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RESTRICTIONS AND TENSOR PRODUCTS OF GENERAL STABLE BUNDLES ON RATIONAL SURFACES AND THE PROJECTIVE SPACE*

BY

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Abstract. Let X be either a smooth projective rational surface or $\mathbf{P}^3$, E and F "general" stable bundles on X and $Z \subset X$ a suitable zero-dimensional scheme. Here we give conditions for the surjectivity of the restriction map $H^0(X, E) \to H^0(Z, E|Z)$ and for the vanishing of $H^1(X, E \otimes F)$.

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Key words: stable bundle, stable vector bundle, vector bundle on a rational surface, vector bundle on a projective space.

1. Introduction. Let X be either a smooth projective rational surface or $\mathbf{P}^3$, E and F " general " stable bundles on X and $Z \subset X$ a suitable zero-dimensional scheme. Here we give conditions for the surjectivity of the restriction map $H^0(X, E) \to H^0(Z, E|Z)$ and for the vanishing of $H^1(X, E \otimes F)$.

In section 2 we will prove the following result.

Theorem 1. *Fix integers $r \geq 2$ and c_2, a smooth and projective rational surface X, a zero-dimensional scheme $Z \subset X$, $Z \neq \emptyset$, and $L, H \in Pic(X)$ such that H is ample, $c_2 \geq r + \binom{r}{2}L^2$ and $h^1(X, \mathcal{I}_Z \otimes L) = 0$. Let $M(X; r, L^{\otimes r}, c_2; H)$ be the moduli space of all H-stable vector bundles E on X such that $rank(E) = r$, $det(E) \cong L^{\otimes r}$ and $c_2(E) = c_2$. Assume $c_2 - \binom{r}{2}L^2 \leq r(h^0(X, L) - length(Z))$. Then $M(X; r, L^{\otimes r}, c_2; H) \neq \emptyset$*

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and for a general E in the irreducible component $A(X, r, c_2 - \binom{r}{2}L^2; \otimes L)$ of $M(X; r, L^{\otimes r}, c_2; H)$ such that $E \otimes L^*$ is in the closure of the quasi-trivial sheaves in the sense of [1] the restriction map $\rho_{E,Z} : H^0(X, E) \to H^0(Z, E|Z)$ has maximal rank. Furthermore, $h^1(X, \mathcal{I}_Z \otimes E) = 0$.

By [4] the algebraic set $M(\mathbf{P}^2; r, L^{\otimes r}, c_2; H)$ is non-empty and irreducible and its general member has the expected cohomology. Call $M(\mathbf{P}^2; r, x, y)$ the set of all stable rank r vector bundles with degree x and $c_2 = y$.

In section 2 we will use Theorem 1 for $X = \mathbf{P}^2$ to prove the following result.

Theorem 2. *Fix positive integers integers $r \geq 2, t > 0, x > 0, c_2, d_2$ such that $c_2 \geq r + \binom{r}{2}L^2$, $(t-1)(t-2)/2 < d_2 \leq (t+1)(t+2)/2$ and $r\binom{x+t+2}{2} - c_2 + \binom{r}{2}L^2 > rd_2$. Then $h^1(X, E \otimes F) = h^2(X, E \otimes F) = 0$ for a general pair $(E, F) \in M(\mathbf{P}^2; r, x, c_2) \times M(\mathbf{P}^2; 2, t, d_2)$.*

We can check in a very similar way Theorem 2 even in the case $\text{rank}(F) = 3$. We do not the general case. we leave to the interested reader to check Theorem 2 for some (or all ?) smooth rational surfaces.

