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ON THE EXISTENCE INTERVAL IN PEANO'S THEOREM*

BY

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Abstract. A reformulation of the celebrated Peano Existence Theorem is given accompanied by an overview of relevant literature. By a carefully selected example, it is shown that the estimate of interval of existence for the solution provided by this reformulation is the best possible one.

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1. History and further development. Lying at the foundation of qualitative theory of ordinary differential equations, the celebrated Peano Existence Theorem (1886 [22], 1890 [23], cf. [26, Remark (a), p. 172]) was originally written in logical symbols. Its first accessible presentation [17], in German language, appeared in 1893 as a contribution of mathematician G. Mie (cf. [18, footnote, p. 206]). The problem of extending Peano's existence theorem to the case of infinite dimensional Banach spaces was answered in the negative in 1973 by A. N. Godunov [8].

There are several proofs of this theorem [14, Satz 3, p. 85], [3, Theorem 1.2, p. 6], [1, Theorem 12, pp. 124-126], [11, Theorem 2.1, Exercises 2.1, 2.2, pp. 10-11]. A proof that relies on the Schauder-Tikhonov fixed point theorem can be found in [10, Theorem 1.1, pp. 14-15]. An interesting, constructive proof of Peano's existence theorem was given by W. Walter in 1959 (see [26, Remark (d), p. 173]).

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Allow us to present next a typical statement of the theorem. Introduce $a \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, $f : (a, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ a continuous function and fix the numbers $t_0 > a$, $c > 0$ such that $t_0 - c > a$.

Theorem 1. (Peano's existence theorem) *Consider $b > 0$ fixed. Then, the problem below*

$$\left\{ \begin{array}{c} x' = f(t,x) \\ \|x(t) - x_0\| \leq b, \qquad |t - t_0| \leq c \\ x(t_0) = x_0 \end{array} \right. \tag{1}$$

has a solution in $[t_0 - \delta, t_0 + \delta]$, where $\delta = \min\{c, \frac{b}{M}\}$.

Here, $M = \sup\{\|f(t,x)\| : |t - t_0| \leq c,\ \|x - x_0\| \leq b\}$.

The question that was raised in the years after the publication of Peano's existence theorem is the following one: *is the estimate $\frac{b}{M}$ the best choice or it can be improved and, if so, which is the best possible value that can replace this bound in the formula of δ without having to add any restrictions of the function f?*

According to A. Wintner [28, p. 539], "*efforts have been made to improve the lower estimate* (1) *of* $\Gamma(a, b.M)$ [*i.e.*, the estimate given by $\frac{b}{M}$, our note]. *In reality, these efforts succeeded only by imposing additional restrictions to* $f(z, w)$." In 1935, Wintner establishes by means of an example that this estimation is the best possible one when the function f is analytic.

Theorem 2. ([28, p. 540]) *Let b, $M > 0$, r be given numbers, $r > \frac{b}{M}$. Then, there exists a function f which is independent of z and possess the following properties:*

(i) *$f(w)$ is regular-analytic* (*i.e.*, holomorphic, cf. [27, p. 55]) *and bounded in the circle $|w| < b$;*

(ii) *the least upper bound of $|f(w)|$ in the circle is M;*

(iii) *the function $w = w(z)$ for which $\frac{dw}{dz} = f(w)$ and $w(0) = 0$ has in the circle $|w| < r$ a singularity.*

The function in Wintner's example has the formula

$$f(w) = M \sqrt[n]{\frac{1}{2}\left(1 + \frac{w}{b}\right)},$$

where n is large enough. The solution of the problem reads as

$$w(z) = b\left[\left(1 + \frac{z}{C_n}\right)^{\frac{n}{n-1}} - 1\right] \qquad C_n = \frac{n\sqrt[n]{2}}{n-1}\frac{b}{M}.$$

It is straightforward that $z = -C_n$ is a singular point of $w(z)$ and tends to $-\frac{b}{M}$ as $n \to +\infty$.

Very interesting estimates of the maximal interval of existence in case of Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}^n$ can be found in [12, Examples 1-2, p. 36] whereas for an analytic f examples are given in [25, VII-VIII, pp. 76-80].

