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A MATHEMATICAL STUDY OF THE SPATIAL BEHAVIOUR OF THE TIME-HARMONIC OSCILLATIONS IN A THERMOELASTIC RECTANGULAR PLATE

BY

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Abstract. The present paper is concerned with the linear thermoelastic plate model based on a Mindlin-type assumption on the displacements. The bending of a Mindlin-type thermoelastic rectangular (semi-infinite) plate is studied when the boundary end data are time-harmonic with angular frequency ω and the lateral sides are clamped and thermally insulated and sufficient time has elapsed for the system to have reached a steady-state. A line-integral cross-sectional measure is associated with the amplitude of the resulting harmonic oscillation and then a first-order differential inequality is established in terms of the measure, provided that the angular frequency ω is lower than of an explicit critical frequency ω_m. An integration leads to a spatial estimate describing the spatial exponential decay of the amplitude with the distance to the excited end.

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1. Introduction. Theories of plates with transverse shear deformation are an improvement on Kirchhoff's classical model in the sense that they yield more information about the physical quantities involved in the process of bending. Some reasons for taking into account the shear deformability and rotational inertia are given by Mindlin [1] and Reissner [2] (see also Constanda [3]). Such theories take into consideration the effects of transverse shear forces and so allow for the physically expected three boundary conditions instead of only two satisfied by the unknown deflection.

On the other hand, the thermal effects in a plate can contribute to plate deformation causing changes in stiffness, frequency of vibration and,

in some cases, buckling. Such features become increasingly important in the present state of powerful computers, stringent demands of accuracy and design sophistication.

In this connexion recently Schiavone and Tait [4] have derived a linear thermoelastic plate model based on a Mindlin-type assumption on the displacements. The bending of a Mindlin-type thermoelastic plate has been examined by Schiavone and Tait [5] when the source terms are time-harmonic with angular frequency ω, and sufficient time has elapsed for the system to have reached a steady-state. By formulating appropriate radiation conditions they establish some uniqueness results for exterior boundary value problems subject to certain regularity assumptions and a condition on the angular frequency of oscillation.

Some appropriate time-weighted line-integral measures are used by D'Apice and Chirita [6] in order to study the spatial behaviour of the transient solutions in the bending of a Mindlin-type thermoelastic plate. On the other hand, the steady time-harmonic oscillations are an important state in their own right, frequently occurring in practical applications, and they are also very significant in the study of full dynamic problems [7] (see also [8] for a complete view on the wave propagation).

In the present paper we study the spatial behaviour of steady-state solutions of thermoelastic plate bending in the linear thermoelastic thin plate model developed in [4]. We consider a thermoelastic (semi-infinite) plate of a rectangular form and we assume that one end of plate is subjected to boundary data which are time-harmonic with angular frequency ω, while the lateral sides (and the other end) are clamped and thermally insulated. We assume that sufficient time has elapsed for the system to have reached a steady-state. We are interested into describing the spatial behaviour of the resulting oscillations within the plate. In this aim we associate with the amplitude of the oscillation a line-integral cross-sectional measure and then, provided the frequency of vibration is lower than a critical value, we establish a first-order differential inequality in terms of the measure in question. By an integration, we establish a spatial estimate describing the spatial exponential decay of the amplitude.

2. Basic formulation. Steady time- harmonic oscillations. Throughout this paper Greek and Latin subscripts take the values $1, 2$, and $1, 2, 3$, respectively, summation is carried out over repeated indices, $x = (x_1, x_2)$ and $x = (x_1, x_2, x_3)$ are generic points referred to orthogonal

Cartesian coordinates in R^2 and R^3, respectively. A superposed dot denotes $\frac{\partial}{\partial t}$ and the suffix ", k" denotes $\frac{\partial}{\partial x_k}$.

Let $\overline{S}\times\left[-\frac{h_0}{2},\frac{h_0}{2}\right]$ be the region occupied by a homogeneous thin elastic plate, where S is a domain in R^2 bounded by a simple closed C^2-curve ∂S and $0 < h_0 = const << diam\ S$ is the thickness. We assume that, in addition to mechanical loads, the plate is subject to an unknown temperature distribution $\tau(x_1, x_2, x_3, t)$ (t is the time variable) measured from a reference state of uniform temperature distribution τ_0, at which temperature the plate is free from thermal stresses and strains. It is further assumed that the plate is elastically and thermally isotropic.

