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ON HOLOMORPHIC THEORY OF A CLASS OF LEFT BOL LOOPS

BY

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Abstract. Conditions for which the holomorph of a left Bol loop will form a left Bol loop is established. It is also proved that the left translation of a K-loop is an automorphism using the holomorph of a k-loop.

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1. Introduction. The study of holomorphic theory has been considered in the works of Bruck and Paige [3], Bruck [2], Robinson [7], and Chiboka and Solarin [4]. It is the intension of this present effort to extend this study to the class of left Bol loops which satisfies the automorphic inverse property. This class of loops are also called K-loops. Studies on this type of loops includes that of Kiechle [5].

If $(Q,.)$ is a loop and $x \in Q$ then the permutations R_x and L_x called the right and the left multiplications respectively are defined by $aR_x = a.x$ and $aL_x = x.a$ for all $a \in Q$.

A loop $(Q,.)$ satisfies the right inverse property if there exists a function $\rho : Q \to Q$ such that $(x.y)y\rho = x$ for all $x, y \in Q$. If there exist a function $\lambda : Q \to Q$ such that $y\lambda(y.x) = x$ for all $x, y \in Q$ then $(Q,.)$ is said to satisfy the left inverse property. If $(Q,.)$ satisfies both the left and right inverse properties then $(Q,.)$ is called an inverse property loop. This implies that for each element $x \in Q$ there exist $x^{-1} \in Q$ such that $x.x^{-1} = x^{-1}.x = 1$. This also implies that in $(Q,.)$ $R_a^{-1} = R_{a^{-1}}$ and $L_a^{-1} = L_{a^{-1}}$.

If $(Q,.)$ is a loop, the triple $< U,V,W >$ of permutations of Q is called an autotopism of Q if and only if $xU.yV = (x.y)W$ for all $x,y \in Q$. The set of all autotopism of a loop forms a group under componentwise multiplication. Bruck [1], pp. 112.

Let $(Q,.)$ and $(G,\circ)$ be two loops. $(G,\circ)$ is said to be an isotope of $(Q,.)$ if and only if there exist a triple $< U,V,W >$ where $U,V\&W$ are one-to-one mappings of Q onto G such that $xU \circ yV = (x.y)W$ for all $x,y \in Q$. If G has the same elements as Q and W is the identity mapping of Q the $(G,\circ)$ is called a principal isotope of $(Q,.)$.

Bruck [2] has proved that every isotope of a loop is isomorphic to a principal isotope of the loop and every principal isotope $(Q,\circ)$ of a loop $(Q,.)$ is defined by

$$x \circ y = xR_{b^{-1}}.yL_{a^{-1}}$$

for all $x,y \in Q$ and some $a,b \in Q$.

A loop $(Q,.)$ is said to be a left Bol loop if for all $x,y.z \in Q$ the identity

$$y(z.yx) = (y.zy)y \tag{1}$$

holds.

For a loop $(Q,.)$ satisfying (1) $x\lambda = x\rho$ for all $x \in Q$ therefore $R_x^{-1} = R_{x^{-1}}$ and $L_x^{-1} = R_{x^{-1}}$ for all $x \in Q$.

A loop $(Q,.)$ is said to be an automorphic inverse property loop provided that for all $x,y \in Q$

$$x^{-1}y^{-1} = (xy)^{-1} \tag{2}$$

A loop $(Q,.)$ is called a K-loop if both (1) and (2) holds in $(Q,.)$.

(1) implies that $< R_xL_x, L_{x^{-1}}, L_x >$ is an autotopism for all $x \in Q$.

If $(Q,.)$ is a loop, let A be any group automorphisms of $(Q,.)$. then the holomorph $A(Q)$ of Q by A or the A-holomorph of Q is the set $H = A \times Q$ with the binary operation $*$ is defined by

$$(\alpha,x) * (\beta,y) = (\alpha\beta, x\beta.y)$$

for all $x,y \in Q$ and $\alpha,\beta \in A$.

2. Main Results

Theorem 2.1. *Let $(Q,.)$ be a left Bol loop, A a group of automorphisms of Q, and H the A-holomorph of Q. Then $(H,*)$ is a left Bol loop if and*

only if the triple $< R_{x\psi}, L_{(x\psi)^{-1}}, I >$ *is autotopism* Q *for all* $x \in Q$ *and all* $\psi \in A$.

