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INDUCTIVE SYSTEMS OF VECTOR BUNDLES ON PROJECTIVE VARIETIES*

BY

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Abstract. Let $\{X_n\}_{n\geq 1}$ be a sequence of projective varieties with $X_n \subset X_{n+1}$ and E_n a vector bundle on X_n such that E_n is a subbundle of $E_{n+1}|X_n$. We allow the case $\lim_n \operatorname{rank}(E_n) = +\infty$. Here we study a few elementary cases (e.g. $X_n \subset \mathbf{P}^{n+k}$ smooth complete intersection with X_n hyperplane section of X_{n+1}) in which each E_n is a direct sum of line bundles.

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1. Introduction. The main aim of this note is the introduction of the following concept of inductive system of vector bundles (with increasing ranks) on an inductive system of algebraic varieties. Let $\{(X_n, i_n)\}_{n\geq 1}$ be a sequence of pairs such that each X_n is an integral variety and $i_n : X_n \to X_{n+1}$ is a closed embedding. We will say that $(X_n, i_n)_{n\geq 1}$ is an inductive sequence of varieties. Let $\{(E_n, j_n)\}_{n\geq 1}$ be a sequence of pair such that each E_n is a vector bundle on X_n and $j_n : E_n \to i_n^*(E_{n+1})$ is an inclusion of sheaves. We will say that $\{(E_n, j_n)\}_{n\geq 1}$ is an inductive system of locally free sheaves on the inductive sequence of varieties $\{(X_n, i_n)\}_{n\geq 1}$. If each j_n is an embedding of vector bundles, i.e. each coherent sheaf $\operatorname{Coker}(j_n)$ is locally free, then we will say that $\{(E_n, j_n)\}_{n\geq 1}$ is an inductive system of vector bundles. The sequence of integers $\{\operatorname{rank}(E_n)\}_{n\geq 1}$ will be called the rank sequence of $\{(E_n, j_n)\}_{n\geq 1}$. Since j_n is injective, $\operatorname{rank}(E_n) \leq \operatorname{rank}(E_{n+1}$ for

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every n. When each j_n is an isomorphism, $\{(E_n, j_n)\}_{n\geq 1}$ is just a vector bundle with finite rank in the sense of [6] on the inductive system $\{(X_n, i_n)\}_{n\geq 1}$.

Theorem 1. *Let $i_n : \mathbf{P}^{n+1} \to \mathbf{P}^{n+2}$ be any inclusion as a hyperplane and $\{(E_n, j_n)\}_{n\geq 1}$ an inductive system of vector bundles on $\{(\mathbf{P}^{n+1}, i_n)\}_{n\geq 1}$. Set $r_n := rank(E_n)$ and assume $r_1 = 1$ and $r_{n-1} \leq r_n \leq r_{n-1} + 1$ for all $n \geq 2$. Then each E_n is a direct sum of line bundles.*

Remark 1. Theorem 1 is not true if $r_n = n + 1$ for all n: just take $E_n = T\mathbf{P}^{n+1}(t)$, $t \in \mathbf{Z}$.

Theorem 2. *Let $Q_n \subset \mathbf{P}^{n+3}$ denote any smooth quadric hypersurface, $i_n : Q_n \to Q_{n+1}$ be any inclusion as a smooth hyperplane section and $\{(E_n, j_n)\}_{n\geq 1}$ an inductive system of vector bundles on $\{(\mathbf{P}^n, i_n)\}_{n\geq 1}$. Set $r_n := rank(E_n)$ and assume either $\sup_{n\geq 1}\{r_n\} < +\infty$ or $r_1 = 1$ and $r_n \leq r_{n+1} \leq r_n + 1$ for all $n \geq 1$. Then each E_n is a direct sum of line bundles.*

For all integers $v > r > 0$ let $G(r, v)$ denote the Grassmannian of all $(v-r)$-dimensional linear subspaces of the vector space $\mathbf{K}^{\oplus v}$. Hence $G(r, v)$ is a connected and projective smooth variety of dimension $r(v - r)$. Fix integers $b \geq a \geq 0$ such that $b > 0$. Fix linear subspaces E, W, B of $\mathbf{K}^{\oplus(v+b)}$ such that $A \cup W = E \cap (W + A) = \emptyset$, $\dim(A) = a$, $\dim(E) = b - a$ and $\dim(W) = v$. For any $(v - r)$-dimensional linear subspace M of W the linear subspace $M + A + E$ of $\mathbf{K}^{\oplus(v+b)}$ has codimension $r + a$. We obtain in this way an embedding $i_{[r,v;r+a,v+b]} : G(r, v) \to G(r + a, v + b)$ which will be called a linear embedding of $G(r, v)$ into $G(r + a, v + b)$. Up to a projective transformation of $G(r + a, v + b)$ the linear embedding $i_{[r,v;r+a,v+b]}$ does not depend from the choices of the linear subspaces E, W and B. In the next theorem we will use any of these projectively equivalent embeddings to see $G(a_n, b_n)$ as a closed subvariety of $G(a_{n+1}, b_{n+1})$.