In the absence of the body forces and heat sources, the field equations for the bending of a Mindlin-type thermoelastic plate are [4]

$$h^2(\lambda+\mu)u_{\rho,\rho\alpha}+\mu(h^2u_{\alpha,\rho\rho}-u_\alpha-u_{3,\alpha})-u_{4,\alpha}=\rho h^2\ddot{u}_\alpha, \tag{2.1}$$

$$\mu(u_{3,\rho\rho}+u_{\rho,\rho})=\rho\ddot{u}_3, \tag{2.2}$$

$$u_{4,\rho\rho}-\frac{1}{K}\dot{u}_4-\eta\alpha h^2(3\lambda+2\mu)\dot{u}_{\rho,\rho}=0. \tag{2.3}$$

Here we have

$$\begin{aligned} &u_\alpha=u_\alpha(x_\rho,t),\quad u_3=u_3(x_\rho,t),\\ &u_4=u_4(x_\rho,t)=\frac{3\lambda+2\mu}{h_0}\int_{-\frac{h_0}{2}}^{\frac{h_0}{2}}x_3\varepsilon_\tau dx_3\ ,\quad h^2=\frac{h_0^2}{12}, \end{aligned} \tag{2.4}$$

λ and μ are the Lamé constants, ε_τ denotes the thermal strain, α is the coefficient of thermal expansion,

$$\eta=\frac{(3\lambda+2\mu)\alpha\tau_0}{\lambda_0},\quad K=\frac{\lambda_0}{\rho c}, \tag{2.5}$$

ρ and c are, respectively, the mass density and specific heat of the plate, and $\lambda_0 > 0$ is the (constant) coefficient of thermal conductivity. In accordance with the plate assumptions [4] u_α and u_3 characterize displacement and u_4 the resultant "thermal moment" on the plate's middle surface. In what follows we assume that

$$\rho>0,\ c>0,\ \alpha>0,\ \mu>0,\ 3\lambda+2\mu>0,\ \tau_0>0. \tag{2.6}$$

We associate with the field equations (2.1) to (2.3) the following initial conditions

$$
\begin{aligned}
&u_i(x_\rho,0) = u_i^0(x_\rho), \quad \dot{u}_i(x_\rho,0) = \dot{u}_i^0(x_\rho),\\
&\theta(x_\rho,0) = \theta^0(x_\rho), \quad x=(x_1,x_2)\in \bar{S},
\end{aligned}
\tag{2.7}
$$

where u_i^0, $\dot{u}_i^0$ and θ^0 are prescribed functions.

Throughout in this paper we assume that S is the strip defined by

$$
S = \Big\{(x_1,x_2)\in R^2 : x_1\in[0,L],\ x_2\in[0,l]\Big\},\ l>0,\ L>0. \tag{2.8}
$$

We further assume that the lateral sides of the strip are clamped and thermally insulated, that is we have

$$
\begin{aligned}
&u_i(x_1,0,t) = 0, \quad u_4(x_1,0,t) = 0,\\
&u_i(x_1,l,t) = 0, \quad u_4(x_1,l,t) = 0, \quad x_1\in[0,L],\ t\in[0,\infty).
\end{aligned}
\tag{2.9}
$$

If the strip is finite, that is $L<\infty$, then we adjoin the conditions that the end located at $x_1 = L$ is clamped and thermally insulated, i.e.

$$
u_i(L,x_2,t) = 0, \quad u_4(L,x_2,t) = 0, \quad x_2\in[0,l],\ t\in[0,\infty). \tag{2.10}
$$

We suppose that all boundary terms on the end of strip located at $x_1 = 0$ are separable with respect to space and time and that the time dependency is periodic. Thus, we consider the following end boundary conditions

$$
\begin{aligned}
&u_r(0,x_2,t) = \tilde{u}_r(x_2)\exp(-i\omega t),\\
&u_4(0,x_2,t) = \tilde{u}_4(x_2)\exp(-i\omega t), \quad x_2\in[0,l],
\end{aligned}
\tag{2.11}
$$

where $\omega\in R^+$ is the prescribed angular frequency of oscillation and $\tilde{u}_r$ and $\tilde{u}_4$ are prescribed continuous functions.

It is easy to see that we can write

$$
\begin{aligned}
u_r(x,t) &= U_r(x,t) + v_r(x)\exp(-i\omega t),\\
u_4(x,t) &= U_4(x,t) + v_4(x)\exp(-i\omega t),
\end{aligned}
\tag{2.12}
$$

where $[U_\alpha, U_3, U_4]$ absorbs the initial conditions (2.7) and satisfies the null boundary conditions and the equations (2.1) to (2.3), while $[v_\alpha, v_3, v_4]$ satisfies the boundary value problem defined by the equations

$$
h^2(\lambda+\mu)v_{\rho,\rho\alpha} + \mu(h^2 v_{\alpha,\rho\rho} - v_\alpha - v_{3,\alpha}) - v_{4,\alpha} + \rho h^2\omega^2 v_\alpha = 0\ , \tag{2.13}
$$

