ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul L, s.I a, Matematică, 2004, f.1.

TWISTED ENDOMORPHISMS OF UNSTABLE VECTOR BUNDLES ON CURVES*

BY

E. BALLICO

Abstract. Let X be a smooth projective curve, E a vector bundle on X and $M \in \mathrm{Pic}(X)$ such that $\deg(M) < 0$. Take a general $f \in H^0(X, Hom(E, E))$. Here we study the " iterates " of f when E is either a direct sum of stable bundles or an extension of a stable bundle by a stable bundle with higher slope

1991 Mathematics Subject Classification : 14H60.

Key words and phrases: stable vector bundle, line bundle, vector bundles on curves, stable vector bundles on curves,endomorphisms,slope, unstable vector bundle, semistable vector bundle.

1. Introduction. Let X be a smooth and connected projective curve of genus $g \geq 2$, $M \in \mathrm{Pic}(X)$ and E a rank r vector bundle on X such that there is $f : E \to E \otimes M$, $f \neq 0$. In [1] we defined the twisted endomorphism algebra $A(E, M) := \oplus_{u,t \in \mathbb{Z}, t \geq 0} H^0(X, Hom(E \otimes M^{\otimes u}, M^{\otimes(u+t)}))$ of the pair (E, M). In [1] we studied $A(E, M)$ for several vector bundles E when $\deg(M) > 0$. Here we will study the case $\deg(M) \leq 0$, $M \neq \mathcal{O}_X$. If $t := \deg(M) < 0$, then E cannot be semistable. E has the best possible stability properties compatible with the existence of $f \neq 0$ if $\mathrm{Ker}(f)$ is stable and $\mathrm{Im}(f)$ is saturated in $E \otimes M$ (i.e. $\mathrm{Coker}(f)$ has no torsion and hence it is locally free) and $\mathrm{Im}(f)$ is stable. However, if $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are semistable and $\deg(M) < 2 - 2g$, then the exact sequence

$$(1) \qquad 0 \to \mathrm{Ker}(f) \to E \to \mathrm{Coker}(f) \to 0$$

*The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

splits (see Remark 1). Even in the range $2-2g \le \deg(M) < 0$ the existence of $f \neq 0$ gives very strong restrictions on the twisted endomorphism algebra $A(E,L)$. However, we are able to study it only for very special unstable bundles. In section 3 we will consider the case $E \cong E_1 \oplus \cdots \oplus E_s$ with E_i stable and " general " for every i. In section 4 we will consider the case in which E is an extension of a stable bundle by a stable bundle.

To apply [3] we work over an algebraically closed field K with $\mathrm{char}(K) = 0$.

2. Preliminary results. For any vector bundle E on X let $\mu(E) := \deg(E)/\mathrm{rank}(E)$ denotes its slope. For all integers r, d such that $r > 0$ let $M(X;r,d)$ be the moduli scheme of all stable vector bundles on X with rank r and degree d. $M(X;r,d)$ is an irreducible variety of dimension $r^2(g-1)+1$. For any $R, M \in \mathrm{Pic}(X)$ and any $h : E \to E \otimes M$, let $h_R : E \otimes R \to E \otimes M \otimes R$ be the induced homomorphism.

Remark 1. Take semistable vector bundles A, B on X such that $\mu(B) < \mu(A) + 2 - 2g$. Hence $\mu(A^*) > \mu(B^* \otimes \omega_X)$. Since A^* and $B^* \otimes \omega_X$ are semistable, we have $h^0(X, \mathit{Hom}(A,B) \otimes \omega_X) = 0$. Hence by Serre duality every extension of B by A splits.

Remark 2. Let E be a rank r vector bundle on X and $M \in \mathrm{Pic}(X)$ such that $E \cong E \otimes M$. Hence $\det(E) \cong \det(E \otimes M)$. Since $\det(E \otimes M) \cong \det(E) \otimes M^{\otimes r}$, we obtain $M^{\otimes r} \cong \mathcal{O}_X$.

The proof of the next lemma uses in an essential way the assumption $g \ge 2$.

