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ON THE GEOMETRY OF SEMI–RIEMANNIAN DISTRIBUTIONS

BY

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We would like to dedicate this paper to Academician Radu Miron on the occasion of his 75th birthday

Abstract. We prove the existence and uniqueness of a linear connection on a semi–Riemannian distribution which satisfies similar conditions as the Levi–Civita connection on a semi–Riemannian manifold. When the ambient manifold is semi–Riemannian too, we study its geometrical properties related to two orthogonal complementary distributions. Finally, we construct a non–holonomic manifold on which a bundle–like metric exists.

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Introduction. The theory of manifolds endowed with non–integrable distributions (non–holonomic manifolds) was introduced in the first half of last century by VRANCEANU [10] and it was successfully used for studying the geometry of non–holonomic mechanical systems. Since then there have been published a large amount of papers on both integrable and non–integrable distributions, the former class being now well known under the name of foliations. One of the main problems of the theory was to construct linear connections which are adapted to the geometry of distributions. In this respect there have been obtained several important results by S. Ianus [3], J.A. Schouten [8], I. Vaisman [9], Gh. Vranceanu [10], A.G. Walker [11], T.J. Willmore [12], etc.

In the present paper we consider a semi–Riemannian distribution $(\mathcal{D}, g)$ on a smooth manifold M and prove the existence and uniqueness of a linear connection D on $\mathcal{D}$ whose properties are very similar to those of the Levi–Civita connection on a semi–Riemannian manifold (Theorem 1.2). When $(\mathcal{D}, g)$ is a semi–Riemannian distribution on a semi–Riemannian manifold (M, g), the Levi–Civita connection on (M, g) induces a liner connection ∇ on $\mathcal{D}$. We obtain geometrical characterizations of semi–Riemannian distributions with $\nabla = D$ (Theorems 2.1, 2.3) and deduce the local feature of a semi–Riemannian manifold endowed with two orthogonal complementary distributions that satisfy the above condition (Corollary 2.2). Finally, we construct a non–holonomic manifold in sense of Vranceanu [10], on which a bundle–like metric exists.

1. The Levi–Civita connection on a semi–Riemannian distribution. Let M be a real $(m + n)$–dimensional paracompact differentiable manifold of class C^∞ endowed with a distribution $\mathcal{D}$ of rank m, that is, the fibres $\mathcal{D}_x$ of $\mathcal{D}$ are m-dimensional subspaces of T_xM for any $x \in M$. Denote by $L^2_s(\mathcal{D}_x, \mathbb{R})$ the real vector space of all symmetric bilinear mappings on $\mathcal{D}_x$. Then we say that $g_x \in L^2_s(\mathcal{D}_x, \mathbb{R})$ is non–degenerate if whenever $g_x(v, w) = 0$ for all $w \in \mathcal{D}_x$, we have $v = 0$. Let p be the dimension of the largest subspace W_x of $\mathcal{D}_x$ on which g_x is negative definite, i.e., $g_x(w, w) < 0$ for any non zero vector $w \in W_x$. Then we say that g_x is of index p.

Next, we consider the vector bundle

$$L^2_s(\mathcal{D}, \mathbb{R}) = \bigcup_{x \in M} L^2_s(\mathcal{D}_x, \mathbb{R}),$$

over M, and give the following definition. A **semi–Riemannian metric of index** p on $\mathcal{D}$ is a differentiable section $g : x \to g_x$ of $L^2_s(\mathcal{D}, \mathbb{R})$ such that each g_x is non–degenerate and of index p, for any $x \in M$. When $p = 0$ we say that g is a **Riemannian metric** on $\mathcal{D}$.

If in particular $\mathcal{D} = TM$ then g becomes a semi–Riemannian metric on M. In this case there exists a unique torsion–free linear connection $\widetilde{\nabla}$ on M such that g is parallel with respect to $\widetilde{\nabla}$. This is the well–known Levi–Civita connection on M (cf. O'Neill [5], p. 61). The purpose of this section is to prove the existence and uniqueness of a linear connection on $\mathcal{D}$ whose properties are very similar to those of the Levi–Civita connection on a semi–Riemannian manifold.