$$\mu(v_{3,\rho\rho} + v_{\rho,\rho}) + \rho\omega^2 v_3 = 0 \ , \tag{2.14}$$

$$v_{4,\rho\rho} + \frac{i\omega}{K} v_4 + i\omega\eta\alpha h^2(3\lambda + 2\mu) v_{\rho,\rho} = 0 \ , \tag{2.15}$$

in $S \equiv [0, L] \times [0, l]$, and the boundary conditions

$$\begin{aligned} &v_i(x_1, 0) = 0, \quad v_4(x_1, 0) = 0, \\ &v_i(x_1, l) = 0, \quad v_4(x_1, l) = 0, \quad x_1 \in [0, L], \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} &v_\alpha(0, x_2) = \tilde{u}_\alpha(x_2), \ \ v_3(0, x_2) = \tilde{u}_3(x_2), \\ &v_4(0, x_2) = \tilde{u}_4(x_2), \ \ x_2 \in [0, l]. \end{aligned} \tag{2.17}$$

If the strip is finite, then we have to adjoin the following conditions

$$v_\alpha(L, x_2) = 0, \quad v_3(L, x_2) = 0, \quad v_4(L, x_2) = 0, \quad x_2 \in [0, l]. \tag{2.18}$$

Physically the term $[U_\alpha, U_3, U_4]$ represents the transient part of the solution and its spatial behaviour can be studied by using the procedure described in [6]. As we can see the initial conditions influence the system only through the transient part, while the boundary conditions are assigned to the forced oscillation $[v_\alpha, v_3, v_4] \exp(-i\omega t)$ of the solution. If sufficient time has elapsed for the transient part of any solution to (2.1) to (2.3) to have vanished, then we can assume that the system has reached the steady state.

In this paper we are interested to study the spatial behaviour of the amplitude $[v_\alpha, v_3, v_4]$ of the steady thermoelastic oscillations with respect to the distance x_1 to the excited end.

3. Spatial behaviour. Let us consider $[v_\alpha, v_3, v_4]$ be a solution of the boundary value problem defined by the field equations (2.13) to (2.15) and the boundary conditions (2.16) to (2.18). We proceed to study the behaviour of the steady-state solution $[v_\alpha, v_3, v_4]$ with respect to the distance x_1 to the excited end. In this aim we associate with $[v_\alpha, v_3, v_4]$ the following line-integral cross-sectional measure

$$\begin{aligned} D(x_1) = -\int_0^l \Big\{ &h^2\mu(v_\rho \bar{v}_{\rho,1} + \bar{v}_\rho v_{\rho,1}) + h^2(\lambda+\mu)(v_1 \bar{v}_{\rho,\rho} + \bar{v}_1 v_{\rho,\rho}) + \\ &+ [\mu(v_1 + v_{3,1})\bar{v}_3 + (\bar{v}_1 + \bar{v}_{3,1}) v_3] + \delta(v_4 \bar{v}_{4,1} + \bar{v}_4 v_{4,1}) - \\ &- (v_4 \bar{v}_1 + \bar{v}_4 v_1) \Big\} dx_2, \ \ x_1 \in [0, L] \end{aligned} \tag{3.19}$$

where the superposed bar denotes complex conjugate and δ is a positive parameter that will be specified later.

It is easy to see that we have

(3.20)
$$D'(x_1) = -2\int_0^l (h^2\mu v_{\rho,1}\bar{v}_{\rho,1} + \mu v_{3,1}\bar{v}_{3,1})dx_2 - 2\delta\int_0^l v_{4,1}\bar{v}_{4,1}dx_2 - \mathcal{I}(x_1),$$

where

(3.21)
$$\begin{aligned}\mathcal{I}(x_1) = \int_0^l \Big\{ & h^2(\lambda+\mu)(v_{1,1}\bar{v}_{\rho,\rho} + \bar{v}_{1,1}v_{\rho,\rho} + v_1\bar{v}_{\rho,\rho 1} + \bar{v}_1 v_{\rho,\rho 1}) + \\ & + h^2\mu(v_\rho\bar{v}_{\rho,11} + \bar{v}_\rho v_{\rho,11}) + + \mu\Big[(v_{1,1}+v_{3,11})\bar{v}_3 + (\bar{v}_{1,1}+\bar{v}_{3,11})v_3 + \\ & + (v_1\bar{v}_{3,1} + \bar{v}_1 v_{3,1})\Big] + \delta(v_4\bar{v}_{4,11} + \bar{v}_4 v_{4,11}) - (v_{4,1}\bar{v}_1 + v_4\bar{v}_{1,1} + \\ & + \bar{v}_{4,1}v_1 + \bar{v}_4 v_{1,1})\Big\}dx_2.\end{aligned}$$