In section 3 we will consider general rank two vector bundles on $\mathbf{P}^3$. For any n-dimensional variety M, any integer $m > 0$ and any $P \in M_{reg}$ let mP denote the infinitesimal neighborhood of order $m-1$ of P in M, i.e. the closed subscheme of M with $(\mathcal{I}_{P,M})^m$ as ideal sheaf. Hence $(mP)_{red} = \{P\}$ and $\text{length}(mP) = \binom{n+m-1}{n}$. Let ψ be any rational map from M into a projective space $\mathbf{P}^N$. Hence ψ is defined in a neighborhood of P and hence it is defined the linear space $\langle mP \rangle \subseteq \mathbf{P}^N$ which is usually called the osculating space of order m of the closure of $\text{Im}(\phi)$. The sequence of integers $\dim(\langle mP \rangle)$, $m \geq 1$, for general $P \in X$ encodes important projective properties of the map ψ; the integer $\dim(\langle mP \rangle)$ may be called the m-rank of X. We have $\dim(\langle mP \rangle) \leq \min\{\binom{n+m-1}{n} - 1; N\}$. If $\dim(\langle mP \rangle) = \binom{n+m-1}{n} - 1$ the map ψ is called ordinary up to order m. Now take instead of ψ a morphism ϕ from X into a Grassmannian $G(r, N+1)$ of all $(N+1-r)$-dimensional linear subspaces of the vector space $\mathbf{K}^{\oplus(N+1)}$. ϕ is associated to a pair (E, V), where E is a rank r vector bundle on X and $V \subseteq H^0(X, E)$ is a linear subspace spanning E at each point of X and with $\dim(V) = N+1$. The concept equivalent to the m-rank of ϕ is the integer $\dim(\text{Im}(\rho)$, where $\rho : V \to H^0(mP, E|mP)$ is the restriction map and P is general in X. This integer may be called the m-rank or the generic m-rank of the pair (E, V). We have $\dim(\text{Im}(\rho)) \leq \min\{N+1, \text{rank}(E)\binom{n+m-1}{n}\}$. If $V = H^0(X, E)$ and $h^0(X, E)$ is known, then computing $\dim(\text{Im}(\rho))$ is equivalent to compute

$h^0(X, \mathcal{I}_{mP} \otimes E)$ for a general $P \in X$.

Let E be a rank two vector bundle on $\mathbf{P}^3$. E is said to have natural cohomology ([3]) if for every integer t at most one of the three integers $h^i(\mathbf{P}^3, E(t))$, $1 \le i \le 3$, is non-zero. For all integers c_1 and c_2 let $M(c_1, c_2)$ denote the moduli scheme of all rank two stable bundles on $\mathbf{P}^3$ with c_1 and c_2 as Chern classes. $M(c_1, c_2) \neq \emptyset$ if and only if $c_1^2 - 4c_2 < 0$ and $c_1 \cdot c_2$ is even (see e.g. the introduction of [3]). For all integers c_1, c_2 such that $c_1^2 - 4c_2 < 0$ and $c_1 \cdot c_2$ is even let $A(c_1, c_2)$ denote the generically reduced irreducible component of $M(c_1, c_2)$ described in [3], Th. 0.1; if c_1 is even, then $A(c_1, c_2)$ is an irreducible component of $M(c_1, c_2)$ containing certain instanton bundles ([3], Example 1.6.1). If $c_1^2 - 4c_2$ and either c_1 is even or $c_1^2 - 4c_2 \le 25$, then a general element of $A(c_1, c_2)$ has natural cohomology ([3], Th. 0.1). In section 3 we will give an outline of the proof of the following results.

Theorem 3. *Fix integers $y > 0$, c_1 and c_2 such that there is an integer $t < c_1$ with $\binom{t+3}{3} > (c_2 - c_1^2/4)(t+1)$ and $\binom{c_1+3}{3} - \binom{t+3}{3} \ge y$. Fix a general $E \in A(c_1, c_2)$ and then take a general $S \subset \mathbf{P}^3$ such that card$(S) = y$. Then the restriction map $\rho_{E,S} : H^0(\mathbf{P}^3, E) \to H^0(S, E|S)$ is surjective. Furthermore, $h^1(\mathbf{P}^3, \mathcal{I}_S \otimes E) = 0$.*

Theorem 4. *Fix integers $c_1 > m > 0$ and $c_2 > 0$ such that $c_1 \dot{c}_2$ is even and $2\binom{c_1+3}{3} \ge (c_2 + c_1)(c_1 + 1)$. Fix $P \in \mathbf{P}^3$ and a general $E \in A(c_1, c_2)$. Then $h^1(\mathbf{P}^3, \mathcal{I}_{mP} \otimes E) = 0$ and the restriction map $\rho_{E,mP} : H^0(\mathbf{P}^3, E) \to H^0(mP, E|mP)$ is surjective.*

We stress that Theorem 4 is rather weak: one would like to check that a general $E \in A(c_1, c_2)$ is m-ordinary (i.e. the restriction map $\rho_{E,mP} : H^0(\mathbf{P}^3, E) \to H^0(mP, E|mP)$ has maximal rank) for all m. It is straightforward to use [3] to get result sharper than Theorem 4 without introducing neither new ideas nor new tricks, but we were unable to come close to the conjectural bound in this way.