Let now $t_0 = 0$, $\psi : \mathbb{R}_+ \to (0, +\infty)$ be a continuous function such that

$$\|f(t,x)\| \leq \psi(\|x\|), \qquad t \geq 0, x \in \mathbb{R}^n, \tag{2}$$

and

$$T_\infty = \int_{\|x_0\|}^{+\infty} \frac{du}{\psi(u)} \leq +\infty. \tag{3}$$

A famous result established by A. Wintner in 1945 asserts that if $T_\infty = +\infty$ then all solutions of the Cauchy problem

$$\begin{cases} x' = f(t,x), t \geq 0 \\ x(0) = x_0 \end{cases} \tag{4}$$

exist in $[0, T]$ for all $T > 0$ (see [29, claim (iii*), p. 283]Wintner1945). On the other hand, if $T_\infty < +\infty$ then there would be solutions of problem (4) that cease to exist in finite time, cf. [29, assertion (iii bis), p. 284].

Replacing inequality (2) with the more permissive one below

$$\|f(t,x)\| \leq \alpha(t)\psi(\|x\|), \qquad t \geq 0, x \in \mathbb{R}^n, \tag{5}$$

F. Brauer showed in 1963 that all solutions of the Cauchy problem (4) exist in $[0, T]$ provided that

$$\int_0^T \alpha(s)ds < T_\infty \leq +\infty, \tag{6}$$

without any discussion about the maximality of the existence interval (see [2, p. 38]).

Following the same line of research, J. Lee and D. O'Regan [16, Theorem 2.3, p. 250] have established in 1988 that the Cauchy problem (4), (2) has a solution in $C^1([0, T], \mathbb{R}^n)$, where $T < T_\infty$, and the latter inequality gives the *best possible estimate* of the existence interval. The second claim is justified via the example below

$$\begin{cases} x' = (\psi(\|x\|), 0, ..., 0) \\ x(0) = 0. \end{cases}$$

In case of $f : \mathbb{C}^2 \to \mathbb{C}$ being analytic, Frigon [5] has obtained the estimate $T < \frac{1}{\sqrt{2}} T_\infty$. If $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is only a Carathéodory function then the best possible estimate is still $T < T_\infty$ even if ψ is a Borel function (see [9, p. 165]).

In a Banach space X, where $f : \mathbb{R}_+ \times X \to X$, the estimation given by (6) for the upper bound of the existence interval to problem (4), (5) holds true provided that ψ is nondecreasing [7]. Here, $\alpha \in L^1(\mathbb{R}_+, \mathbb{R}_+)$.

There are also variants of these results for differential inclusions [6] and fuzzy differential equations [21]. Theorems relying on inequalities (2), (5) are almost unanimously called nowadays *Wintner-type results* (see [21, p. 405]).

To end this section, let us mention also that an interesting formulation of Peano's existence theorem, much in the spirit of (6), can be found in E. Kamke's book [15, Satz 4, p. 87] while a proof of Peano's existence theorem in the case of f being analytic that uses Cauchy's method of majorizing functions can be read in F. Tricomi's book [24, V, Section 42, p. 205 and the following].

2. A reformulation of Peano existence theorem and its proof. Let us begin by pointing out two aspects that are essential for understanding our investigation. First, the presence of a nondecreasing eventually positive function ψ in (2) seems unavoidable. It can be related rather easily with mathematical machineries such as the technique of majorizing functions [24, pp. 203, 205] or even more complicated "skin" methods as in 5 (we are using this terminology in the spirit of T. Ważewski's work). Second, in our opinion, it would be better to put in the statement of a result concerning only *local* existence of solutions of a certain Cauchy problem a condition similar to $T < T_\infty$ but which cannot be regarded instantly as *non-local*. As opposed to theorems presented in the first section, we have no intention here in bringing together both local and non-local properties.

Two auxiliary lemmas are needed in the sequel.

Lemma 3. *Consider $d \in [0, c]$ and the linear space $X = C([t_0 - d, t_0 + d], \mathbb{R}^n)$ endowed with the Chebyshev norm ($\sup -$ norm). Introduce the operator $T : X \to X$ by the formula*

$$(Tx)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \qquad |t - t_0| \le d, x \in X,$$

and assume further that $B \subset X$ is a bounded set.

Then,

(i) $T : B \to X$ is uniformly continuous;

(ii) TB is bounded and equicontinuous.

Proof. See [10, proof of Theorem 1.1, p. 15]. $\square$

The next lemma provides a successful example of majorizing function ψ.

Lemma 4. *Consider $R > 0$. Then, the function $\psi : [0, R] \to [0, +\infty)$ given by the formula*

$$\psi(r) = \sup\{\|f(t,x)\| : |t - t_0| \le c,\ \|x - x_0\| \le r\}$$

is (uniformly) continuous in $[0, R]$.