In view of the lateral boundary conditions (2.16), we can write the relation (3.21) in the form

(3.22)
$$\begin{aligned}\mathcal{I}(x_1) = \int_0^l \Big\{ & h^2(\lambda+\mu)(v_{1,1}\bar{v}_{\rho,\rho} + \bar{v}_{1,1}v_{\rho,\rho}) - h^2(\lambda+\mu)(v_{1,2}\bar{v}_{2,1} + \\ & + \bar{v}_{1,2}v_{2,1}) + \mu(v_1\bar{v}_{3,1} + \bar{v}_1 v_{3,1}) - (v_{4,1}\bar{v}_1 + v_4\bar{v}_{1,1} + \\ & + \bar{v}_{4,1}v_1 + \bar{v}_4 v_{1,1})\Big\}dx_2 + \mathcal{K}(x_1),\end{aligned}$$

where

(3.23)
$$\begin{aligned}\mathcal{K}(x_1) = \int_0^l \Big\{ & h^2(\lambda+2\mu)(v_1\bar{v}_{1,11} + \bar{v}_1 v_{1,11}) + h^2\mu(v_2\bar{v}_{2,11} + \bar{v}_2 v_{2,11}) + \\ & + \mu[v_3(\bar{v}_{1,1} + \bar{v}_{3,11}) + \bar{v}_3(v_{1,1} + v_{3,11})] + \delta(v_4\bar{v}_{4,11} + \bar{v}_4 v_{4,11})\Big\}dx_2.\end{aligned}$$

Further, we use the field equations (2.13) to (2.15) and the lateral boundary conditions (2.16) in order to obtain

$$\begin{aligned}\mathcal{K}(x_1) = \int_0^l \Big\{ & 2\mu h^2 v_{1,2}\bar{v}_{1,2} + 2(\lambda+2\mu)h^2 v_{2,2}\bar{v}_{2,2} + 2\mu v_{3,2}\bar{v}_{3,2} + \\ & + 2\delta v_{4,2}\bar{v}_{4,2} + h^2(\lambda+\mu)(v_{1,2}\bar{v}_{2,1} + \bar{v}_{1,2}v_{2,1} + \bar{v}_{1,1}v_{2,2} + v_{1,1}\bar{v}_{2,2}) +\end{aligned}$$

(3.24)
$$+\mu\Big[v_1(\bar v_1+\bar v_{3,1})+\bar v_1(v_1+v_{3,1})+v_2(\bar v_2+\bar v_{3,2})+\bar v_2(v_2+v_{3,2})-$$
$$-v_3\bar v_{2,2}-\bar v_3 v_{2,2}\Big]+(v_1\bar v_{4,1}+\bar v_1 v_{4,1}+v_2\bar v_{4,2}+\bar v_2 v_{4,2})-$$
$$-2\rho h^2\omega^2 v_\rho\bar v_\rho-2\rho\omega^2 v_3\bar v_3+i\delta\omega\eta\alpha h^2(3\lambda+2\mu)(v_4\bar v_{\rho,\rho}-\bar v_4 v_{\rho,\rho})\Big\}dx_2.$$

If we substitute the relation (3.24) into (3.22), then we get
(3.25)
$$\mathcal{I}(x_1)=\int_0^l\Big\{2\mu h^2 v_{\rho,2}\bar v_{\rho,2}+2\mu(v_\alpha+v_{3,\alpha})(\bar v_\alpha+\bar v_{3,\alpha})-2\mu v_{3,1}\bar v_{3,1}+$$
$$+2(\lambda+\mu)h^2 v_{\alpha,\alpha}\bar v_{\rho,\rho}+2\delta v_{4,2}\bar v_{4,2}-2\rho h^2\omega^2 v_\rho\bar v_\rho-2\rho\omega^2 v_3\bar v_3-[1-$$
$$-i\delta\omega\eta\alpha h^2(3\lambda+2\mu)]v_4\bar v_{\rho,\rho}-[1+i\delta\omega\eta\alpha h^2(3\lambda+2\mu)]\bar v_4 v_{\rho,\rho}\Big\}dx_2.$$

Now, we substitute the relation (3.25) into (3.20) so that we obtain
(3.26)
$$D'(x_1)=-\int_0^l\Big\{2\mu h^2 v_{\rho,\alpha}\bar v_{\rho,\alpha}+2\mu(v_\alpha+v_{3,\alpha})(\bar v_\alpha+\bar v_{3,\alpha})+$$
$$+2\delta v_{4,\alpha}\bar v_{4,\alpha}+2(\lambda+\mu)h^2 v_{\alpha,\alpha}\bar v_{\rho,\rho}-2\rho h^2\omega^2 v_\rho\bar v_\rho-2\rho\omega^2 v_3\bar v_3-[1-$$
$$-i\delta\omega\eta\alpha h^2(3\lambda+2\mu)]v_4\bar v_{\rho,\rho}-[1+i\delta\omega\eta\alpha h^2(3\lambda+2\mu)]\bar v_4 v_{\rho,\rho}\Big\}dx_2.$$

