ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" IAŞI
Tomul LI, s.I, Matematică, 2005, f.1

QUASICONFORMAL MAPPINGS ON GLOBALLY QUASICONFORMALLY m-COLLARED DOMAINS ON THE BOUNDARY

BY

ANCA ANDREI

Abstract. In this note we generalize the definition of globally quasiconformally collared domains on the boundary. We give a boundary extension theorem for quasiconformal mappings on globally quasiconformally m-collared domains.

Mathematics Subject Classification 2000: 32A07, 33E99.

Key words: quasiconformal mapping, globally quasiconformally m-collared domains.

This note is a continuation to [1], in which we proved that each quasiconformal mapping $f : D \to D^{'}$ can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D}^{'}$ if $D, D^{'}$ are domains in $\overline{R}^n$, globally quasiconformally collared on the boundary. We use the notation and terminology of [1]. Remember that all the sets considered in this note are assumed to lie in $\overline{R}^n = R^n \cup \{\infty\}, n \geq 2$. Mention must be of the fact that all neighborhoods considered are supposed to be open and the sets $D, D^{'}$ are domains in $\overline{R}^n$. We denote by $\Delta(E, F; G)$ the family of all paths which join E and F in G. If Γ is a family of paths in $\overline{R}^n$, we denote by $F(\Gamma)$ the family of all Borel functions $\rho : \overline{R}^n \to [0, \infty]$ for which $\int\limits_{\gamma} \rho ds \geq 1$ for every rectifiable path $\gamma \in \Gamma$. The modulus of Γ is defined as $M(\Gamma) = \inf\limits_{\rho \in F(\Gamma)} \int\limits_{R^n} \rho^n dm$.

A homeomorfism $f : D \to D^{'}$ is said to be K-quasiconformal, $1 \leq K < \infty$, if it satisfies the double inequality :

$$\frac{1}{K}M(\Gamma) \leq M(f(\Gamma)) \leq KM(\Gamma)$$

for each path family Γ in D. Here $f(\Gamma) = \{f \circ \alpha, \alpha \in \Gamma\}$.

A homeomorfism f is said to be quasiconformal if it is K-quasiconformal for some K. We remember the following definitions :

Definition 1. D is locally connected at $b \in \partial D$ if b has arbitrarily small neighborhoods U such that $U \cap D$ is connected, that is, for every neighborhood V of b, there exists a nighborhood U of b, $U \subset V$, such that $U \cap D$ is connected

Definition 2. D is m-connected at $b \in \partial D$, $(m = 1, 2, ...)$ if m is the least integer for which there exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of m components.

Definition 3. $b \in \partial D$ is quasiconformally accessible from D if, given any neighborhood U of b, there exists a nondegenerate continuum $F \subset D$ and a number $\delta > 0$ such that $M(\Delta(F, E; D)) \geq \delta$ whenever E is a connected subset of D with $b \in \overline{E}$ and $E \cap \partial U \neq \Phi$.

Definition 4. D is globally quasiconformally collared on the boundary if there exists a neighborhood U of ∂D and a homeomorphism

$$g : U \cap \overline{D} \to \{x \in R^n, a < |x| \leq 1\}, a \geq 0$$

such that $g/_{U \cap D}$ is quasiconformal.

The last definition has been introduced by Gehring [3].

Definition 5. D is globally quasiconformally m-collared on the boundary $(m = 1, 2, ...)$ if there exists a neighborhood U of ∂D such that $U \cap D$ consists of m components $E_1, E_2, .., E_m$ and for each $i \in \{1, ..., m\}$, there exists a homeomorphism

$$g_i : U \cap \overline{E}_i \to \{x \in R^n, a < |x| \leq 1\}, a \geq 0$$

such that the restriction $g_i/_{E_i}$ is quasiconformal.