To this end we recall that Otsuki [6] has defined a generalization of a linear connection of a vector bundle as follows. Let E be a vector bundle over M and $P : E \to E$ be a vector bundle morphism of E. Denote by $\mathcal{F}(M)$

the algebra of smooth functions on M and by $\Gamma(E)$ the $\mathcal{F}(M)$-module of smooth sections of E. Then according to Abe [1], an **Otsuki connection** (**general connection**) on E with respect to the vector bundle morphism P is a mapping $\widetilde{\nabla}$ which assigns to any vector field X on M the differential operator

$$\widetilde{\nabla}_X : \Gamma(E) \longrightarrow \Gamma(E),\ S \longrightarrow \widetilde{\nabla}_X S,\ \forall\, S \in \Gamma(E),$$

satisfying the following conditions:

$$\widetilde{\nabla}_{fX+Y}(S) = f\widetilde{\nabla}_X S + \widetilde{\nabla}_Y S,$$

and

$$\widetilde{\nabla}_X(fS + S') = f\widetilde{\nabla}_X S + X(f)P(S) + \widetilde{\nabla}_X S',$$

for any $f \in \mathcal{F}(M)$, $X, Y \in \Gamma(TM)$ and $S, S' \in \Gamma(E)$. Clearly, when P is the identity morphism on E, then $\widetilde{\nabla}$ becomes a linear connection on E.

We note that contrary to the case of linear connections, for an Otsuki connection, the covariant derivative of the identity morphism I_E on E does not vanish identically on E. More precisely we have

$$(\widetilde{\nabla}_X I_E)(S) = \widetilde{\nabla}_X P(S) - P(\widetilde{\nabla}_X S),\ \forall\, X \in \Gamma(TM),\ S \in \Gamma(E).$$

The curvature form $\widetilde{\Omega}$ of $\widetilde{\nabla}$ is defined as follows (cf. Abe [1])

$$\begin{aligned}\widetilde{\Omega}(X,Y)S = \widetilde{\nabla}_X\widetilde{\nabla}_Y P(S) - \widetilde{\nabla}_Y\widetilde{\nabla}_X P(S) - P(\widetilde{\nabla}_{[X,Y]}P(S))\\ -(\widetilde{\nabla}_X I_E)(\widetilde{\nabla}_Y S) + (\widetilde{\nabla}_Y I_E)(\widetilde{\nabla}_X S),\end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $S \in \Gamma(E)$. For the particular case $E = TM$, an Otsuki connection $\widetilde{\nabla}$ has a torsion tensor field $\widetilde{T}$ given by (cf. Nemoto [4])

$$\widetilde{T}(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - P([X,Y]),\ \forall\, X, Y \in \Gamma(TM). \tag{1.1}$$

Finally, we show that starting with a linear connection ∇ on a vector bundle E we can always obtain an Otsuki connection $\widetilde{\nabla}$ on a vector bundle G which is larger than E such that $\widetilde{\nabla} = \nabla$ on E. Indeed, suppose $G = E \oplus F$, where F is another vector bundle over M and denote by P the projection morphism of G on E. Then for any $X \in \Gamma(TM)$ we define the differential operator

$$\widetilde{\nabla}_X : \Gamma(G) \longrightarrow \Gamma(G) :\ \widetilde{\nabla}_X S = \nabla_X PS,\ \forall\, S \in \Gamma(G). \tag{1.2}$$

It is easy to check that $\widetilde{\nabla}$ is an Otsuki connection on G with respect to the vector bundle morphism P. Moreover, the following has been proved.

Theorem 1.1. (Bejancu–Otsuki [2]) *The restriction of the curvature form $\widetilde{\Omega}$ of $\widetilde{\nabla}$ to the sections of E is equal to the curvature form Ω of ∇.*

Now, we apply the above theory to the study of the geometry of a semi–Riemannian distribution $(\mathcal{D}, g)$ on M. First, we consider a complementary distribution $\mathcal{D}'$ to $\mathcal{D}$ in TM, that is, we have

$$TM = \mathcal{D} \oplus \mathcal{D}'. \tag{1.3}$$

The existence of $\mathcal{D}'$ is guaranteed by the paracompactness of M.