Lemma 1. *Fix integers r and d such that $r > 0$ and take a general $E \in M(X;r,d)$. Then $E \otimes M \ncong E$ for all $M \in Pic(X)$ such that $M \ncong \mathcal{O}_X$.*

Proof. By Remark 2 it is sufficient to show that $E \otimes M \ncong E$ for all $M \in \mathrm{Pic}(X)$ such that $M^{\otimes r} \cong \mathcal{O}_X$. Since the set of all such line bundles is finite, while $M(X;r,d)$ is irreducible, it is sufficient to show that for every degree zero $M \in \mathrm{Pic}(X)$ such that $M \ncong \mathcal{O}_X$ there is $F \in M(X;r,d)$ such that $F \otimes M \ncong F$. Since the case $r = 1$ is trivial, we may assume $r \ge 2$ and that the result is true for all ranks at most $r-1$. Twisting by a line bundle we may reduce to the case $0 \le d < r$. Fix $P \in X$. By the inductive assumption there is $G \in M(X;r-1,d-1)$ such that $G \otimes M \ncong G$. Let F be a general extension of $\mathcal{O}_X(P)$ by G. Since $g \ge 2$ it is easy to check that

G is the unique stable subbundle of F with maximal slope and hence that $G \otimes M$ is the unique stable subbundle of $F \otimes M$ with maximal slope. Hence the assumption $G \otimes M \ncong G$ implies $F \otimes M \ncong F$.

Lemma 2. *Fix integers a, b, $r > 0$ and $s > 0$. Fix a general $(A, B) \in M(X; r, a) \times M(X; s, b)$. Then*

(i) We have $h^0(X, Hom(A, B)) = \max\{0, (rb - sa + rs)(1 - g)\}$. In particular we have $h^0(X, Hom(A, B)) \neq 0$ if and only if $b/s - a/r > g - 1$.

(i) We have $h^1(X, Hom(A, B)) = \max\{0, (sa - rb + rs)(g - 1)\}$. In particular we have $h^1(X, Hom(A, B)) = 0$ if and only if $b/s - a/r \geq g - 1$.

Proof. Part (i) is [2], §4 (see [3], Th. 1.2, for a published proof). Part (ii) is equivalent to part (i) by Serre duality.

3. Direct sums of stable bundles. We fix integers $t \leq 0$, $s \geq 0$, $r_i > 0$, $1 \leq i \leq s$, and d_i, $1 \leq i \leq s$ and $M \in \mathrm{Pic}^t(X)$. We assume $d_1/r_1 > \cdots > d_s/r_s$. After fixing M we fix a general $(E_1, \ldots, E_s) \in M(X; r_1, d_1) \times \cdots \times M(X; r_s, d_s)$ and set $E := E_1 \oplus \cdots \oplus E_s$. By the generality of each pair $(E_i, E_j \otimes M)$ and Lemma 2 the values $h^0(X, Hom(E, E \otimes M))$ are known and depend only on the integers t, r_i and d_i, $1 \leq i \leq s$. In particular we have the following observation.

Remark 3. $h^0(X, Hom(E, E \otimes M)) \neq 0$ if and only if $d_s/r_s < d_1/r_1 + 1 - g + t$.

For each integer i with $1 \leq i \leq s$ set $j(i) = 0$ if $d_i/r_i \geq d_1/r_1 + 1 - g + t$. Hence $j(1) = 0$. If $d_i/r_i < d_1/r_1 + 1 - g + t$ let $j(i)$ be the maximal integer $j \leq s$ such that $d_i/r_i < d_j/r_j + 1 - g + t$. Hence $j(i) < i$. Set $j(0) := 0$. For any integer $k \geq 2$ and any integer $i \in \{0, \ldots, s\}$ define inductively the integer $j^k(i)$ by the formula $j^k(i) := j(j^{k-1}(i))$. Hence $j^k(i) = 0$ if $i \leq k$.

Theorem 1. *Fix an integer $t \leq 0$, $M \in Pic^t(X)$ and integers $s \geq 2$, r_i and d_i, $1 \leq i \leq s$, such that $M \ncong \mathcal{O}_X$, $0 < r_1 \leq \cdots \leq r_s$ and $d_1/r_1 \geq \cdots \geq d_s/r_s$. Fix a general $(E_1, \ldots, E_s) \in M(X; r_1, d_1) \times \cdots \times M(X; r_s, d_s)$ and set $E := E_1 \oplus \cdots \oplus E_s$.*

(a) We have $h^0(X, Hom(E, E \otimes M)) \neq 0$ if and only if $d_s/r_s < d_1/r_1 + 1 - g + t$.

(b) *Assume* $d_s/r_s < d_1/r_1+1-g+t$ *and take a general* $f \in H^0(X, Hom(E, E \otimes M))$. *We have* $f^k = 0$ *and* $f^{k-1} \neq 0$, $k \geq 2$, *if and only if* k *is the maximal integer such that there are* k *integers* j_i, $1 \leq i \leq s$, *such that* $1 \leq j_1 < \cdots < j_k \leq s$ *and* $d_{j_{i+1}}/r_{j_{i+1}} < d_{j_i}/r_{j_i} + 1 - g + t$ *for all* $1 \leq i < k$.