Remark 1. Väisälä ([5], p.120) has the following definition of domains locally quasiconformally bi-collared at a boundary point: "A domain D is

said to be locally quasiconformally bi-collared at a boundary point b, if there is a neighborhood U of b and a quasiconformal mapping $g : U \to B^n$ such that $g(U \cap D)$ is the upper half ball B^n_+". As you can see, this definition is different of my definition from above for $m = 2$.

Remark 2. We observe that every E_i in the Definition 5 is locally connected at each point of $\partial E_i \cap \partial D$.

We want to give a boundary extension theorem for quasiconformal mappings when one of the domains in question is globally quasiconformally m -collared on the boundary. First, we will prove some preliminary results.

Theorem 1. *A domain D in $\overline{R}^n$ is globally quasiconformally m-collared on boundary if and only if there exist arbitrarily small neighborhoods U of ∂D such that $U \cap D$ consists of m components $E_1, ..., E_m$ and for each $i \in \{1, ..., m\}$, there exists a homeomorphism $g_i : U \cap \overline{E}_i \to \{x \in R^n, a < |x| \leq 1\}, a \geq 0$, such that the restriction $g_i/_{E_i}$ is quasiconformal.*

Proof. The sufficiency is obviously. We prove the necessity.

Suppose that D is globally quasiconformally m-collared on the boundary and hence there exists a neighborhood U of ∂D such that $U \cap D$ consists of m components $E_1, ..., E_m$ and for each $i \in \{1, ..., m\}$, there exists a homeomorphism $g_i : U \cap \overline{E}_i \to \{x \in R^n, a < |x| \leq 1\}, a \geq 0$ such that the restriction $g_i/_{E_i}$ is quasiconformal. We denote by $C = \{x \in R^n, a < |x| < 1\}$ and hence $\{x \in R^n, a < |x| \leq 1\} = C \cup S^{n-1}$.

Let V be a neighborhood of ∂D. There exist the real numbers b_i, c_i such that $a < b_i < 1 < c_i$ and $g_i^{-1}(C_i \cap (C \cup S^{n-1})) \subset V \cap U \cap \overline{E}_i$ and $g_i^{-1}(C_i \cap C) \subset V \cap U \cap E_i$, where $C_i = \{x \in R^n, b_i < |x| < c_i\}$.

The set $W = (V - \overline{D}) \cup (\bigcup_{i=1}^{m} g_i^{-1}(C_i \cap (C \cup S^{n-1})))$ is a neighborhood of ∂D, $W \subset V$ and $W \cap D$ consists of the components $g_1^{-1}(C \cap C_1), ..., g_m^{-1}(C \cap C_m)$.We consider the homeomorphism $h_i^{'}(x) = |x|^{c-1} x$, where $c = \frac{\log a}{\log b_i}$ which maps $\{x \in R^n, b < |x| \leq 1\}$onto$\{x \in R^n, a < |x| \leq 1\}$ and its restriction to $\{x \in R^n, b_i < |x| < 1\}$ is quasiconformal. The mapping $h_i = h_i^{'} \circ g_i : W \cap \overline{g_i^{-1}(C \cap C_i)} \to \{x \in R^n, a < |x| \leq 1\}$ is a homeomorphism and since the restriction $g_i/_{E_i}$ is quasiconformal and $W \cap g_i^{-1}(C \cap C_i) = g_i^{-1}(C \cap C_i)$ it follows that $h_i/_{g_i^{-1}(C \cap C_i)}$ is quasiconformal mapping. □

Theorem 2. *Let D be a domain in $\overline{R}^n$ globally quasiconformally m-collared on the boundary. Then D is quasiconformally accessible on the*

boundary.