We denote by Q and Q' the projection morphisms of TM on $\mathcal{D}$ and $\mathcal{D}'$ respectively. Then suppose that ∇ is a linear connection on $\mathcal{D}$, and consider the Otsuki connection $\widetilde{\nabla}$ on TM with respect to the decomposition (1.3), such that $\widetilde{\nabla} = \nabla$ on $\mathcal{D}$. Then according to (1.2) we have

$$\widetilde{\nabla}_X Y = \nabla_X QY;\ \forall\, X, Y \in \Gamma(TM). \tag{1.4}$$

Taking into account the relation between curvature forms of $\widetilde{\nabla}$ and ∇ stated in Theorem 1.1, we may define a torsion tensor field T of ∇ as the restriction of the torsion tensor field $\widetilde{T}$ of $\widetilde{\nabla}$ to $\Gamma(TM) \times \Gamma(\mathcal{D})$. Thus by using (1.1) and (1.4) we deduce that

$$T(X, QY) = \widetilde{T}(X, QY) = \nabla_X QY - \nabla_{QY} QX - Q[X, QY], \forall X, Y \in \Gamma(TM). \tag{1.5}$$

We note that the torsion tensor field of ∇ depends on the choice of the complementary distribution $\mathcal{D}'$.

Next, we say that ∇ is a $\mathcal{D}'$–**torsion–free linear connection** on $\mathcal{D}$ if its torsion tensor field vanishes identically on M. Also, we say that g is $\mathcal{D}$–**parallel** with respect to the linear connection ∇ on $\mathcal{D}$ if we have

$$\begin{aligned} (\nabla_{QX} g)(QY, QZ) = QX(g(QY, QZ)) - g(\nabla_{QX} QY, QZ) \\ - g(QY, \nabla_{QX} QZ) = 0,\ \forall\, X, Y, Z \in \Gamma(TM). \end{aligned} \tag{1.6}$$

Now we can state the following theorem.

Theorem 1.2. *Let $(\mathcal{D}, g)$ be a semi–Riemannian distribution on M and $\mathcal{D}'$ be a complementary distribution to $\mathcal{D}$ in TM. Then there exists a* *unique $\mathcal{D}'$–torsion–free linear connection D on $\mathcal{D}$ such that g is $\mathcal{D}$–parallel with respect to D.*

Proof. Define the differential operator $D : \Gamma(TM) \times \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D})$ by

$$\begin{aligned} 2g(D_{QX}QY, QZ) &= QX(g(QY, QZ)) + QY(g(QZ, QX)) \\ &- QZ(g(QX, QY)) + g(Q[QX, QY], QZ) \\ &- g(Q[QY, QZ], QX) + g(Q[QZ, QX], QY), \end{aligned} \tag{1.7}$$

and

$$D_{Q'X}QY = Q[Q'X, QY], \tag{1.8}$$

for any $X, Y \in \Gamma(TM)$. It is easy to verify that D given by (1.7) and (1.8) is a $\mathcal{D}'$-torsion-free linear connection on $\mathcal{D}$ and g is parallel with respect to D. Next, suppose that ∇ is another linear connection that satisfies the conditions from the theorem. Since ∇ is $\mathcal{D}'$-torsion-free, from (1.5) we deduce that

$$\nabla_{Q'X}QY = Q[Q'X, QY], \tag{1.9}$$

and

$$\nabla_{QX}QY - \nabla_{QY}QX - Q[QX, QY] = 0, \tag{1.10}$$

for any $X, Y \in \Gamma(TM)$. Now we use (1.6) and (1.10) and obtain

$$\begin{aligned} 0 &= (\nabla_{QX}g)(QY, QZ) + (\nabla_{QY}g)(QZ, QX) - (\nabla_{QZ}g)(QX, QY) \\ &= QX(g(QY, QZ)) + QY(g(QZ, QX)) - QZ(g(QX, QY)) \\ &+ g(Q[QX, QY], QZ) - g(Q[QY, QZ], QX) \\ &+ g(Q[QZ, QX], QY) - 2g(\nabla_{QX}QY, QZ). \end{aligned} \tag{1.11}$$