We work over an algebraically closed field $\mathbf{K}$.

2. Bundles on a rational surface

Remark 1. Let F be a torsion free sheaf on a smooth projective surface X. Then $\chi(F) = c_1(F) \cdot (c_1(F) - K_X)/2 - c_2(F) + \mathrm{rank}(F) \cdot \chi(\mathcal{O}_X)$ (Riemann - Roch). Take X, Z and L as in the statement of Theorem 1. Hence $\chi(\mathcal{O}_X) = 1$. Since Z is zero-dimensional, we have $h^1(Z, L|Z) = 0$. Since

$h^1(X, \mathcal{I}_Z \otimes L) = 0$, we obtain $h^1(X, L) = 0$. Since $Z \neq \emptyset$, $h^0(Z, L|Z) \neq 0$. Since $h^1(X, \mathcal{I}_Z \otimes L) = 0$, we obtain $h^0(X, L) \geq h^0(Z, L|Z) \neq 0$. Since $p_g(X) = 0$ and $h^0(X, L) \neq 0$, Serre duality gives $h^2(X, L) = 0$. Hence $h^0(X, L) = \chi(L) = L^2/2 - K_X \cdot L/2 + 1$.

Proof of Theorem 1. By [1], Th. 1.3, $A(X, r, c_2 - \binom{r}{2}L^2; \otimes L) \neq \emptyset$. By a standard exact sequence if $h^1(X, \mathcal{I}_Z \otimes E) = 0$, then the restriction map $H^0(X, E) \to H^0(Z, E|Z)$ is surjective. Hence it is sufficient to prove the last assertion of Theorem 1. Set $y := c_2 - \binom{r}{2}L^2$. By assumption $y > r$. For any finite $B \subset X$ let T_B the skyscraper sheaf supported by B and whose stalk over each $P \in B$ is a one-dimensional vector space. Thus $h^0(X, T_B) = \mathrm{card}(B)$ and T_B is uniquely determined by B. Fix a general $S \subset X$ such that $\mathrm{card}(S) = y$. Let $\eta : L^{\oplus r} \to T_S$ be a general surjection; η is uniquely determined choosing for each $P \in S$ a surjection $\eta_P : L^{\oplus r}|\{P\} \to T_P$ and hence essentially giving y $r-ples$ $(a_{1P}, \dots, a_{rP})$ of elements of $\mathbf{K}$ such that for each P there is $j(P)$ with $a_{j(P)P} \neq 0$. Set $G := \mathrm{Ker}(\eta)$. Hence G is a rank r torsion-free sheaf with $c_1(G) \cong L^{\otimes r}$ and $c_2(G) = y + \binom{r}{2}L^2 = c_2$. We have an exact sequence

$$0 \to G \to L^{\oplus r} \xrightarrow{\eta} T_S \to 0 \tag{1}$$

By the generality of S we may assume $Z_{red} \cap S = \emptyset$. Hence by tensoring (1) with $\mathcal{I}_Z$ we obtain the following exact sequence

$$0 \to \mathcal{I}_Z \otimes G \to (\mathcal{I}_Z \otimes L)^{\oplus r} \to T_S \to 0 \tag{2}$$

By the generality of S and η from (2) we obtain (e.g. use induction on the integer $\mathrm{card}(S)$) $h^0(X, \mathcal{I}_Z \otimes G) = \max\{0, h^0(X, \mathcal{I}_Z \otimes L^{\oplus r}) - \mathrm{card}(S)\} = \max\{0, r(h^0(X, \mathcal{I}_Z)) - y\}$. By [1], Th. 1.3, G is a flat limit of H-stable vector bundles on X. Hence we conclude by the semicontinuity of cohomology.

Proof of Theorem 2. By [2], Remarque 3.6, Prop. 4.2 and 4.3 and Th. 5.1, for a general $F \in$ we have $h^1(\mathbf{P}^2, F) = h^2(\mathbf{P}^2, F) = 0$, $h^0(\mathbf{P}^2, F) = (t+2)(t+1)/2 - d_2 > 0$ and there is an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2} \to F \to \mathcal{I}_S(t) \to 0 \tag{3}$$

in which S is a general union of d_2 points of $\mathbf{P}^2$. Take a general E and tensorize (3) with E. Hence to prove the theorem it is sufficient to obtain $h^1(\mathbf{P}^2, E(t)) = h^1(\mathbf{P}^2, \mathcal{I}_S \otimes E(t)) = h^2(\mathbf{P}^2, E) = h^2(\mathbf{P}^2, \mathcal{I}_S \otimes E(t)) =$

0. Since S is general, to apply Theorem 1 it is sufficient to notice that $\deg((E(t))) = rt + x$ and $c_2(E(t)) = c_2 - \binom{r}{2}x^2 + \binom{r}{2}(x+t)^2$.