Proof. Let b a boundary point of D and let U be a neighborhood of b. Performing a preliminary inversion if necessary, we may assume that $b \neq \infty$. By Theorem 1, there exists a neighborhood V of ∂D such that $dist(b, \partial V) < dist(b, \partial U), V \cap D$ consists of the components $E_1, ..., E_m$ and for each $i \in \{1, ..., m\}$, there exists a homeomorphism $g_i : V \cap \overline{E}_i \to \{x \in R^n, a < |x| \leq 1\}, a \geq 0$, such that the restriction $g_i/_{E_i}$ is quasiconformal. Since V is a neighborhood of b and $b \in \partial D$ it follows that there exists $i \in \{1, ..., m\}$ such that $b \in \partial E_i$. On the other hand $V \cap U$ is a neighborhood of b and hence there exists $0 < a_1 < 1 - a$ such that $g_i^{-1}(B^n(g_i(b), a_1) \cap C_1) \subset \overline{E_i \cap V \cap U}$, where $C_1 = \{x \in R^n, a < |x| < 1\}$. Set $I^{'}$ a radial segment of $\overline{B^n(g_i(b), a_1)}$ with length $\frac{a_1}{2}$ and one endpoint in $S^{n-1}(g_i(b), a_1)$, which is lie in $\overline{B^n(g_i(b), a_1) \cap C_1}$. Denote by $I = g_i^{-1}(I^{'})$ and consider a continuum F in D such that $I \subset F$.

We show that the modulus condition in the Definition 3 for quasiconform accessibility of b from D is satisfied by these F, U and $\delta = \frac{b_n \log 2}{K}$, where b_n is a positive constant depending only on n (Theorem 10.2 [5]) and $K = K(g_i)$.

Let E be a connected set in D with $b \in \overline{E}$ and $E \cap \partial U \neq \Phi$. Set $\Gamma^{'} = \Delta(g_i(E_i \cap E), g_i(E_i \cap F); C_1)$.

If $g_i(E_i \cap E) \cap g_i(E_i \cap F) \neq \Phi$ it follows that $M(\Gamma^{'}) = \infty$ and hence $M(\Delta(E, F; D)) > \delta$.

Suppose that $g_i(E_i \cap E) \cap g_i(E_i \cap F) = \Phi$ and let $\rho \in F(\Gamma^{'})$. By the choice of a_1, F and since E is connected, we infer that the sphere $S^{n-1}(g_i(b), r)$ meets both $g_i(E_i \cap E), g_i(E_i \cap F)$ for $\frac{a_1}{2} < r < a_1$.

By Fubini's theorem and by Theorem 10.2[4] we have

$$\int_{R^n} \rho^n dm \geq \int_{\frac{a_1}{2}}^{a_1} dt \int_{S^{n-1}(g_i(b), t)} \rho^n dm_{n-1} \geq \int_{\frac{a_1}{2}}^{a_1} \frac{b_n}{t} dt = b_n \log 2.$$

Since the modulus is monotone function and g_i is K-quasiconformal and hence g_i^{-1} is K-quasiconformal, we obtain:

$$M(\Delta(E, F; D)) \geq M(g_i^{-1}(\Gamma^{'})) \geq \frac{1}{K} M(\Gamma^{'}) \geq \frac{1}{K} b_n \log 2 = \delta.$$

□

Theorem 3. *Let $f : D \to D^{'}$ be a quasiconformal mapping and $D, D^{'}$ globally quasiconformally m-collared domains on the boundary such that the sets $C(f, b), C(f^{-1}, b^{'})$ are connected whenever $b \in \partial D$ and $b^{'} \in \partial D^{'}$. Then f can be extented to a homeomorphism $f^{*} : \overline{D} \to \overline{D}^{'}$.*

Proof. Since $D, D^{'}$ are globally quasiconformally m-collared domains on the boundary, by Theorem 2, $D, D^{'}$ are quasiconformally accessible on the boundary Let b be a boundary point of D. By Theorem 1, there exist arbitrarily small neighborhoods U of ∂D such that $U \cap D$ consists of m components $E_1, ..., E_m$. Obviously, such a U is a neighborhood of b and there exists an E_i such that $b \in \overline{E}_i$. Let us prove that $C(f/_{E_i}, b)$ contains a single point.