Finally, comparing (1.9) and (1.11) with (1.8) and (1.7) respectively, we conclude that $\nabla = D$, which proves the uniqueness of D. □

In particular, if $\mathcal{D} = TM$ then $\mathcal{D}' = \{0\}$ and the liner connection from Theorem 1.2 is just the Levi–Civita connection on M with respect to g. For this reason we call D given by (1.7) and (1.8) the **$\mathcal{D}'$-Levi–Civita connection** on the semi-Riemannian distribution $(\mathcal{D}, g)$.

2. Semi–Riemannian distributions on semi–Riemannian manifolds. Let (M, g) be an $(m+n)$-dimensional semi–Riemannian manifold, where g is a semi–Riemannian metric of index p (cf. O'Neill [5], p.54). Suppose $\mathcal{D}$ is a non–degenerate distribution of rank m on M, that is g induces on $\mathcal{D}$ a semi-Riemannian metric which we denote by the same symbol g. Hence $(\mathcal{D}, g)$ is a semi–Riemannian distribution on the semi–Riemannian manifold (M, g). Then by the theory we developed in the first section, for any complementary distribution $\mathcal{D}'$ to $\mathcal{D}$ in TM we have the $\mathcal{D}'$–Levi–Civita connection on $\mathcal{D}$. In the present case we take the complementary orthogonal distribution $\mathcal{D}^{\perp}$ to $\mathcal{D}$ in TM and obtain the $\mathcal{D}^{\perp}$–**Levi–Civita connection** on $\mathcal{D}$ which we denote by D. In a similar way, g induces a semi–Riemannian metric on $\mathcal{D}^{\perp}$ which we denote also by g. Thus $(\mathcal{D}^{\perp}, g)$ is a semi–Riemannian distribution and hence there exists a $\mathcal{D}$–**Levi–Civita connection** on $\mathcal{D}^{\perp}$ which we denote by $D^{\perp}$.

On the other hand, the Levi–Civita connection $\widetilde{\nabla}$ on (M, g) induces some linear connections on $\mathcal{D}$ and $\mathcal{D}^{\perp}$. Thus it is interesting to see if these connections coincide with the Levi–Civita connections on $(\mathcal{D}, g)$ and $(\mathcal{D}^{\perp}, g)$. We will show that, in general, they do not coincide, but when this happens the manifold must be locally a Riemannian product.

To this end we consider the decomposition

$$TM = \mathcal{D} \oplus \mathcal{D}^{\perp}, \tag{2.1}$$

and denote by Q and Q' the projection morphisms of TM on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. Then according to (2.1) we write

$$\widetilde{\nabla}_X QY = \nabla_X QY + B(X, QY), \tag{2.2}$$

and

$$\widetilde{\nabla}_X Q'Y = B'(X, Q'Y) + \nabla'_X Q'Y, \tag{2.3}$$

where we set

$$\text{(a)}\ \ \nabla_X QY = Q\widetilde{\nabla}_X QY, \qquad \text{(b)}\ \ \nabla'_X Q'Y = Q'\widetilde{\nabla}_X Q'Y, \tag{2.4}$$

and

$$\text{(a)}\ \ B(X, QY) = Q'\widetilde{\nabla}_X QY, \qquad \text{(b)}\ \ B'(X, Q'Y) = Q\widetilde{\nabla}_X Q'Y, \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$. It is easy to check that ∇ and ∇' are linear connections on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively, while B and B' are $\mathcal{F}(M)$–bilinear mappings:

$$B : \Gamma(TM) \times \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D}^{\perp}),\; B' : \Gamma(TM) \times \Gamma(\mathcal{D}^{\perp}) \longrightarrow \Gamma(\mathcal{D}).$$