3. Bundles on the projective space. Set $\mathcal{O} := \mathcal{O}_{\mathbf{P}^3}$.

Proof of Theorem 3. Set $c_1 = c_1' + 2x$ with x integer and $c_1' \in \{-1, 0\}$. Set $a = 2$ if $c_1' = 0$ and $a = 3$ if $c_1' = -1$. We recall the following results proved in [3]:

(i) Let $Y \subset \mathbf{P}^3$ be a general union of $b \geq 1$ lines (resp. $b \geq 4$ conics) and let $a = 2$ (resp. $a = 3$). Then for a general $\beta : \mathcal{O}(-a) \to \mathcal{O}_Y$ and every integer z the map $H^0(\mathbf{P}^3, \mathcal{O}(z-a)) \oplus \mathcal{O}_Y(z))$ induced in cohomology by the map $(\beta, 1)$ has maximal rank ([3], Prop. 5.1).

(ii) Fix an integer $z \geq 0$. Let $Y \subset \mathbf{P}^3$ be a general union of $[(z^2+2z+3)/3]$ lines and $(z+1)((z^2+2z+3)/3 - [(z^2+2z+3)/3])$ collinear points. Then for a general $\beta : \mathcal{O}(-2) \to \mathcal{O}_Y$ and every integer z the map $H^0(\mathbf{P}^3, \mathcal{O}(z-2) \oplus \mathcal{O}(z)) \to H^0(Y, \mathcal{O}_Y(z))$ induced in cohomology by the map $(\beta, 1)$ has maximal rank ([3], assertion H_z proved in [3], §8).

(iii) Fix an integer $z \geq 4$. Let $Y \subset \mathbf{P}^3$ be a general union of $[(z^2+z+6)/6]$ smooth conics and $(2z+1)((z^2+z+6)/6 - [(z^2+z+6)/6])$ points lying on a smooth conic. Then for a general $\beta : \mathcal{O}(-3) \to \mathcal{O}_Y$ and every integer z the map $H^0(\mathbf{P}^3, \mathcal{O}(z-3) \oplus \mathcal{O}(z)) \to H^0(Y, \mathcal{O}_Y(z))$ induced in cohomology by the map $(\beta, 1)$ has maximal rank ([3], assertion H_z' proved in [3], §9).

Now we define Assertions $A_{z,c}$ and $A_{z,c}'$, $c \geq 0$, which generalize respectively Assertion H_z and Assertion H_z'.

$A_{z,c}$: Fix integers $z \geq 0$ and $c \geq 0$. Let $Y \subset \mathbf{P}^3$ be a general union of $[(z^2+2z+3)/3]$ lines and $(z+1)((z^2+2z+3)/3 - [(z^2+2z+3)/3])$ collinear points and a general $S \subset \mathbf{P}^3$ such that $\mathrm{card}(S) = [(\binom{z+3+c}{3} - \binom{z+3}{3} + \binom{z+1+c}{3} - \binom{z+1}{3})/2] - c$. Then for a general $\beta : \mathcal{O}(-2) \to \mathcal{O}_Y$ the map $H^0(\mathbf{P}^3, \mathcal{I}_S(z-2) \oplus \mathcal{I}_S(z)) \to H^0(Y, \mathcal{O}_Y(z))$ induced in cohomology by the map $(\beta, 1)$ has maximal rank.

$A_{z,c}'$: Fix integers $z \geq 4$ and $c \geq 0$. Let $Y \subset \mathbf{P}^3$ be a general union of $[(z^2+z+6)/6]$ smooth conics and $(2z+1)((z^2+z+6)/6 - [(z^2+z+6)/6])$ points lying on a smooth conic and a general $S \subset \mathbf{P}^3$ such that $\mathrm{card}(S) = [(\binom{z+3+c}{3} - \binom{z+3}{3} + \binom{z+c}{3} - \binom{z}{3})/2] - c$. Then for a general $\beta : \mathcal{O}(-3) \to \mathcal{O}_Y$

the map $H^0(\mathbf{P}^3, \mathcal{I}_S(z-3)\oplus\mathcal{I}_S(z)) \to H^0(Y, \mathcal{O}_Y(z))$ induced in cohomology by the map $(\beta, 1)$ has maximal rank.