Suppose that there exist two distinct points $b_i^{'}$,$b^{'} \in C(f/_{E_i}, b) \subset \partial D^{'}$ and by Theorem 2, $b_i^{'}, b^{'}$ are quasiconformally accessible from $D^{'}$. We may assume that $b_i^{'}, b^{'}, b$ are finite composing f with an auxiliary inversion if necessary. Let $0 < 2r \leq \left| b_i^{'} - b^{'} \right|$ and choose a nondegenerate continuum $A^{'}$ in $D^{'}$. Since $b_i^{'}$ is quasiconformally accessible from $D^{'}$, by Theorem 1.16 [4], there exists $\delta_i > 0$ such that

(1) $M(\Delta(A^{'}, E^{'}; D^{'})) \geq \delta_i$ whenever $E^{'}$ is a connected set in $D^{'}, b_i^{'} \in \overline{E}^{'}$ and $E^{'} \cap S^{n-1}(b_i^{'}, r) \neq \Phi$.

Since $b^{'}$ is quasiconformally accessible from $D^{'}$, there exists $\delta > 0$ such that

(2) $M(\Delta(A^{'}, E_1^{'}; D^{'})) \geq \delta$ whenever $E_1^{'}$ is a connected set in $D^{'}, b^{'} \in \overline{E}_1^{'}$ and $E_1^{'} \cap S^{n-1}(b^{'}, r) \neq \Phi$.

Let us denote $\delta^{'} = \min\{\delta_i, \delta\}$ and $d = dist(A, \partial D) > 0$, where $A = f^{-1}(A^{'})$. Since $b_i^{'}, b^{'} \in C(f/_{E_i}, b)$ we may choose two sequences $(x_{in}), (x_n)$ in E_i such that $x_{in} \to b, x_n \to b$ and $f(x_{in}) \to b_i^{'}$, $f(x_n) \to b^{'}$. Fix $\varepsilon, 0 < \varepsilon < d$, and since E_i is 1 -connected at b by Theorem 1.10 [4] there exists a component E of $B^n(b, \varepsilon) \cap E_i$ which contains two subsequences of (x_{in}) and (x_n). The set $E^{*} = f(E)$ is connected and $b_i^{'}, b^{'} \in \overline{E}^{*}$. By (1), (2) and the choice of $\delta^{'}$ and since $E^{*} \cap S^{n-1}(b_i^{'}, r) \neq \Phi, E^{*} \cap S^{n-1}(b^{'}, r) \neq \Phi$ we have

(3) $M(\Delta(A^{'}, E^{*}; D^{'})) \geq \delta^{'}$.

On the other hand, the path family $\Gamma = \Delta(A, E; D)$ is minorized by the family $\Delta(S^{n-1}(b, \varepsilon), S^{n-1}(b, d); D)$, and hence

(4) $M(\Gamma) \leq \omega_{n-1}(\log \frac{d}{\varepsilon})^{1-n}$.

Letting $\varepsilon \to 0$ and using (3), we contradict the fact that f is a quasiconformal mapping. Therefore, $C(f/_{E_i}, b) = \{b_i^{'}\}$.

Using the fact that $C(f,b) = \cup C(f/_{E_i}, b)$ where the union is taken over all i for that $b \in \overline{E}_i$ (Remark 1, [2]) and since $C(f,b)$ is connected we obtain that $C(f,b)$ to reduces a single point and hence there exists $\lim_{x\to b} f(x)$. Considering likewise the inverse mapping, f^{-1} has a limit at each $b^{'} \in \partial D^{'}$. Therefore, f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D}^{'}$. □

Theorem 4. *Let $f : D \to D^{'}$ be a quasiconformal mapping and D a globally quasiconformally m-collared domain on the boundary. Then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D}^{'}$ if and only if $D^{'}$ is globally quasiconformally m-collared on the boundary and the sets $C(f,b), C(f^{-1}, b^{'})$ are connected whenever $b \in \partial D, b^{'} \in \partial D^{'}$.*