We call ∇ and ∇' the **induced connections** by $\widetilde{\nabla}$ on$(\mathcal{D}, g)$ and $(\mathcal{D}^{\perp}, g)$ respectively. Also, we call

$$B : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D}^{\perp}) \text{ and } B' : \Gamma(\mathcal{D}^{\perp}) \times \Gamma(\mathcal{D}^{\perp}) \longrightarrow \Gamma(\mathcal{D})$$

given by

$$\text{(2.6)} \quad \text{(a) } B(QX, QY) = Q'\widetilde{\nabla}_{QX}QY \text{ and (b) } B'(Q'X, Q'Y) = Q\widetilde{\nabla}_{Q'X}Q'Y,$$

the **second fundamental forms** of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. Next, for any $Q'X \in \Gamma(\mathcal{D}^{\perp})$ and $QX \in \Gamma(\mathcal{D})$ we define the linear operators

$$A_{Q'X} : \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D}) \;\text{ and }\; A'_{QX} : \Gamma(\mathcal{D}^{\perp}) \longrightarrow \Gamma(\mathcal{D}^{\perp})$$

$$\text{(2.7)} \quad \text{(a) } A_{Q'X}QY = -\,B'(QY, Q'X) \text{ and (b) } A'_{QX}Q'Y = -\,B(Q'Y, QX).$$

According to the theory of submanifolds, we call $A_{Q'X}$ and A'_{QX} the **shape operators** of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ with respect to the normal sections $Q'X$ and QX respectively. Taking into account that g is parallel with respect to $\widetilde{\nabla}$ we deduce that the second fundamental forms and shape operators are related by

$$g(B(QX, QY), Q'Z) = g(A_{Q'Z}QX, QY), \tag{2.8}$$

and

$$g(B'(Q'X, Q'Y), QZ) = g(A'_{QZ}Q'X, Q'Y). \tag{2.9}$$

Finally, we write (2.2) and (2.3) by using second fundamental forms and shape operators as follows

$$\begin{aligned} &\text{(a)} \quad \widetilde{\nabla}_{QX}QY = \nabla_{QX}QY + B(QX, QY), \\ &\text{(b)} \quad \widetilde{\nabla}_{Q'X}QY = -A'_{QY}Q'X + \nabla_{Q'X}QY, \end{aligned} \tag{2.10}$$

$$\text{(2.11)}\qquad \begin{aligned} &\text{(a)}\quad \widetilde{\nabla}_{QX}Q'Y = -A_{Q'Y}QX + \nabla'_{QX}Q'Y,\\ &\text{(b)}\quad \widetilde{\nabla}_{Q'X}Q'Y = B'(Q'X, Q'Y) + \nabla'_{Q'X}Q'Y. \end{aligned}$$

Now, we prove the following.

Lemma 2.1. (i) *The semi–Riemannian metric g is parallel with respect to the induced connection ∇.*

(ii) *The torsion tensor field T of ∇ is given by*

$$\text{(2.12)}\qquad T(X, QY) = \nabla_{Q'X}QY - Q[Q'X, QY],\ \forall X, Y \in \Gamma(TM).$$

(iii) *$\mathcal{D}$ is an integrable distribution if and only if any one of the following two conditions is satisfied:*

(a) *The second fundamental form B of $\mathcal{D}$ is symmetric.*

(b) *The shape operator $A_{Q'Z}$ of $\mathcal{D}$ is symmetric with respect to g for any $Q'Z \in \Gamma(\mathcal{D}^{\perp})$.*

Proof. (i) As g is parallel with respect to $\widetilde{\nabla}$, we have

$$(\widetilde{\nabla}_X g)(QY, QZ) = 0,\ \forall X, Y, Z \in \Gamma(TM),$$

which via (2.4a) proves our assertion.

(ii) By direct calculations in (1.5) using (2.4a) and taking into account that $\widetilde{\nabla}$ is torsion–free, we obtain (2.12).

(iii) By using (2.5a) and taking into account again that $\widetilde{\nabla}$ is torsion–free we deduce that

$$B(QX, QY) - B(QY, QX) = Q'[QX, QY],\ \forall X, Y \in \Gamma(TM)$$

which proves that $\mathcal{D}$ is integrable if and only if the second fundamental form of $\mathcal{D}$ is symmetric.