Let us now fix $\varepsilon\in(0,1)$. Then we can write
(3.27)
$$D'(x_1)=-\int_0^l\Big\{2\mu h^2 v_{\rho,\alpha}\bar v_{\rho,\alpha}+2\mu(v_\alpha+v_{3,\alpha})(\bar v_\alpha+\bar v_{3,\alpha})+$$
$$+2(1-\varepsilon)(\lambda+\mu)h^2 v_{\alpha,\alpha}\bar v_{\rho,\rho}+2\varepsilon(\lambda+\mu)h^2\Big[v_{\alpha,\alpha}\bar v_{\rho,\rho}-$$
$$-\frac{1-i\delta\omega\eta\alpha h^2(3\lambda+2\mu)}{2\varepsilon(\lambda+\mu)h^2}v_4\bar v_{\rho,\rho}-\frac{1+i\delta\omega\eta\alpha h^2(3\lambda+2\mu)}{2\varepsilon(\lambda+\mu)h^2}\bar v_4 v_{\rho,\rho}\Big]+$$
$$+2\delta v_{4,\alpha}\bar v_{4,\alpha}-2\rho h^2\omega^2 v_\rho\bar v_\rho-2\rho\omega^2 v_3\bar v_3\Big\}dx_2,$$

and hence we have

$$D'(x_1)=-\int_0^l\Big\{2\mu h^2 v_{\rho,\alpha}\bar v_{\rho,\alpha}+2\mu(v_\alpha+v_{3,\alpha})(\bar v_\alpha+\bar v_{3,\alpha})+2(1-\varepsilon)(\lambda+$$

(3.28)
$$+\mu)h^2 v_{\alpha,\alpha}\bar{v}_{\rho,\rho}+2\varepsilon(\lambda+\mu)h^2\Big[v_{\alpha,\alpha}-\frac{1-i\delta\omega\eta\alpha h^2(3\lambda+2\mu)}{2\varepsilon(\lambda+\mu)h^2}v_4\Big]\Big[\bar{v}_{\rho,\rho}-$$
$$-\frac{1+i\delta\omega\eta\alpha h^2(3\lambda+2\mu)}{2\varepsilon(\lambda+\mu)h^2}\bar{v}_4\Big]-\frac{1+\delta^2\omega^2\eta^2\alpha^2h^4(3\lambda+2\mu)^2}{2\varepsilon(\lambda+\mu)h^2}v_4\bar{v}_4+$$
$$+2\delta v_{4,\alpha}\bar{v}_{4,\alpha}-2\rho h^2\omega^2 v_\rho\bar{v}_\rho-2\rho\omega^2 v_3\bar{v}_3\Big\}dx_2.$$

Since $v_i(x_1,0)=0$, $v_4(x_1,0)=0$, $v_i(x_1,l)=0$, $v_4(x_1,l)=0$ for all $x_1\in[0,L]$, we have

$$\int_0^l v_{r,2}\bar{v}_{r,2}dx_2\geq\frac{\pi^2}{l^2}\int_0^l v_r\bar{v}_r dx_2\quad r=1,2,3,4\quad\text{(not summed on } r). \tag{3.29}$$

We use now the relation (3.29) for $r=2$ in order to deduce that

$$\begin{aligned}&\int_0^l\Big[h^2v_{2,2}\bar{v}_{2,2}+(v_2+v_{3,2})(\bar{v}_2+\bar{v}_{3,2})\Big]dx_2\geq\\&\geq\int_0^l\Big[(\frac{\pi^2h^2}{l^2}+1)v_2\bar{v}_2+v_2\bar{v}_{3,2}+\bar{v}_2v_{3,2}+v_{3,2}\bar{v}_{3,2}\Big]dx_2\geq\\&\geq m_0\int_0^l\Big[(\frac{\pi^2h^2}{l^2}+1)v_2\bar{v}_2+v_{3,2}\bar{v}_{3,2}\Big]dx_2,\end{aligned} \tag{3.30}$$

where

$$m_0=1-\frac{l}{\sqrt{\pi^2h^2+l^2}}\in[0,1]. \tag{3.31}$$

We conclude from the relations (3.29) for $r=3$ and (3.30) that we have

$$\begin{aligned}&\int_0^l\Big[h^2v_{2,2}\overline{v}_{2,2}+(v_2+v_{3,2})(\overline{v}_2+\overline{v}_{3,2})\Big]dx_2\geq\\&\geq m_0\int_0^l\Big[\Big(\frac{\pi^2h^2}{l^2}+1\Big)v_2\overline{v}_2+\frac{\pi^2}{l^2}v_3\overline{v}_3\Big]dx_2.\end{aligned} \tag{3.32}$$

