_n^a}''(f;x) = \left(\frac{n}{\varphi(x)}\right)^2 \sum_{k=0}^{\infty} r_{k,n}(x,a) f\left(\frac{k}{n}\right) p_{n,k}(x,a), \tag{4.2}$$

where

$$\begin{aligned} r_{k,n}(x,a) = &\left(\frac{k}{n} - x\right)^2 - \left\{\frac{1+2x}{n} + \frac{2ax}{n(1+x)}\right\}\left(\frac{k}{n} - x\right) - \\ &- \left\{\frac{x(1+x)}{n} + \frac{a^2x^2}{n^2(1+x)^2}\right\} \end{aligned} \tag{4.3}$$

Second representation is:

$$B_n^{a''}(f;x) = n(n+1)\sum_{k=0}^{\infty}\Delta_{1/n}^2 f\left(\frac{k}{n}\right)p_{n+2,k}(x,a) \tag{4.4}$$

where

$$\begin{aligned} p_{n+2,k}(x,a) = \frac{p_k(n,a)}{n(+1)k!}&\left[\frac{a^2}{(1+x)^2}+\frac{2a(n+k+1)}{(1+x)}+\right.\\ &\left.+(n+k)(n+k+1)\right]\frac{x^k}{(1+x)^{n+k+2}}. \end{aligned} \tag{4.5}$$

From (3.4), we obtain (Ref. [2])

$$|\Delta_{1/n}^2 f_\delta(t)| \le \frac{9(n\delta)^{-2}\omega_p^2(f;\delta)}{w_p\left(\frac{t+2}{n}\right)}. \tag{4.6}$$

Therefore, by (4.2), (4.4) and (4.6), we get for $x, h > 0$

$$\begin{aligned} &w_p(x)|B_n^{a''}(f;x)| \le w_p(x)|B_n^{a''}(f-f_\delta;x)| + w_p(x)|B_n^{a''}(f_\delta;x)| \le \\ &\le w_p(x)\left(\frac{n}{\varphi(x)}\right)^2\sum_{k=0}^{\infty}|r_{k,n}(x,a)|\left|f\left(\frac{k}{n}\right)-f_\delta\left(\frac{k}{n}\right)\right|p_{n,k}(x,a)+ \\ &+w_p(x)n(n+1)\sum_{k=0}^{\infty}\Delta_{1/n}^2 f_\delta\left(\frac{k}{n}\right)p_{n+2,k}(x,a), \end{aligned}$$

with $n(n+1) \le 2n^2$ and (4.6), we have,

$$\begin{aligned} &w_p(x)|B_n^{a''}(f;x)| \le \\ &\le \|f-f_\delta\|w_p(x)\left(\frac{n}{\varphi(x)}\right)\sum_{k=0}^{\infty}|r_{k,n}(x,a)|\left(w_p\left(\frac{k}{n}\right)\right)^{-1}p_{n,k}(x,a)+ \\ &+w_p(x)2n^2\frac{9}{(n\delta)^2}\omega_p^2(f;\delta)\sum_{k=0}^{\infty}\left(w_p\left(\frac{k+2}{n}\right)\right)^{-1}p_{n+2,k}(x,a), \end{aligned}$$

making use of (3.4), we have,

$$w_p(x)|B_n^{a''}(f;x)| \le \omega_p^2(f;\delta)\left[U\left(\frac{n}{\varphi(x)}\right)^2 + V\frac{18}{\delta^2}\right]. \tag{4.7}$$

Now,

$$U = w_p(x)\left(\frac{n}{\varphi(x)}\right)^2 \sum_{k=0}^{\infty} |r_{k,n}(x,a)| \left(w_p\left(\frac{k}{n}\right)\right)^{-1} p_{n,k}(x,a) \le$$

$$\le w_p(x) B_n^a\left(\frac{(t-x)^2}{w_p(t)}; x\right) + \left|\left\{\frac{1+2x}{n} + \frac{2ax}{n(1+x)}\right\} w_p(x) B_n^a\left(\frac{t-x}{w_p(t)}; x\right)\right| +$$

$$+ \left\{\frac{x(1+x)}{n} + \frac{a^2x^2}{n^2(1+x)^2}\right\} w_p(x) B_n^a\left(\frac{1}{w_p(t)}; x\right),$$

using (2.7), (2.9) and (2.10), we get

$$U \le \frac{\varphi(x)}{n} M_p(a)\left\{1 + \frac{2ax}{n(1+x)(1+2x)} + \frac{a^2x}{n(1+x)^3}\right\}. \tag{4.8}$$

And using (2.6), we obtain,

$$V = w_p(x) \sum_{k=0}^{\infty} \left(w_p\left(\frac{k+2}{n}\right)\right)^{-1} p_{n+2,k}(x,a) \le M_p(a). \tag{4.9}$$

Substituting (4.8) and (4.9) in (4.7), we get (4.1).