The equivalence of (a) and (b) is a consequence of (2.8). □

The assertion (ii) states that, in general, the induced connection ∇ has non zero torsion tensor field, so it does not coincide with the $\mathcal{D}^{\perp}$–Levi–Civita connection D on $\mathcal{D}$. The following theorem sheds more light upon this issue.

Theorem 2.1. *Let $(\mathcal{D}, g)$ be a semi–Riemannian distribution on the semi–Riemannian manifold (M, g). Then the following assertions are equivalent:*

(i) *The induced connection* ∇ *on* $\mathcal{D}$ *coincides with the* $\mathcal{D}^{\perp}$*–Levi–Civita connection* D *on* $\mathcal{D}$.

(ii) *The second fundamental form of* $\mathcal{D}$ *vanishes identically on* M.

(iii) $\mathcal{D}$ *is integrable and its leaves are totally geodesic immersed in* (M, g).

Proof. By Theorem 1.2 we see that D is the only linear connection on $\mathcal{D}$ which is torsion–free and with respect to which g is $\mathcal{D}$–parallel. Taking into account that g is $\mathcal{D}$–parallel with respect to ∇ (cf. (i) of Lemma 2.1), and by using (2.12) and (2.4a) we deduce that $\nabla = D$ if and only if

$$Q\widetilde{\nabla}_{Q'X}QY = Q[Q'X, QY], \; \forall\, X, Y \in \Gamma(TM). \tag{2.13}$$

Next, we recall that $\widetilde{\nabla}$ is given by (cf. O'Neill [5], p. 61)

$$\begin{aligned} 2g(\widetilde{\nabla}_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &+ g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y), \; \forall\, X, Y, Z \in \Gamma(TM). \end{aligned} \tag{2.14}$$

By using (2.14) we compute $2g(Q\widetilde{\nabla}_{Q'X}QY, QZ)$ and infer that (2.13) is equivalent to

$$\begin{aligned} &Q'X(g(QY, QZ)) - g([Q'X, QY], QZ) - g([QY, QZ], Q'X) \\ &\quad + g([QZ, Q'X], QY) = 0, \; \forall\, X, Y, Z \in \Gamma(TM). \end{aligned} \tag{2.15}$$

Taking into account that $\widetilde{\nabla}$ is torsion–free and g is parallel with respect to $\widetilde{\nabla}$, we infer that (2.15) is equivalent to

$$g(\widetilde{\nabla}_{QY}QZ, Q'X) = 0, \; \forall\, X, Y, Z \in \Gamma(TM). \tag{2.16}$$

Next, by using (2.5a) we obtain from (2.16) that $\nabla = D$ if and only if

$$B(QY, QZ) = 0, \; \forall\, Y, Z \in \Gamma(TM), \tag{2.17}$$

which proves the equivalence of (i) and (ii). Finally, from (2.10a), by using the assertion (iii) a of Lemma 2.1 we deduce that $\mathcal{D}$ is integrable and its leaves are totally geodesic immersed in (M, g) if and only if (2.17) is satisfied. This proves the equivalence of (ii) and (iii), and thus completes the proof of the theorem. □

Clearly, both Lemma 2.1 and Theorem 2.1 are true for $\mathcal{D}^{\perp}$ too. Then applying Theorem 2.1 for both $\mathcal{D}$ and $\mathcal{D}^{\perp}$ we obtain the following

Corollary 2.1. *The induced connections ∇ and ∇' coincide with the Levi–Civita connections D and $D^{\perp}$ on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively, if and only if, both distributions are integrable and M is locally a Riemannian product of leaves of $\mathcal{D}$ and $\mathcal{D}^{\perp}$.*

As in general the second fundamental form B of $\mathcal{D}$ is not symmetric we define the **symmetric second fundamental form**
H by

$$H(QX,QY) = \frac{1}{2}(B(QX,QY) + B(QY,QX)),\ \forall\, X,Y,Z \in \Gamma(TM). \tag{2.18}$$

Also, we say that a vector field **X** is a $\mathcal{D}$–**Killing vector field**, if

$$\begin{aligned}(L_X g)(QY,QZ) = g(\widetilde{\nabla}_{QY}X,QZ) + g(\widetilde{\nabla}_{QZ}X,QY) = 0,\\ \forall\, X,Y,Z \in \Gamma(TM).\end{aligned} \tag{2.19}$$

where L is Lie derivative on M.