Notice that $A_{z,0} = H_z$ and $A'_{z,0} = H'_z$. Hence $A_{z,0}$ and $A'_{z,0}$ are true. To check $A_{z,c}$ and $A'_{z,c}$ as in [3]. §8 and §9, by induction on c it is sufficient (as in [3], §7) to check the corresponding statements in lower dimension. Since our statements of $A_{z,c}$ and $A'_{z,c}$ are not optimal, due to the term "$-c$" in the definition of $\mathrm{card}(S)$, it is sufficient to have a result only in the plane (contrary to the case $c = 0$ considered in [3], §7). We leave the easy (but long) details to the interested reader.

Proof of Theorem 4. Fix a hyperplane $H \in \mathbf{P}^3$ and $P \in H$. Thus $xP|H$ is the infinitesimal neighborhood of order $x-1$ of P in H and $\mathrm{Res}_H(xP) = (x-1)P$ with the convention $0P = \emptyset$. We introduce the following assertions B_z and B'_z similar respectively to assertions H_z and H'_z of [3].

B_z: Fix an integers $z \geq 1$. Let $Y \subset \mathbf{P}^3$ be a general union of $[(z^2 + 2z + 3)/3] - z$ lines and $(z+1)((z^2+2z+3)/3 - [(z^2+2z+3)/3])$ collinear points. Then for a general $\beta : \mathcal{O}(-2) \to \mathcal{O}_Y$ and evey integer z the map $H^0(\mathbf{P}^3, \mathcal{I}_{zP}(z-2)\oplus\mathcal{I}_{zP}(z)) \to H^0(Y, \mathcal{O}_Y(z))$ induced in cohomology by the map $(\beta, 1)$ is surjective.

B'_z: Fix integers $z \geq 4$ and $c \geq 0$. Let $Y \subset \mathbf{P}^3$ be a general union of $[(z^2+z+6)/6]-z$ smooth conics and $(2z+1)((z^2+z+6)/6-[(z^2+z+6)/6])$ points lying on a smooth conic. Then for a general $\beta : \mathcal{O}(-3) \to \mathcal{O}_Y$ and evey integer z the map $H^0(\mathbf{P}^3, \mathcal{I}_{zP}(z-2)\oplus\mathcal{I}_{zP}(z)) \to H^0(Y, \mathcal{O}_Y(z))$ induced in cohomology by the map $(\beta, 1)$ is surjective.

These assertions are much weaker than the corresponding assertions in [3] because it is added a term "$-z$" in the number of components of Y which weight much more than the term associated to zP. Assertions B_z and B_z may be easily proved by induction on z using the hyperplane H and that $\mathrm{Res}_H(zP) = (z-1)P$ (see respectively [3], Lemma 7.2 and 7.2), adding at this step one line (resp. conic) less than in [3]. As in [3] the theorem follows from B_z and B_{z+1} (case c_1 even) and B'_z and B'_{z+1} (case c_1 odd), where z is respectively the maximal integer such that $[(z^2+2z+3)/3] \leq c_2 + 1 - c_1^2/4$ and $2[(z^2+z+6)/6] \leq c_2 + 2 - c_1^2/4 + 2$.

REFERENCES

1. Artamkin, I.V. – *Stable bundles with $c_1 = 0$ on rational surfaces,* Izv. Akad. Nauk SSSR Ser. Mat. 54 (1990), no. 2, pp. 227–241, translation in: Math. USSR-Izv. 36 (1991), no. 2, pp. 231–246.
2. Brun, J. – *Les fibrés de rang deux sur $\mathbb{P}_2$ et leurs sections,* Bull. Soc. Math. France 107 (1979), pp. 457–473.
3. Hartshorne, R.; Hirschowitz, A. – *Cohomology of a general instanton bundle,* Ann. Sient Éc. Norm. Sup. 15 (1982), pp. 365–390.
4. Hirschowitz, A.; Lazlo, Y. – *Fibrés générique sur le plan projectif,* Math. Ann. 297 (1993), no. 1, pp. 85–112.

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