By using the relations (3.29) for $r=1,4$, and (3.32) into the relation

(3.28), we can get

(3.33)

$$-D'(x_1)\geq \int_0^l \Big\{2\mu h^2 v_{\rho,1}\overline{v}_{\rho,1}+2(1-\varepsilon)(\lambda+\mu)h^2 v_{\alpha,\alpha}\overline{v}_{\rho,\rho}+2\delta v_{4,1}\overline{v}_{4,1}+$$
$$+2\mu(v_1+v_{3,1})(\overline{v}_1+\overline{v}_{3,1})+\Big(\frac{2\mu\pi^2h^2}{l^2}-2\rho h^2\omega^2\Big)v_1\overline{v}_1+$$
$$+\Big[2\mu m_0\Big(\frac{\pi^2h^2}{l^2}+1\Big)-2\rho h^2\omega^2\Big]v_2\overline{v}_2+\Big(\frac{2\mu m_0\pi^2}{l^2}-2\rho\omega^2\Big)v_3\overline{v}_3+$$
$$+\Big[\frac{2\delta\pi^2}{l^2}-\frac{1+\delta^2\omega^2\eta^2\alpha^2h^4(3\lambda+2\mu)^2}{2\varepsilon h^2(\lambda+\mu)}\Big]v_4\overline{v}_4\Big\}dx_2.$$

At this stage of our analysis we choose the arbitrary parameter δ in our measure $D(x_1)$ in such a way to have

$$\delta > \frac{l^2}{4\varepsilon\pi^2h^2(\lambda+\mu)}. \tag{3.34}$$

Moreover, we define

$$\omega_m^2=\min\Big\{\frac{\mu m_0\pi^2}{\rho l^2},\ \frac{4\delta\varepsilon\pi^2h^2(\lambda+\mu)-l^2}{\delta^2l^2\eta^2\alpha^2h^4(3\lambda+2\mu)^2}\Big\}. \tag{3.35}$$

Throughout in the remainder of this paper we assume that the angular frequency of the oscillation is lower than the critical value ω_m defined by the relation (3.35). That means that we assume that

$$\omega<\omega_m. \tag{3.36}$$

With these choices the relation (3.33) implies that

(3.37)

$$-D'(x_1)\geq\int_0^l\Big\{M_1h^2v_{\rho,1}\overline{v}_{\rho,1}+M_2h^2v_{\alpha,\alpha}\overline{v}_{\rho,\rho}+M_3v_{4,1}\overline{v}_{4,1}+M_4(v_1+$$
$$+v_{3,1})(\overline{v}_1+\overline{v}_{3,1})+M_5v_1\overline{v}_1+M_6v_2\overline{v}_2+M_7v_3\overline{v}_3+M_8v_4\overline{v}_4\Big\}dx_2,$$

where

(3.38)

$$M_1=2\mu,\quad M_2=2(1-\varepsilon)(\lambda+\mu),\quad M_3=2\delta,\quad M_4=2\mu,$$
$$M_5=\frac{2\mu\pi^2h^2}{l^2}\Big(1-\frac{\omega^2}{\omega_m^2}\Big),\quad M_6=2\mu m_0\Big(\frac{\pi^2h^2}{l^2}+1\Big)\Big(1-\frac{\omega^2}{\omega_m^2}\Big),$$
$$M_7=\frac{2\mu m_0\pi^2}{l^2}\Big(1-\frac{\omega^2}{\omega_m^2}\Big),\quad M_8=\frac{4\delta\varepsilon\pi^2h^2(\lambda+\mu)-l^2}{2\varepsilon l^2h^2(\lambda+\mu)}\Big(1-\frac{\omega^2}{\omega_m^2}\Big).$$

It is worth to note that under the assumption (3.36), the relation (3.37) implies that

$$D^{'}(x_1) \leq 0, \quad \text{for all } x_1 \in [0, L], \tag{3.39}$$

and hence $D(x_1)$ is a non-increasing function with respect to x_1.