To prove the inverse theorem we also make use of the following Lemma proved by Becker [2].

Lemma 4.2. *Let $\varphi(x) = x(1+x)$. Then one has for $x \ge 0, 0 < h \le 1$.*

$$\int^h \int_0 \frac{ds dt}{\varphi(x+s+t)} \le \frac{Mh^2}{\varphi(x+2h)}.$$

Theorem 4.1. *Let $p \in W$. If $f \in C_p$ satisfies for some $\alpha \in (0,2)$, there holds for all $n \in N, x \ge 0$*

$$w_p(x)|B_n^a(f;x) - f(x)| \le M_p(a)\left(\frac{\varphi(x)}{n}\right)^{\alpha/2} + w_p(x) f'(x) \frac{ax}{n(1+x)} \tag{4.10}$$

then $f \in \mathrm{Lip}_p^2\, \alpha$.

Proof. To prove inverse part it is sufficient to prove that

$$\omega_p^2(f;h) \le M_p(a)\left[\delta^\alpha\left\{1 + 3w_p(x)f'(x)\frac{ax}{n(1+x)}\right\} + \left(\frac{h}{\delta}\right)^2 \omega_p^2(f;\delta)\right].$$

Fix $0 < h, \delta \leq 1, \delta < \sqrt{h}, x \geq 0$.

Also form [2] we have,

$$\frac{w_p(x)}{w_p(x+2h)} \leq 3^p \text{ as } h \leq 1. \tag{4.11}$$

Using Lemma (4.1) and (4.2), it follows from (4.10) and (4.11) for all $n \in N$

$$w_p(x)|\Delta_h^2 f(x)| \leq w_p(x)|f(x+2h) - B_n^a(f; x+2h)| + 2w_p(x)|f(x+h)-$$
$$-B_n^a(f; x+h)| + w_p(x)|f(x) - B_n^a(f; x)| + w_p(x)|\Delta_h^2 B_n^a(f; x)| \leq$$
$$\leq \frac{w_p(x)}{w_p(x+2h)}\left[M_p(a)\left(\frac{\varphi(x+2h)}{n}\right)^{\alpha/2} + w_p(x+2h)f_h'(x)\frac{ax}{n(1+x)}\right]+$$
$$+\frac{w_p(x)}{w_p(x+h)}\left[M_p(a)\left(\frac{\varphi(x+h)}{n}\right)^{\alpha/2} + w_p(x+h)f_h'(x)\frac{ax}{n(1+x)}\right]+$$
$$+M_p(a)\left(\frac{\varphi(x)}{n}\right)^{\alpha/2} + w_p(x)f_h'(x)\frac{ax}{n(1+x)}+$$
$$+w_p(x)\int^h \int_0 \left|B_n^{a''}(f; x+s+t)\right| dsdt \leq$$
$$\leq M_p(a)\left(\frac{\varphi(x+2h)}{n}\right)^{\alpha/2}\left[\frac{w_p(x)}{w_p(x+2h)} + \frac{w_p(x)}{w_p(x+h)} + 1\right.$$
$$\left.+3w_p(x)f'(x)\frac{ax}{n(1+x)}\right] + w_p(x)\int^h \int_0 \left|B_n^{a''}(f; x+s+t)\right| dsdt \leq$$
$$\leq M_p(a)\left(\frac{\varphi(x+2h)}{n}\right)^{\alpha/2}\left[1 + 3w_p(x)f'(x)\frac{ax}{n(1+x)}\right]+$$
$$+3^p\omega_p^2(f;\delta)\left[\frac{M_p(a)Mnh^2}{\varphi(x+2h)} + 18M_p(a)\frac{h^2}{\delta^2}\right].$$

Now choose n such that

$$\sqrt{\varphi(x+2h)/n} \leq \delta < \sqrt{2}\sqrt{\varphi(x+2h)/n}$$

implies $\dfrac{n}{\varphi(x+2h)} < \dfrac{2}{\delta^2}$ and $\left(\dfrac{\varphi(x+2h)}{n}\right)^{\alpha/2} \leq \delta^\alpha$ and hence,

$$w_p(x)|\Delta_h^2 f(x)| \leq$$
$$\leq M_p(a)\left[\delta^\alpha\left\{1 + 3w_p(x)f'(x)\frac{ax}{n(1+x)}\right\} + \left(\frac{h}{\delta}\right)^2 \omega_p^2(f;\delta)\right].$$

This completes the proof.

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