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SOME REMARKS ON AN INTEGRAL INEQUALITY

BY

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Abstract. The integral equation

$$u(x) = \alpha(x) + a(x)\int_0^x b(s)u(s)ds + a_1(x)\int_0^A c(s)u(s)ds \tag{1}$$

is solved under the hypothesis that the functions α, a, a_1, b, c are continuous on $[0, A]$. Under some additional conditions, is proved that from

$$f(x) \leq \alpha(x) + a(x)\int_0^x b(s)f(s) + a_1(x)\int_0^A c(s)f(s)ds \tag{2}$$

it follows $f(x) \leq u(x)$, where $u(x)$ is the solution of (1).

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In [1], p. 11 the following linear integral inequality is considered

$$f(x) \leq \alpha + \int_0^x b(s) f(s)\, ds + \int_0^A c(s) f(s)\, ds, \quad 0 \leq x \leq A \tag{1}$$

where f is continuous, b and c are continuous nonnegative functions on $[0, A]$ and α is a constant. Under the additional assumption that

$$q = \int_0^A c(s) \exp\left(\int_0^s b(\tau)\, d\tau\right) ds < 1,$$

it is proved that

$$f(x) \leq \frac{\alpha}{1-q}\, e^{\int_0^x b(s)ds}, \quad 0 \leq x \leq A. \tag{2}$$

Because the upper bound in (2) is the solution of the corresponding integral equation

$$u(x)=\alpha+\int_0^x b(s)\,u(s)\,ds+\int_0^A c(s)\,u(s)\,ds,\quad 0\le x\le A, \tag{3}$$

we can ask when is it possible to obtain $f(x)\le u(x)$ on $[0,A]$, where $u=u(x)$ is the solution of equation (3). The following example shows that, in general, the assertion $f(x)\le u(x)$ is false, even in the case when the functions f, b, c and the constant α are nonnegative. By taking $f(x)=e^{2x}$, $b(x)=2$ and $c(x)=2x$, the inequality (1) is equivalent with $0\le\alpha-1/2+(A-1/2)\,e^{2A}$, which is true if we suppose $\alpha\ge 1/2$, $A\ge 1/2$. But the solution of equation (3) is $u(x)=\left(\alpha/\left(1/2-(A-1/2)\,e^{2A}\right)\right)e^{2x}$, which is negative on $[0,A]$ if A is sufficiently large.

In what follows, we find sufficient conditions to obtain, for a function which satisfies a more general integral inequality than (1), an upper bound given by the solution of the corresponding integral equation. Throughout this paper, it is supposed that the continuous functions a, a_1, b, c are not identically zero. We will use the notation $C[0,A]$ for the set of all continuous functions defined on $[0,A]$.

We consider the inequality

$$\begin{aligned} f(x)\le{}&\alpha(x)+a(x)\int_0^x b(s)\,f(s)\,ds+\\ &+a_1(x)\int_0^A c(s)\,f(s)\,ds,\quad 0\le x\le A \end{aligned} \tag{4}$$

and the corresponding integral equation

$$\begin{aligned} u(x)={}&\alpha(x)+a(x)\int_0^x b(s)\,u(s)\,ds+\\ &+a_1(x)\int_0^A c(s)\,u(s)\,ds,\quad 0\le x\le A \end{aligned} \tag{5}$$

for which we search a continuous solution $u=u(x)$. Denoting $\int_0^A c(s)\,u(s)\,ds=m=\text{const}$, it follows that u must satisfy an integral equation of Volttera type, of the form

$$u(x)=\alpha(x)+ma_1(x)+a(x)\int_0^x b(s)\,u(s)\,ds,\quad 0\le x\le A \tag{6}$$

where the constant m must be determined. Using the resolvent kernel $r(x,s) = a(x)\,b(s)\exp\left(\int_s^x a(\tau)\,b(\tau)\,d\tau\right)$, we obtain that

$$u(x) = \alpha(x) + ma_1(x) + a(x)\int_0^x b(s)\,e^{\int_s^x a(\tau)b(\tau)d\tau}\left(\alpha(s) + ma_1(s)\right)ds \tag{7}$$

for $0 \le x \le A$. Hence, we have $u(x) = \varphi(x) + m\psi(x)$, where

$$\begin{aligned} \varphi(x) &= \alpha(x) + a(x)\int_0^x \alpha(s)\,b(s)\,e^{\int_s^x a(\tau)b(\tau)d\tau}ds, \\ \psi(x) &= a_1(x) + a(x)\int_0^x a_1(s)\,b(s)\,e^{\int_s^x a(\tau)b(\tau)d\tau}ds \end{aligned} \tag{8}$$

for $0 \le x \le A$. Putting $u(x) = \varphi(x) + m\psi(x)$ in (5), it follows that

$$\begin{aligned} &m\left(\psi(x) - a(x)\int_0^x b(s)\,\psi(s)\,ds - a_1(x)\int_0^A c(s)\,\psi(s)\,ds\right) = \\ &= \alpha(x) - \varphi(x) + a(x)\int_0^x b(s)\varphi(s)ds + \\ &+ a_1(x)\int_0^A c(s)\varphi(s)ds, \quad 0 \le x \le A. \end{aligned} \tag{9}$$