On the other hand, by means of the Schwarz's inequality and the arithmetic - geometric mean inequality, from the relations (2.6) and (3.19), we deduce that

(3.40)

$$|D(x_1)| \leq \int_0^l \Big\{ M_1^* h^2 v_{\rho,1}\overline{v}_{\rho,1} + M_2^* h^2 v_{\alpha,\alpha}\overline{v}_{\rho,\rho} + M_3^* v_{4,1}\overline{v}_{4,1} + M_4^*(v_1 + \\ + v_{3,1})(\overline{v}_1 + \overline{v}_{3,1}) + M_5^* v_1\overline{v}_1 + M_6^* v_2\overline{v}_2 + M_7^* v_3\overline{v}_3 + M_8^* v_4\overline{v}_4 \Big\} dx_2,$$

where

$$M_1^* = \frac{\mu l}{\pi}, \quad M_2^* = \frac{(\lambda+\mu)l}{\pi}, \quad M_3^* = \frac{\delta l}{\pi}, \quad M_4^* = \frac{\mu\pi}{l}, \\ M_5^* = \frac{h^2\pi(\lambda+2\mu)}{l} + \frac{l}{\delta\pi}, \quad M_6^* = \frac{h^2\mu\pi}{l}, \quad M_7^* = \frac{\mu l}{\pi}, \quad M_8^* = \frac{2\delta\pi}{l}. \tag{3.41}$$

By combining the relations (3.37) and (3.40), we get the following first-order differential inequality

$$\sigma|D(x_1)| + D'(x_1) \leq 0, \quad x_1 \in [0, L], \tag{3.42}$$

where

$$\frac{1}{\sigma} = \max\Big\{ \frac{M_1^*}{M_1}, \frac{M_2^*}{M_2}, \frac{M_3^*}{M_3}, \frac{M_4^*}{M_4}, \frac{M_5^*}{M_5}, \frac{M_6^*}{M_6}, \frac{M_7^*}{M_7}, \frac{M_8^*}{M_8} \Big\}. \tag{3.43}$$

The integration of the above first-order differential inequality furnishes the spatial estimate describing the spatial decay of the amplitude $[v_\alpha, v_3, v_4]$ measured by $D(x_1)$. Our analysis further considers separately the bounded and unbounded plates.

Let us consider first the case when the middle surface section S of the plate is a finite strip, that is we have $L < \infty$. Then by taking into account the boundary conditions (2.18) and the relation (3.19), we deduce that

$$D(L) = 0. \tag{3.44}$$

Since $D(x_1)$ is a non-increasing function of x_1 it follows from (3.44) that we have

$$D(x_1) \geq 0, \quad \text{for all } x_1 \in [0, L]. \tag{3.45}$$

Then the relation (3.42) implies that

$$\sigma D(x_1) + D'(x_1) \leq 0, \quad x_1 \in [0, L], \tag{3.46}$$

and hence we have

$$\left[e^{\sigma x_1} D(x_1)\right]' \leq 0, \quad x_1 \in [0, L]. \tag{3.47}$$

By an integration of the relation (3.47), we obtain the following spatial estimate of Saint-Venant type

$$0 \leq D(x_1) \leq D(0) \exp(-\sigma x_1), \quad x_1 \in [0, L]. \tag{3.48}$$

Let us now consider the case of a semi-infinite plate. Hence we have $L = \infty$ and in order to integrate the first-order differential inequality (3.42) we have to restrict the class of amplitudes. In this aim we introduce the double-integral measure of the amplitude by

(3.49)

$$\mathcal{E}(x_1) = \int_{x_1}^{\infty} d\eta \int_0^l \Big\{ M_1^* h^2 v_{\rho,1} \overline{v}_{\rho,1} + M_2^* h^2 v_{\alpha,\alpha} \overline{v}_{\rho,\rho} + M_3^* v_{4,1} \overline{v}_{4,1} +$$

$$+ M_4^* \cdot (v_1 + v_{3,1})(\overline{v}_1 + \overline{v}_{3,1}) + M_5^* v_1 \overline{v}_1 + M_6^* v_2 \overline{v}_2 + M_7^* v_3 \overline{v}_3 + M_8^* v_4 \overline{v}_4 \Big\} dx_2.$$

Then, in the class of steady-state oscillations whose amplitudes make $\mathcal{E}(x_1)$ to be bounded, we have

$$D(\infty) \equiv \lim_{x_1 \to \infty} D(x_1) = 0. \tag{3.50}$$

Since $D(x_1)$ is a non-increasing function of x_1, the relation (3.50) proves that

$$D(x_1) \geq 0, \quad \text{for all } x_1 \in [0, \infty). \tag{3.51}$$

Therefore, the first-order differential inequality (3.42) takes the form (3.47) and hence, by an integration we deduce that

$$0 \leq D(x_1) \leq D(0) \exp(-\sigma x_1), \quad \text{for all } x_1 \in [0, \infty). \tag{3.52}$$

We conclude that the relations (3.48) and (3.52) describe the spatial decay of the amplitudes $[v_\alpha, v_3, v_4]$ of the time-harmonic oscillations as measured by $D(x_1)$, provided that the angular frequency ω is lower than the critical value ω_m defined by the relation (3.35).