Proof. $(H, *)$ is a left Bol loop if and only if $x \in Q$ and $\alpha \in A$ the triple

$$< R_{*(\alpha,x)}L_{*(\alpha,x)}, L_{*(\alpha,x)^{-1}}, L_{*(\alpha,x)} >$$

implies that for $(\beta, y), (\gamma, z) \in A$ we have

$$(\beta, y)R_{*(\alpha,x)}L_{*(\alpha,x)} * (\gamma, z)L_{*(\alpha,x)^{-1}} = ((\beta, y) * (\gamma, z))L_{*(\alpha,x)}$$

$$(\alpha, x)[(\beta, y) * (\alpha, x)] * (\gamma, z)L_{*(\alpha,x)^{-1}} = ((\beta, y) * (\gamma, z))L_{*(\alpha,x)}$$

$$(\alpha, x) * (\beta\alpha, y\alpha.x) * (\gamma, z)L_{*(\alpha,x)^{-1}} = (\alpha, x) * (\beta\gamma, y\gamma.z)$$

$$(\alpha\beta\alpha, x\beta\alpha(y\alpha.x)) * (\gamma, z)L_{*(\alpha,x)^{-1}} = (\alpha\beta\gamma, x\beta\gamma(y\gamma.z))$$

Now suppose

$$(\kappa, u) = (\gamma, z)L_{*(\alpha,x)^{-1}}$$

which implies that

$$(\alpha, x) * (\kappa, u) = (\gamma, z)$$

therefore

$$(\alpha\kappa, x\kappa.u) = (\gamma, z)$$

therefore $\alpha\kappa = \gamma \Rightarrow \kappa = \alpha^{-1}\gamma$ and $z = x\kappa.u = x\alpha^{-1}\gamma.u \Rightarrow u = zL_{(x\alpha^{-1}\gamma)^{-1}}$

Substituting these we have

$$(\alpha\beta\alpha, x\beta\alpha.(y\alpha.x))(\alpha^{-1}\gamma, zL_{(x\alpha^{-1}\gamma)^{-1}}) = (\alpha\beta\gamma, x\beta\gamma.(y\gamma.z))$$

$$(\alpha\beta\gamma, \{x\beta\alpha.(y\alpha.x)\}\alpha^{-1}\gamma.zL_{(x\alpha^{-1}\gamma)^{-1}}) = (\alpha\beta\gamma, x\beta\gamma.(y\gamma.z))$$

therefore

$$\{x\beta\alpha.(y\alpha.x)\}\alpha^{-1}\gamma.zL_{(x\alpha^{-1}\gamma)^{-1}} = x\beta\gamma.(y\gamma.z)$$

$$x\beta\gamma.(y\gamma.x\alpha^{-1}\gamma).zL_{(x\alpha^{-1}\gamma)^{-1}} = x\beta\gamma.(y\gamma.z)$$

let $\alpha^{-1}\gamma = \psi$ and $y\gamma = y$ then we obtain

$$x\beta\gamma.(\overline{y}.x\psi).zL_{(x\psi)^{-1}} = x\beta\gamma(\overline{y}z)$$

$$(\overline{y}R_{x\psi}.zL_{(x\psi)^{-1}} = \overline{y}zI$$

hence the triple

$$< R_{x\psi}, L_{(x\psi)^{-1}}, I >$$

is an autotopism for all $x \in Q$ and $\psi \in A$. □

Corollary 2.1. *Let A be a group of automorphisms of a left Bol loop $(Q, .)$. Then the holomorph $(H, *)$ is a left Bol loop if and only if every automorphism of $(Q, .)$ is nuclear.*

Proof. From the preceding theorem $(H, *)$ is a left Bol loop if and only if

$$< R_{x\psi}, L_{(x\psi)^{-1}}, I >$$

is an autotopism for all $x \in Q$ and $\psi \in A$. Therefore for all $a, b \in Q$

$$aR_{x\psi}.bL_{(x\psi)^{-1}} = (a.b)I$$

if $b = L_{(x\psi)}$ we have

$$aR_{x\psi}.b = a.bL_{x\psi}$$

$$(a.x\psi).b = a.(x\psi.b)$$

therefore $x\psi \in N_\mu$. But for a left Bol loop N_μ and N_λ coincide hence ψ is nuclear. □

Lemma 2.1. *Let $(Q, .)$ be a K-loop . Then $< R_{x^{-1}}L_{x^{-1}}, L_x, L_{x^{-1}} >$ is an autotopism for all $x \in Q$.*

Proof. Since Q is a K-loop then Q satisfies identities (1) and (2). By the virtue of (2), Q also satisfy

$$x^{-1}(z^{-1}.(xy)^{-1}) = (x^{-1}.(zx)^{-1})y^{-1} \tag{3}$$

for all $x, y, z \in Q$.