Proof. Part (a) is just Remark 3. Assume $d_s/r_s < d_1/r_1+1-g+t$ and take a general $f \in H^0(X, Hom(E, E\otimes M))$. Call $f_{ij} : E_i \to E_j \otimes M$ the maps induced by f. Similarly, for all integers $m > 0$ call $(f^m)_{ij} : E_i \to E_j \otimes M^{\otimes m}$ the components of the iterated maps. By Remark 3 or Lemma 2 and the generality of all pairs $(E_i, E_j \otimes M^{\otimes m})$, $1 \leq m \leq k$, it is clear that all the components $(f^k)_{ij}$ vanishes and hence $f^k = 0$. Notice that f is general if and only if all its component f_{ij} are general in $H^0(X, Hom(E_i, E_j \otimes M))$. Since $r_{j_{1+1}} \geq r_{j_i}$ and $d_{j_{i+1}}/r_{j_{i+1}} < d_{j_i}/r_{j_i} + 1 - g + t$, $1 \leq i < k$ and $f_{j_{i+1}j_i}$ is general, $f_{j_{i+1}j_i}$ has rank r_{j_i} ([3], maximal rank assertion of Th. 0.3) and hence all iterates of f up to f^{k-1} are non-zero.

The easiest part of the proof of part (b) of Theorem 1 gives without any modification the following result.

Theorem 2. *Fix an integer* $t \leq 0$, $M \in Pic^t(X)$ *and integers* $s \geq 2$, r_i *and* d_i, $1 \leq i \leq s$, *such that* $M \ncong \mathcal{O}_X$, $0 < r_1 \leq \cdots \leq r_s$ *and* $d_1/r_1 \geq \cdots \geq d_s/r_s$. *Fix a general* $(E_1, \ldots, E_s) \in M(X; r_1, d_1) \times \cdots \times M(X; r_s, d_s)$. *Set* $E := E_1 \oplus \cdots \oplus E_s$ *and take a general* $f \in H^0(X, Hom(E, E \otimes M))$. *Let* k *be the maximal integer such that there are* k *integers* j_i, $1 \leq i \leq k$, *such that* $1 \leq j_1 < \cdots < j_k \leq s$ *and* $d_{j_{i+1}}/r_{j_{i+1}} < d_{j_i}/r_{j_i} + 1 - g + t$ *for all* $1 \leq i < k$. *Then for all* $f_i \in H^0(X, Hom(E, E \otimes M))$ *we have* $(f_1)_{M^{\otimes(k-1)}} \circ \cdots \circ f_k = 0$.

If we drop the assumption $0 < r_1 \leq \cdots \leq r_s$ in the statement of Theorem 2 we still know the exact value of rank(f^x), $x \geq 1$, and in particular the minimal integer k such that $f^k = 0$ for a general $f \in H^0(X, Hom(E, E \otimes M))$, but the statements are much more cumbersone. For simplicity we will state only the last property (see Theorem 3). the first property can be done in the same way using the following result.

Proposition 1. *Fix integers* $t \leq 0$, $m \geq 2$, $a_j > 0$, $\rho_j > 0$, c_j, $1 \leq j \leq m$. *Assume* $c_{j+1}/\rho_{j+1} < g - 1 + ta_j + c_j\rho_j$ *for* $1 \leq j \leq m - 1$. *Fix* $M \in Pic^t(X)$ *such that* $M^{\otimes x} \ncong \mathcal{O}_X$ *for all integers* x *such that* $1 \leq x \leq \sum_{j=1}^m a_j$. *Take a general* $(F_1, \ldots, F_m) \in M(X; \rho_1, c_1) \times \cdots \times M(X; \rho_m, c_m)$ *and general* $h_j \in H^0(X, Hom(F_{j+1}, F_j \otimes M^{\otimes a_j}))$, $1 \leq j \leq m - 1$. *Define* *inductively the morphisms $\phi_j : E_{j+1} \to E_1 \otimes M^{\otimes \beta_j}$, $\beta_j := \sum_{i=1}^{j} a_i$ by the formulas $\phi_1 := f_1$, $\phi_j := \phi_{j-1} \circ h_{j,M^{\otimes c}}$, $c := \sum_{i=1}^{j-1} a_i$, for $j \geq 2$. We have rank$(\phi_1) = \min\{\rho_1, \rho_2\}$ and rank$(\phi_j) = \min\{rank(\phi_{j-1}, \rho_{j-1}\}$ for $2 \leq j \leq m-1$.*