Using the formulas (8), we find that the coefficient of m is given by

$$\begin{aligned} &a_1(x) + a(x)\int_0^x a_1(s)\,b(s)\,e^{\int_0^x a(\tau)b(\tau)d\tau}ds - a(x)\int_0^x a_1(s)\,b(s)\,ds - \\ &- a(x)\int_0^x a(s)\,b(s)\left(\int_0^s a_1(\tau)\,b(\tau)\,e^{\int_\tau^s a(\sigma)b(\sigma)d\sigma}d\tau\right)ds - \\ &- a_1(x)\int_0^A c(s)\,\psi(s)\,ds = a_1(x)\left(1 - \int_0^A c(s)\,\psi(s)\,ds\right) \end{aligned} \tag{10}$$

because the sum of the other three terms in (10) is identically zero. The r. h. s. of (9) is given by

$$\begin{aligned} &-a(x)\int_0^x \alpha(s)\,b(s)\,e^{\int_s^x a(\tau)b(\tau)d\tau}ds + a(x)\int_0^x \alpha(s)\,b(s)\,ds + \\ &+ a(x)\int_0^x a(s)\,b(s)\left(\int_0^s \alpha(\tau)\,b(\tau)\,e^{\int_\tau^s a(\sigma)b(\sigma)d\sigma}d\tau\right)ds + \\ &+ a_1\int_0^A c(s)\,\varphi(s)\,ds = a_1(x)\int_0^A c(s)\,\varphi(s)\,ds, \quad 0 \le x \le A \end{aligned} \tag{11}$$

because the sum of the first three terms in (11) is identically zero. This last assertion is proved by changing the order of integration in the double integral appearing in (11). Hence, we can write

$$m\left(1-\int_0^A c(s)\,\psi(s)\,ds\right)=\int_0^A c(s)\,\varphi(s)\,ds \tag{12}$$

and state the following

Proposition 1. *Assume that the functions a, b, c are continuous and non identically zero on $[0,A]$, α is continuous on $[0,A]$ and φ, ψ are defined by (8). Then*

(*i*) *if $1-\int_0^A c(s)\,\psi(s)\,ds\neq 0$, the equation (5) has a unique continuous solution, namely*

$$u(x)=\varphi(x)+\frac{\int_0^A c(s)\,\varphi(s)\,ds}{1-\int_0^A c(s)\,\psi(s)\,ds}\,\psi(x),\quad 0\leq x\leq A. \tag{13}$$

(*ii*) *if $1-\int_0^A c(s)\,\psi(s)\,ds=0$ and $\int_0^A c(s)\,\varphi(s)\,ds\neq 0$, there exists no solution of (5).*

(*iii*) *if $1-\int_0^A c(s)\,\psi(s)\,ds=\int_0^A c(s)\,\varphi(s)\,ds=0$, all functions $u(x)=\varphi(x)+m\psi(x)$, where m is an arbitrary constant, are solutions of (5).*

We consider some examples (particular cases) of equations of the form (5).

Example 1. Suppose $\alpha=\text{const}$, $a(x)=a_1(x)=1$. In this case we have $\varphi(x)=\alpha\exp\left(\int_0^x b(s)\,ds\right)$, $\psi(x)=\exp\left(\int_0^x b(s)\,ds\right)$. Putting

$$\int_0^A c(s)\exp\left(\int_0^s b(\tau)\,d\tau\right)ds=q\neq 1$$

we find, for equation (3), the solution

$$u(x)=\frac{\alpha}{1-q}\,e^{\int_0^x b(s)ds},\quad 0\leq x\leq A. \tag{14}$$

Example 2. Suppose that, in (5), $\alpha = \text{const}$ and $a_1(x) = 1$. Then we have

$$\begin{aligned} &\varphi(x) = \alpha\psi(x), \\ &\psi(x) = 1 + a(x)\int_0^x b(s)\, e^{\int_s^x a(\tau)b(\tau)d\tau} ds, \quad 0 \le x \le A \end{aligned} \tag{15}$$

and the unique solution u is given by

$$u(x) = \frac{\alpha}{1 - \int_0^A c(s)\,\psi(s)\,ds}\,\psi(x), \quad 0 \le x \le A \tag{16}$$

where, of course, it has been supposed that the denominator is $\neq 0$.

Example 3. Suppose that $\alpha = \text{const}$, $a(x) = 1$, $a_1(x) \neq 1$. This time, we have

$$\begin{aligned} &\varphi(x) = \alpha e^{\int_0^x b(s)ds}, \\ &\psi(x) = a_1(x) + \int_0^x a_1(s)\, b(s)\, e^{\int_s^x b(\tau)d\tau}, \quad 0 \le x \le A \end{aligned} \tag{17}$$

and the unique solution of (5) is given by

$$u(x) = \alpha e^{\int_0^x b(s)ds} + \frac{\alpha\int_0^A c(s)\, e^{\int_0^s b(\tau)d\tau} ds}{1 - \int_0^A c(s)\,\psi(s)\,ds}\,\psi(x), \quad 0 \le x \le A \tag{18}$$

if the denominator is $\neq 0$.