Theorem 3. *Fix integers r_n, a_n, b_n such that $r_1 = 1$, $r_n \leq r_{n+1} \leq r_n$ for all $n \geq 1$, $a_{n+1} \geq a_n$, $b_{n+1} \geq b_n$, $(b_{n+1}, a_{n+1}) \neq (b_n, a_n)$, $a_n \geq 2r_n$, $b_n - a_n \geq 2r_n$. Let $i_n : G(a_n, b_n) \to G(a_{n+1}, b_{n+1})$ any linear embedding and $\{(E_n, j_n)\}_{n\geq 1}$ an inductive system of vector bundles on $\{G(a_n, b_n)_{n\geq 1}$ with rank$(E_n) = r_n$ for every n. $\lim_n a_n = \lim_n b_n - a_n = +\infty$. Assume either $\sup r_n < +\infty$ or $r_1 = 1$ and $r_n \leq r_{n+1} \leq r_n + 1$ for all $n \geq 1$. Then each E_n is isomorphic to a direct sum of line bundles.*

Theorem 4. *Fix smooth complete intersections $X_n \subset \mathbf{P}^{n+k}$, $n \geq 1$, $dim(X_1) \geq 3$, such that, seeing $\mathbf{P}^{n+k}$ as a hyperplane section of $\mathbf{P}^{n+k+1}$ the smooth variety X_n is a hyperplane section of X_{n+1}. Let $\{(E_n, j_n)\}_{n\geq 1}$ an inductive system of vector bundles on this inductive system of smooth varieties. Set $r_n := rank(E_n)$ and assume $r_1 = 1$, $r_n \leq r_{n+1} \leq r_n + 1$ for all $n \geq 1$. Then each E_n is isomorphic to a direct sum of line bundles.*

Remark 2. The conditions "$r_1 = 1$ and $r_n \leq r_{n+1} \leq r_n$ for all $n \geq 1$ " in Theorems 1, 2, 3 and 4 are equivalent to requiring $r_1 = 1$, that the sequence $\{r_n\}_{n\geq 1}$ is a non-decreasing sequence which is either with a maximal element or which contains all positive integers.

We work over an algebraically closed base field $\mathbf{K}$ such that char$(\mathbf{K}) = 0$. In the set-up of analytic geometry, our situation corresponds (roughly speaking, but a direct link may be done) to complex manifolds locally modelled over the locally convex topological vector space $\mathbf{C}^{\oplus \mathbf{N}}$.

2. The proofs.

Proof of Theorems 1,2 and 3 when $\sup r_n < +\infty$. Set $r := \sup_{n\geq 1} r_n$. In this case j_n is an isomorphism for $n \gg 0$, i.e., deleting at most finitely many varieties X_n in the inductive system $\{X_n\}_{n\geq 1}$, the inductive system $\{(E_n, j_n\}$ is a rank r vector bundle on $\{X_n\}_{n\geq 1}$ in the sense of [6]. See [6] or [5] for Theorem 1 and [5] for Theorem 3. For Theorem 2 use the method of [6] to show that each E_n is a uniform vector bundle and then apply [1], Th. 1.

Proof of Theorem 4 and Theorems 1, 2 and 3. For all these statements let $\{(X_n, i_n)\}_{n\geq 1}$ be the corresponding inductive system of varieties. For each $n \geq 1$ let $m(n)$ be the minimal integer such that $r_{m(n)} = n$. Set $Y_n := X_{m(n)}$, $F_n := E_{m(n)}$, $i'_n : 0 : i_{m(n)-1} \circ \cdots i_{m(n)}$ and $u'_n : 0 : j_{m(n)-1} \circ \cdots j_{m(n)}$. Hence rank$(F_n) = n$ for every n and Y_n is a linear section of Y_{n+1}.

Claim. For every line bundle L on X_n there is a unique integer t such that $L \cong \mathcal{O}_{X_n}(t)$. Furthermore, $h^1(X_n, L) = 0$.

Proof of the Claim. Everything is classical (e.g. the arithmetic Cohen - Macaulyness of the Grassmannians). For the cohomology groups of line bundles on complete intersections, see e.g. [2] or [4], Ex. II 8.4 and Ex. III.5.5).

F_1 is a line bundle. $u_1^*(F_2)$ is an extension of a line bundle by a line bundle. By the Claim $u_1^*(F_2)$ is a direct sum of line bundles. By [3], Cor. 1.2 (or its proof as explained in [1], Prop. 2), F_2 is a direct sum of two line bundles. In the same way we obtain that if F_n is a direct sum of n line bundles, then F_{n+1} is a direct sum of $+1n$ line bundles. Hence by induction on n we get that each F_n is a direct sum of line bundles. Since $\{(E_n, j_n)\}_{n\geq 1}$ is an inductive system of vector bundles, j_n is an isomorphism if $r_n = r_{n+1}$. Hence each E_n is a direct sum of n line bundles.

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