Proof. By [3], Th. 0.3, we have $\text{rank}(h_j) = \min\{\rho_{j+1}, \rho_j\}$, i.e. the rank of h_j is maximal. Hence the result is true if $j = 1$ and it is true by induction if $\rho_j \leq \rho_{j+1}$ because in this case h_j is generically surjective. If $\rho_j > \rho_{j+1}$, then the map h_j is injective and the result is equivalent to the assertion that the subsheaves $\text{Ker}(h_{j,M^{\otimes c}})$, $c := \sum_{i=1}^{j-1} a_i$, and $\text{Im}(\phi_{j-1})$ are as skew as possible inside E_{j-1}. This follows inductively by the generality of all maps h_i and the proof of [3], Th. 0.3.

The statement of Proposition 1 means that for general h_i, $1 \leq i \leq j$, the integer $\text{rank}(\phi_j)$ is as large as possible with the constraints given by the ranks of the vector bundles $E_1, \dots, E_{j+1}$.

Theorem 3. *Fix an integer $t \leq 0$, $M \in Pic^t(X)$ and integers $s \geq 2$, $r_i > 0$ and d_i, $1 \leq i \leq s$, such that $M^{\otimes x} \not\cong \mathcal{O}_X$, for all x with $1 \leq x \leq s$, and $d_1/r_1 \geq \cdots \geq d_s/r_s$. Fix a general $(E_1, \dots, E_s) \in M(X; r_1, d_1) \times \cdots \times M(X; r_s, d_s)$. Set $E := E_1 \oplus \cdots \oplus E_s$ and take a general $f \in H^0(X, Hom(E, E \otimes M))$. We have $f^k = 0$ and $f^{k-1} \neq 0$ (with the convention $f^0 = 1$ and the abuse of notation of writing f in a composition of maps instead of $f_{\otimes M^{\otimes \beta}}$) for a suitable β if and only if k is the maximal integer such that there are k integers j_i, $1 \leq i \leq k$, such that:*

(a) $1 \leq j_1 < \cdots < j_k \leq s$;

(b) $d_{j+1}/r_{j+1} < d_j/r_j + (j_{i+1} - j_i)t + 1 - g$;

(c) $\sum_{i=1}^{k-1} \max\{0, r_{j+1} - r_j\} \geq r_{j_1}\}$.

Proof. A map $f \in H^0(X, Hom(E, E \otimes M))$ is uniquely determined by its components $f_{ij} \in H^0(X, Hom(E_i, E_j \otimes M))$ and conversely any set of f_{ij}'s gives a map f. The map f is general if and only if all f_{ij} are general. The components of each f^x is obtained taking x suitable iterates of the components of f. By [3], Th. 1.2, and the generality of each pair $(E_i, E_j \otimes M)$ we have $h^0(X, Hom(E_i, E_j \otimes M)) \neq 0$ if and only if $d_i/r_i < d_j/r_j + t + 1 - g$. Apply Proposition 1.

Remark 4. Let k be the integer described in the statement of Theorem 3 or Theorem 2. Fix an integer x such that $2 \le x < k$. It is very easy to prove the existence of $h \in H^0(X, Hom(E, E \otimes M))$ such that $h^x = 0$ and $h^x \neq 0$: take as h a general twisted endonorphism such that some of the components $h_{ij} \in H^0(X, Hom(E_i, E_j \otimes M))$ are zero, while the other are general and then apply Proposition 1 as in the proof of Theorem 3.

4. Extensions of two stable bundles. Fix an integer $s \ge 2$ and semistable vector bundles $E_1, \dots, E_s$ on X such that $\mu(E_1) > \cdots > \mu(E_s)$. Let $D(E_1, \dots, E_s)$ be the set of all vector bundles (up to isomorphisms) on X which have a Harder - Narasimhan filtration whose graded subquotients are isomorphic to $E_1, \dots, E_s$.