If we replace $y = x^{-1}y$ in (3) then we obtain

$$x^{-1}(z^{-1}y^{-1}) = (x^{-1}.(zx)^{-1})(x^{-1}y^{-1})$$

$$= (x^{-1}.z^{-1}x^{-1})(x^{-1}y^{-1})$$

which implies that

$$z^{-1}y^{-1}L_{x^{-1}} = z^{-1}R_{x^{-1}}L_{x^{-1}}.y^{-1}L_x$$

for all $x, y, z \in Q$ hence the result follows. □

Theorem 2.2. *Let $(Q,.)$ be a K-loop, A a group of automorphisms of Q and H the A-holomorph of Q. Then $< R_{*(\alpha,x)^{-1}} L_{*(\alpha,x)^{-1}}, L_{*(\alpha,x)}, L_{*(\alpha,x)^{-1}} >$ is an autotopism for all $(\alpha,x) \in H$ (i.e. $(H,*)$ is a K-loop) provided $\alpha = I$ (the identity mapping) and $< R_{x^{-1}\alpha}, L_x L_{x\alpha}, I >$ is an autotopism for all $x \in Q$.*

Proof.

$$< R_{*(\alpha,x)^{-1}} L_{*(\alpha,x)^{-1}}, L_{*(\alpha,x)}, L_{*(\alpha,x)^{-1}} >$$

is an autotopism for all $(\alpha,x) \in H$ implies that for all $(\beta,y),(\gamma,z) \in H$ we have

$$(\beta,y)R_{*(\alpha,x)^{-1}} L_{*(\alpha,x)^{-1}} * (\gamma,z)L_{*(\alpha,x)} = (\beta\gamma, y\gamma.z)L_{*(\alpha,x)^{-1}}$$

which implies that

$$(\beta,y)R_{*(\alpha,x)^{-1}} * (\gamma,z)L^2_{*(\alpha,x)} = (\beta\gamma, y\gamma.z) \tag{4}$$

Now suppose

$$(\beta,y)R_{*(\alpha,x)^{-1}} = (\lambda,u)$$

therefore we have

$$(\beta,y) = (\lambda,u) * (\alpha,x) = (\lambda\alpha, u\alpha.x)$$

which implies that $\lambda = \beta\alpha^{-1}$ and $u = yR_{x^{-1}\alpha^{-1}}$. Hence

$$(\beta,y)R_{*(\alpha,x)^{-1}} = (\beta\alpha^{-1}, yR_{x^{-1}\alpha^{-1}})$$

Putting this in (4) we obtain

$$(\beta\alpha^{-1}, yR_{x^{-1}\alpha^{-1}}) * (\alpha^2\gamma, x\alpha\gamma.(x\gamma.z)) = (\beta\gamma, y\gamma.z)$$

$$(\beta\alpha\gamma, yR_{x^{-1}\alpha\gamma.}(x\alpha\gamma.(x\gamma.z))) = (\beta\gamma, y\gamma.z)$$

Therefore $\beta\alpha\gamma = \beta\gamma \Rightarrow \alpha = I$ and

$$yR_{x^{-1}\alpha\gamma.}(x\alpha\gamma.(x\gamma.z)) = y\gamma.z$$

for which using $\alpha = I$ gives

$$yR_{x^{-1}\gamma.}(x\gamma.(x\gamma.z)) = y\gamma.z$$

which implies that

$$yR_{x^{-1}}.(x.(x.z\gamma^{-1})) = y.z\gamma^{-1}$$

If $z\gamma^{-1} = \overline{z}$, we have

$$yR_{x^{-1}}.(x.(x.\overline{z})) = y\overline{z}$$

which implies that

$$yR_{x^{-1}}.\overline{z}L_xL_x = y\overline{z}I$$

and hence $< R_{x^{-1}}, L_xL_x, I >$ is an autotopism for all $x \in Q$. □

Corollary 2.2. *Let $(Q, .)$ be a K-loop then L_x is automorphisn for all $x \in Q$.*

Proof. Since $(Q, .)$ is a Bol loop therefore $< R_xL_x, L_{x^{-1}}, L_x >$ for all $x \in Q$. By theorem 2.2 $< R_{x^{-1}}, L_xL_x, I >$ is also an autotopism for all $x \in Q$, therefore $< L_x, L_x, L_x >$ is an autotopism for all $x \in Q$ hence L_x is an automorphism for all $x \in Q$. □

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