4. Concluding remarks. The decay properties presented in this paper are in accord with the observations made by BOLEY [9] concerning a dynamic version of the Saint-Venant principle and they are similar in nature with the results established by FLAVIN and KNOPS [10] for the classical elastodynamics.

There is no a general rule for obtaining an expression for the considered measure $D(x_1)$. Various forms are used in the literature on the spatial behaviour they depending on the considered context (a complete review can be found by searching the references cited in [11]-[13]). Our measure here is inspired by one introduced by CHIRITA [14] for the study of three-dimensional thermoelastic steady-states. One can obtain an idea on how it was chosen by combining the basic equations (2.13) to (2.15). In fact, we multiply the relation (2.13) by $\bar{v}_\alpha$ and the relation (2.14) by $\bar{v}_3$ and the relation (2.15) by $\delta\bar{v}_4$ ($\delta > 0$) and then add the results. Then we integrate the result over $[x_1, L] \times [0, l]$. Thus, by using the integration by parts, $D(x_1)$ appears like a quasi-work function, on the cross-section $\{x_1\} \times [0, l]$.

On the other hand, by integrating the relation (3.26) with respect to x_1 and by taking into account the relations (3.44) and (3.50), we can write

(4.53)
$$D(x_1) = \int_{S_{x_1}} \Big\{ 2\mu h^2 v_{\rho,\alpha}\bar{v}_{\rho,\alpha} + 2\mu(v_\alpha + v_{3,\alpha})(\bar{v}_\alpha + \bar{v}_{3,\alpha}) + 2\delta v_{4,\alpha}\bar{v}_{4,\alpha} + \\ + 2(\lambda+\mu)h^2 v_{\alpha,\alpha}\bar{v}_{\rho,\rho} - 2\rho h^2\omega^2 v_\rho\bar{v}_\rho - 2\rho\omega^2 v_3\bar{v}_3 - \\ - [1 - i\delta\omega\eta\alpha h^2(3\lambda+2\mu)]v_4\bar{v}_{\rho,\rho} - [1 + i\delta\omega\eta\alpha h^2(3\lambda+2\mu)]\bar{v}_4 v_{\rho,\rho} \Big\} da,$$

where $S_{x_1} = \{y = (y_1, y_2) \in S, \ \ y_1 > x_1\}$. Under the restriction (3.36), the relations (3.37) and (4.53) prove that $D(x_1)$ appears as an energetical measure associated with the domain S_{x_1} for the steady-state solution $[v_\alpha, v_3, v_4]$. Then the estimates (3.48) and (3.52) proves an exponential spatial decay of the energetical measure associated with the domain S_{x_1} with respect to the same measure associated to the entire domain.

In view of the relation (3.19) it follows that $D(x_1)$ can be estimated in terms of the end boundary data. Our analysis in the above proves that one

can consider other types of end boundary conditions, provided that they have to assure a useful estimate for $D(x_1)$ in terms of the given data.

Our spatial estimate is established under the condition (3.36) upon the angular frequency. If the angular frequency ω is greater than or equal to the explicit value ω_m defined by the relation (3.35), then $D'(x_1)$ should be positive for all $x_1 > 0$ and hence $D(x_1)$ should be an increasing function with respect to x_1. In such a case it is not possible to have a spatial decay for the amplitude of the harmonic vibration.

It is worth to mention it that the above results imply a uniqueness theorem. In order to see this we first note that the considered boundary value problem is linear and therefore, to prove uniqueness it suffices to prove that the null end boundary conditions ($\widetilde{u}_r = 0, \widetilde{u}_4 = 0$) imply that the corresponding amplitude $[v_\alpha, v_3, v_4]$ is vanishing. To prove this we note that $\widetilde{u}_r = 0, \widetilde{u}_4 = 0$ implies that (see relation (3.19))

$$D(0) = 0,$$

and hence the relations (3.48) and (3.52) give

$$D(x_1) = 0, \quad \text{for all } x_1 \in [0, L] \quad (x_1 \in [0, \infty)),$$

and therefore,

$$D'(x_1) = 0, \quad \text{for all } x_1 \in [0, L] \quad (x_1 \in [0, \infty)).$$

If we use this into relation (3.37) we obtain that $[v_\alpha, v_3, v_4] = 0$ in $\overline{S}$. We recall that such a result is possible under the assumption that the angular frequency satisfies the relation (3.36). For an infinite plate we have to adjoin that the uniqueness result is valid in the class of amplitudes making $\mathcal{E}(x_1)$ to be bounded.

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