To obtain a bound for a function f satisfying the inequality (4), we need to consider a class of operators acting on $C[0, A]$.

Definition. (**Property (P)**) *We say that an operator T acting on $C[0, A]$ has the property (P) if there exists a metric ρ on $C[0, A]$ such that, for every fixed continuous function g defined on $[0, A]$, the operator T_g given by $T_g u = g + Tu$, $\forall u \in C[0, A]$ is a contraction with respect to the metric ρ.*

Note that T, having the property (P), is a contraction with respect to the metric ρ. For the metric ρ we can take, for example, $\rho(u, v) = \sup_{[0,A]} |u(x) - v(x)|$, $\forall u, v \in C[0, A]$ or a metric of Bielecki type, given by

$\rho(u,v) = \sup_{[0,A]} |u(x)-v(x)| \exp\left(-2\alpha\int_0^x \lambda(s)\,ds\right)$, $\forall u,v \in C[0,A]$, where $\alpha > 1$ and $\lambda(x) \geq 0$ is a fixed continuous function. This last metric was used in [2], in connection with an integral inequality, too.

We denote by T the operator defined by

$$\begin{aligned}(Tu)(x) &= a(x)\int_0^x b(s)\,u(s)\,ds + \\ &+ a_1(x)\int_0^A c(s)\,u(s)\,ds, \quad 0 \leq x \leq A\end{aligned} \tag{19}$$

$\forall u \in C[0,A]$ and we are going to prove the following

Proposition 2. *Assume that the continuous function f satisfies the inequality* (4)*, where a, a_1, b, c are continuous nonnegative functions and α is a continuous function on $[0,A]$. Assume, also, that the operator T defined by* (19) *has the property* (P) *and that the equation* (5) *has a unique solution u, defined by* (13)*. Then we have $f(x) \leq u(x)$ on $[0,A]$, i. e.*

$$f(x) \leq \varphi(x) + \frac{\int_0^A c(s)\,\varphi(s)\,ds}{1-\int_0^A c(s)\,\psi(s)\,ds}\,\psi(x), \quad 0 \leq x \leq A \tag{20}$$

where φ and ψ are given by (8)*.*

Proof. The function f is the unique solution of the equation

$$\begin{aligned}f(x) &= \alpha(x) - \alpha_1(x) + a(x)\int_0^x b(s)\,f(s)\,ds + \\ &+ a_1(x)\int_0^A c(s)\,f(s)\,ds, \quad 0 \leq x \leq A\end{aligned} \tag{21}$$

where $\alpha_1(x) \geq 0$ is a continuous function. Denoting by T_α the operator defined by $T_\alpha v = \alpha + Tv$, $\forall v \in C[0,A]$, and by g the function given by $g(x) = \alpha(x) - \alpha_1(x)$, the equations (21) and (5) may be rewritten under the form $f = T_g f$ and respectively $u = T_\alpha u$. Since the operator T has the property (P), both solutions f and u can be obtained by the method of successive approximations, starting with initial functions $f_0(x) = \alpha(x) - \alpha_1(x)$, $u_0(x) = \alpha(x)$ and putting $f_{n+1} = T_g f_n$, $u_{n+1} = T_\alpha u_n$ for $n \geq 0$. Because $f_0(x) \leq u_0(x)$ on $[0,A]$, it follows that $f_1(x) \leq u_1(x)$ and,

in general, $f_n(x) \leq u_n(x)$ on $[0, A]$ for all $n \geq 0$. Taking $n \to \infty$, we obtain $f(x) \leq u(x)$ on $[0, A]$. □

Remark. The condition that the operator T has the property (P) may be replaced by condition that the (unique) solution of the equation $u = T_g u$ can be obtained by the method of successive approximations.

The method here presented remains also applicable in the case of functions depending of several independent variables. For example, in the case of two independent variables, we use the notations: $x = (x_1, x_2)$, $dx = dx_1 dx_2$, $\int_0^x f(t)\, dt = \int_0^{x_1} \int_0^{x_2} f(t_1, t_2)\, dt_1 dt_2$, $x \leq y \Leftrightarrow x_1 \leq y_1$, $x_2 \leq y_2$.

REFERENCES

1. Bainov, D.; Simeonov, P. – *Integral Inequalities and Applications,* Kluwer Academic Publishers, Volume 57, 1992, pp. 11-12.
2. Corduneanu, A. – *An Integral Inequality and the Corresponding Integral Equation* (to appear).
3. Mureşan, V. – *Functional - Integral Equations,* Editura Mediamira, Cluj-Napoca, 2003, pp. 115-175.
4. Pachpatte, B.G. – *On an Inequality Applicable to General Volttera-Fredholm Integral Equation,* An. şt. Univ. "Al. I. Cuza" - Iaşi, tom XLIX, s. I a, Matematică, 2003, f.2, pp. 221-228.

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