Theorem 4. *Fix integers $r_2 > r_1 > 0$, d_1, d_2 and t such that $t \le 0$ and $d_2/r_2 < d_1/r_1$ and $d_2/r_2 < d_1/r_1 + t + 1 - g$. Fix $M \in Pic^t(X)$ such that $M \ncong \mathcal{O}_X$. Take a general $(E_1, E_2) \in M(X; r_1, d_1) \times M(X; r_2, d_2)$ and fix any $E \in D(E_1, E_2)$. Take a general $f \in H^0(X, Hom(E, E \otimes M))$. There is no $R \in Pic(X)$, $f_1 \in H^0(X, Hom(E, E \otimes R))$ and $f_2 \in H^0(X, Hom(E \otimes R, E \otimes M))$ such that $t \le deg(R) \le 0$, $R \ncong \mathcal{O}_X$, $R \ncong M$ and $f_2 \circ f_1 = f$.*

Proof. By Lemma 2 the composition of $f|E_1$ with the natural surjection $E \otimes M \to E_2 \otimes M$ vanishes. Hence f is induced by $h' : E_2 \to E$. Since any map $E_2 \to E_2 \otimes M$ vanishes, h' is induced by $h : E_2 \to E_1 \otimes M$. Conversely, any such h induces a morphism $E \to E \otimes M$. Hence the generality of f implies that f is induced by a general $h \in H^0(X, Hom(E_2, E_1 \otimes M))$. By the inequalities $d_2/r_2 < d_1/r_1 + t + 1 - g$, $r_2 > r_1$, the generality of the pair $(E_2, E_1 \otimes M)$ and the proof of [3], Th. 0.2, h is surjective. Assume the existence of R, f_1, f_2 as above. By the first part of the proof and Lemma 2 (which guarantees $E_2 \ncong E_2 \otimes R$) $f_1(E) \subset E_1 \otimes R$, $f_1(E_1) = 0$ and hence f_1 induces $h_1 : E_2 \to E_1 \otimes R$. Hence $f_2 \circ f_1$ is uniquely determined by $f_2|E_1 \otimes R : E_1 \otimes R \to E_1 \otimes M$. Since E_1 is stable and $E_1 \otimes R \ncong E_1 \times M$ (Lemma 1), $f_2|E_1 \otimes R$ is not surjective, contradicting the surjectivity of h.

Theorem 5. *Fix integers $r > 0$, $d_1 > d_2$ and $t \le 0$ such that $d_2 < d_1 + t + r(1 - g)$. Fix $M \in Pic^t(X)$ such that $M \ncong \mathcal{O}_X$. Take a general $(E_1, E_2) \in M(X; r, d_1) \times M(X; r, d_2)$ and fix any $E \in D(E_1, E_2)$. Take a general $f \in H^0(X, Hom(E, E \otimes M))$. There is no $R \in Pic(X)$, $f_1 \in H^0(X, Hom(E, E \otimes R))$ and $f_2 \in H^0(X, Hom(E \otimes R, E \otimes M))$ such that $t \le deg(R) \le 0$, $R \ncong \mathcal{O}_X$, $R \ncong M$ and $f_2 \circ f_1 = f$.*

Proof. By the first part of the proof of Theorem 4 f is induced by a general $h \in H^0(X, Hom(E_2, E_1 \otimes M))$. By the inequality $d_2 < d_1+t+1-g$, the assumption $\mathrm{rank}(E_1) = \mathrm{rank}(E_2)$, the generality of the pair $(E_2, E_1 \otimes M)$ and [3], Th. 0.2, h is injective, i.e. Coker(h) is a skyscraper sheaf. Assume the existence of R, f_1, f_2 as above. By the first part of the proof of Theorem 4 and Lemma 2 (which guarantees $E_2 \ncong E_2 \otimes R$) $f_1(E) \subset E_1 \otimes R$, $f_1(E_1) = 0$ and hence f_1 induces $h_1 : E_2 \to E_1 \otimes R$. Hence $f_2 \circ f_1$ is uniquely determined by $f_2|E_1 \otimes R : E_1 \otimes R \to E_1 \otimes M$. Since E_1 is stable and $E_1 \otimes R \ncong E_1 \times M$ (Lemma 1), $f_2|E_1 \otimes R$ is not injective, i.e. it drops rank, contradicting the injectivity of h.

REFERENCES

1. Ballico, E. – *On the twisted endomorphism algebra of a vector bundle on a curve,* Annali Mat. pura app. (IV) 169 (2001), 383–400.
2. Hirschowitz, A. – *Probléme de Brill-Noether en rang superieur*, Prepublicaions Mathematiques n. 91, Nice, 1986.
3. Russo, B. and Teixidor i Bigas, M. – *On a conjecture of Lange,* J. Alegebraic Geom. 8 (1999), 483–496.

Received: 24.III.2003

Dept. of Mathematics,
University of Trento,
38050 Povo (TN), ITALY
ballico@science.unitn.it