Proof. The sufficiency follows by Theorem 3. We prove the necessity. Suppose that f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D}^{'}$. Obviously $C(f,b)$ and $C(f^{-1}, b^{'})$ are connected sets whenever $b \in \partial D$,$b^{'} \in \partial D^{'}$. Since D is globally quasiconformally m-collared on the boundary, there exists a neighborhood U of ∂D such that $U \cap D$ consists of m components $E_1, ..., E_m$ and for each $i \in \{1, ..., m\}$ there exists a homeomorphism $g_i : U \cap \overline{E}_i \to \{x \in R^n, a < |x| \leq 1\}, a \geq 0$, such that the restriction $g_i/_{E_i}$ is a quasiconformal mapping. Choose $V^{'}$ a neighborhood of $\partial D^{'}$ such that $(f^*)^{-1}(V^{'} \cap \overline{D}^{'}) \subset U \cap \overline{D}$ and for each $i \in \{1, ..., m\}$, set $E_i^{'} = f(E_i)$ and since f^* is a homeomorphism, $f^*(\overline{E}_i) = \overline{E}_i^{'}$. The set $U^{'} = V^{'} \cup E_1^{'} \cup ... \cup E_m^{'}$ is a neighborhood of $\partial D^{'}$ and $U^{'} \cap D^{'}$ consists of m components $E_1^{'}, ..., E_m^{'}$.

Setting $f_i = f^*/_{U \cap \overline{E}_i}$, we obtain the homeomorphisms $h_i = g_i \circ f_i^{-1} : U^{'} \cap \overline{E}_i^{'} \to \{x \in R^n, a < |x| \leq 1\}$ and since $g_i/_{E_i}$,$f_i^{-1}/_{E_i^{'}}$ are quasiconformal, it follows that $h_i/_{E_i^{'}}$ is quasiconformal.

The proof of the theorem is complete. □

Theorem 5. *Let $f : D \to D^{'}$ be a quasiconformal mapping and D locally connected on the boundary and globally quasiconformally m-collared on the boundary. Then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D}^{'}$ if and only if $D^{'}$ is locally connected on the boundary and globally quasiconformally m-collared on the boundary.*

Proof. We prove the necessity. Since D is globally quasiconformally m-collared on the boundary, by Theorem 2, D is quasiconformally accessible

on the boundary and by Theorem 4, it follows that $D^{'}$ is globally quasiconformally m-collared on the boundary. On the other hand, D is locally connected on the boundary and by Theorem 3.1.[4], $D^{'}$ is locally connected on the boundary.

We prove the sufficiency. We assume that $D^{'}$ is locally connected and globally quasiconformalyy m-collared on the boundary. By Theorem 2, D, $D^{'}$ are quasiconformally accessible on the boundary. Therefore $D, D^{'}$ are locally connected and quasiconformally accessible on the boundary. We infer from Theorem 3.6(3) [4], that f can be extended to a homeomorphism $f^{*} : \overline{D} \to \overline{D}^{'}$. □

REFERENCES

1. Andrei, A. – *On boundary behavior of quasiconformal mappings in* $\overline{R}^n$, Libertas Mathematica, Tomus XXIV, Arlington, Texas 2004, p. 137-143.
2. Andrei, A. – *On boundary extension theorems for quasiconformal mappings in* $\overline{R}^n$, Analele. Ştiintifice. Univ. "Al. I. Cuza", Tomul L, s. I-a Math. 2004, f.1, p. 151-158.
3. Gehring, F.W. – *Extension theorems for quasiconformal mappings in n-space*, J. Analyse Math. 19(1967), 149-169.
4. Näkki, R. – *Boundary behavior of quasiconformal mappings in n-space*, Ann.Acad. Sci. Fenn, AI 484 (1970), 1-50.
5. Väisälä, J. – *Lectures on n-dimensional quasiconformal mappings*, Springer-Verlag-Berlin-Heidelberg-New York 229, 1971 (XIV), p. 144.

Received: 27.II.2004

Computer Science High-School,
Alexandru cel Bun 12, bl.B1, sc. A, ap.14
Suceava,
ROMÂNIA
anca56@hotmail.com,