Now we remark that, in general, g is not parallel with respect to any of the linear connections D and $D^{\perp}$. More precisely we have

Theorem 2.2. *Let $(\mathcal{D},g)$ be a semi–Riemannian distribution on the semi–Riemannian manifold (M,g). Then the following assertions are equivalent:*

(i) *g is parallel with respect to the $\mathcal{D}^{\perp}$–Levi–Civita connection D on $\mathcal{D}$.*

(ii) *Any $Q'X \in \Gamma(\mathcal{D}^{\perp})$ is a $\mathcal{D}$–Killing vector field.*

(iii) *The symmetric second fundamental form of $\mathcal{D}$ vanishes identically on M.*

Proof. First we note that g is parallel with respect to D if and only if

$$(D_{Q'X}g)(QY,QZ) = 0,\ \forall\, X,Y,Z \in \Gamma(TM), \tag{2.20}$$

since g is $\mathcal{D}$–parallel with respect to D. Then by using (1.8) and the properties of the Levi–Civita connection $\widetilde{\nabla}$ on M we deduce that (2.20) is equivalent to

$$\begin{aligned}0 &= Q'X(g(QY,QZ)) - g([Q'X,QY],QZ) - g(QY,[Q'X,QZ])\\ &= g(\widetilde{\nabla}_{Q'X}QY,QZ) + g(QY,\widetilde{\nabla}_{Q'X}QZ) - g(\widetilde{\nabla}_{Q'X}QY,QZ)\\ &\quad + g(\widetilde{\nabla}_{QY}Q'X,QZ) - g(QY,\widetilde{\nabla}_{Q'X}QZ) + g(QY,\widetilde{\nabla}_{QZ}Q'X)\\ &= g(\widetilde{\nabla}_{QY}Q'X,QZ) + g(QY,\widetilde{\nabla}_{QZ}Q'X),\ \forall\, X,Y,Z \in \Gamma(TM).\end{aligned} \tag{2.21}$$

Thus by (2.19) we obtain the equivalence of (i) and (ii). Finally, by using (2.6a) and (2.18) we infer that (2.21) is equivalent to

$$0 = g(Q'X, \widetilde{\nabla}_{QY}QZ + \widetilde{\nabla}_{QZ}QY) = 2g(Q'X, H(QY, QZ)),$$

which completes the proof of the theorem. □

So far we have obtained characterizations of two important classes of distributions. More precisely we have investigated a semi–Riemannian distribution $(\mathcal{D}, g)$ for which $\nabla = D$ or g is parallel with respect to D. These two conditions can be related as follows.

Theorem 2.3. *Let $(\mathcal{D}, g)$ be a semi–Riemannian distribution on the semi–Riemannian manifold (M, g). Then the following assertions are equivalent:*

(i) *The induced connection ∇ on $\mathcal{D}$ coincides with the $\mathcal{D}^{\perp}$–Levi–Civita connection D on $\mathcal{D}$.*

(ii) *The induced connection ∇ on $\mathcal{D}$ is torsion–free.*

(iii) *$\mathcal{D}$ is integrable and g is parallel with respect to D.*

Proof. (i) $\Longrightarrow$ (ii). As D is torsion–free, it follows that ∇ is torsion–free too. (ii) $\Longrightarrow$ (i). Since ∇ is torsion–free and g is $\mathcal{D}$–parallel with respect to ∇ (cf. (i) of Lemma 2.1), by the uniqueness of D stated in Theorem 1.2 we obtain $\nabla = D$. (i) $\Longleftrightarrow$ (iii). By assertion (iii)a of Lemma 2.1 and assertion (iii) of Theorem 2.2 we deduce that our assertion (iii) holds if and only if the second fundamental form of $\mathcal{D}$ vanishes identically on M. Then apply Theorem 2.1 and obtain the equivalence of (i) and (iii). □

Taking into account that all the above results are also true for the semi–Riemannian distribution $(\mathcal{D}^{\perp}, g)$ and using Corollary 2.1 and Theorem 2.3 we state the following

Corollary 2.2. *Let $(\mathcal{D}, g)$ and $(\mathcal{D}^{\perp}, g)$ be two complementary orthogonal semi–Riemannian distributions on the semi–Riemannian manifold (M, g). Then the following assertions are equivalent:*

(i) *Both distributions are integrable and M is locally a Riemannian product of their leaves.*

(ii) *The induced connections ∇ and ∇' coincide with the Levi–Civita connections D and $D^{\perp}$ on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively.*

(iii) *Both induced connections are torsion–free.*

(iv) *Both distributions are integrable and g is parallel with respect to both linear connections D and $D^{\perp}$.*

From Theorem 2.3 we see that the condition $\nabla = D$ is stronger than the condition for g being parallel with respect to D. The latter condition was first introduced by Reinhart [7] for foliations on manifolds, that is, $\mathcal{D}^{\perp}$ is supposed to be an integrable distribution. A Riemannian metric g satisfying this condition on a foliated manifold was called a **bundle–like metric** and was intensively studied by several authors. It is interesting to see whether bundle–like metrics can be found on manifolds endowed with two non–integrable distributions (non–holonomic manifolds in sense of Vranceanu [10]).

The next example shows that the answer is in the affirmative.

Example. Let $(\mathbb{R}^4, g)$ be the Euclidean space with g given by

$$g(X,Y) = \sum_{i=1}^{4} X^i Y^i.$$

Take the rectangular coordinate system (x^1, x^2, x^3, x^4) on $\mathbb{R}^4$ and define the open submanifold M of $\mathbb{R}^4$ by

$$M = \left\{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : 2x^3 - (x^1)^2 > 0\right\}.$$

Then on the Riemannian manifold (M, g) we consider the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ spanned by

$$\left\{ \begin{array}{l} X_1 = \dfrac{\partial}{\partial x^1} + \sqrt{2x^3 - (x^1)^2}\,\dfrac{\partial}{\partial x^2} + x^1\,\dfrac{\partial}{\partial x^3} \\[2ex] X_2 = \dfrac{\partial}{\partial x^4} + x^1\,\dfrac{\partial}{\partial x^2} - \sqrt{2x^3 - (x^1)^2}\,\dfrac{\partial}{\partial x^3} \end{array} \right\},$$

and

$$\left\{ \begin{array}{l} Y_1 = \dfrac{\partial}{\partial x^2} - \sqrt{2x^3 - (x^1)^2}\,\dfrac{\partial}{\partial x^1} - x^1\,\dfrac{\partial}{\partial x^4} \\[2ex] Y_2 = \dfrac{\partial}{\partial x^3} - x^1\,\dfrac{\partial}{\partial x^1} + \sqrt{2x^3 - (x^1)^2}\,\dfrac{\partial}{\partial x^4} \end{array} \right\},$$

respectively. It is easy to see that $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are complementary orthogonal distributions on (M, g). Moreover, both are not involutive, so by Frobenius theorem they are not integrable. However we show that g is parallel with respect to the $D^\perp$–Levi–Civita connection D on $\mathcal{D}$. To this end we first note that g is $\mathcal{D}$–parallel with respect to D (by the definition of D) and therefore we should verify only (2.20). Taking into account that $\{X_1, X_2\}$ is an orthogonal basis in $\Gamma(\mathcal{D})$, from the first equality in (2.21) we deduce that g is parallel with respect to D if and only if

$$g([Y_i, X_1], X_2) + g([Y_i, X_2], X_1) = 0,\ i \in \{1, 2\}.$$

By direct calculations it follows that these equalities are satisfied and hence g is a bundle–like metric on M. Thus we constructed an example of a non–holonomic manifold $(M, g, \mathcal{D}, \mathcal{D}^\perp)$ (i.e. both $\mathcal{D}$ and $\mathcal{D}^\perp$ are non integrable distributions), on which g is a bundle–like metric.

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