Proof. Since $f : [t_0 - c, t_0 + c] \times \{x \in \mathbb{R}^n : \|x - x_0\| \le R\} \to \mathbb{R}^n$ is uniformly continuous, for every $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon, R) > 0$ such that

$$\|f(t,x) - f(t,y)\| \le \varepsilon, \qquad |t - t_0| \le c$$

for all x, y satisfying $\|x - x_0\|$, $\|y - x_0\| \le R$ and $\|x - y\| \le \eta$. Further, for $0 \le r_1 \le r_2 \le R$, we have

$$\sup_{\|x-x_0\|\le r_2} \|f(t,x)\| - \sup_{\|x-x_0\|\le r_1} \|f(t,x)\| \le \sup_{r_1\le\|x-x_0\|\le r_2} \|f(t,x)\| - \|f(t,x_2)\|\,,$$

where $x_2 \in [x_0, x_1]$ is an element of $\mathbb{R}^n$ with $\|x_2 - x_0\| = r_1$ and $\sup\{\|f(t,x)\| : r_1 \le \|x - x_0\| \le r_2\} = \|f(t,x_1)\|$. Of course, $x_i = x_i(t)$; $i = \overline{1,2}$. We have obtained that

$$\psi(r_2) - \psi(r_1) \le \sup_{|t-t_0|\le c} \|f(t, x_1(t)) - f(t, x_2(t))\| \le \varepsilon,$$

where $\|x_1(t) - x_2(t)\| \le r_2 - r_1 < \eta$. $\square$

The proof of Peano's existence theorem (reformulated) given in the following relies on the Leray-Schauder Alternative [4]. According to it, if X and T are as in Lemma 3 then either the set $E(T) = \{u \in X : u = \lambda T u,$ $\lambda \in (0,1)\}$ is unbounded or T has a fixed point. The Alternative was used recently with success to obtain general existence results (see [20], [19]).

Using the transformation $y(t) = x(t) - x_0$, we can reduce the study of existence of a solution to (1) to the situation below.

Theorem 5. *There exists a* C^1*−function* $y(t)$ *defined in* $[t_0 - \delta, t_0 + \delta]$ *which verifies the conditions*

$$\begin{cases} y' = g(t, y) \\ \|y(t)\| \le b, \qquad |t - t_0| \le c, \\ y(t_0) = 0 \end{cases}$$

where $g(t, y) = f(t, y + x_0)$, $\delta = \min\{c, \frac{b}{M}\}$.

It is easy to see that

$$\sup\{\|g(t, y)\| : |t - t_0| \le c, \|y\| \le b\} = M.$$

In other words, we are allowed to restrict the proof of Peano's existence theorem to case $x_0 = 0$.

We notice also that $\psi(0) = 0$, where the function ψ is given in Lemma 4, if and only if problem (1) has the zero solution in $[t_0 - c, t_0 + c]$. In such a case, of course, the result is superfluous. On the other hand, if $\psi(0) > 0$ then we can introduce the function $G : [0, +\infty) \to [0, +\infty)$ by the formula

$$G(x) = \int_0^x \frac{du}{\psi(u)}, \qquad x \ge 0.$$

Theorem 6. (Peano's existence theorem reformulated) *Consider* $b > 0$ *fixed and the problem below*

$$\begin{cases} x' = f(t, x) \\ \|x(t) - x_0\| \le b, \qquad |t - t_0| \le c \\ x(t_0) = x_0. \end{cases} \tag{7}$$

Then, only one of the following statements is true

(i) $\psi(0) = 0$ *and* $x(t) = x_0$ *is a solution of problem (7) in* $[t_0 - c, t_0 + c]$;

(ii) $\psi(0) > 0$ *and problem (7) has a solution in* $[t_0 - \tau, t_0 + \tau]$*, where* $\tau = \min\{c, G(b)\}$.

Proof. We assume that $x_0 = 0$, and $\psi(0) > 0$. Then, since $\psi(u)$ is monotone nondecreasing, we may write

$$\frac{b}{M} \le \int_0^b \frac{du}{\psi(u)} = G(b), \qquad \delta \le \tau, \tag{8}$$

according to the hypotheses in Theorem 1.

Introduce $d = \tau$ and X, T as in Lemma 3. We claim that *the set* $E(T)$ *is bounded.*

Indeed, the identity below $(0 < \lambda < 1)$

$$u(t) = \lambda \int_{t_0}^{t} f(s, u(s))ds, \qquad |t - t_0| \le \tau,$$

implies that

$$\|u(t)\| \le \int_{t_0}^{t} \psi(\|u(s)\|)ds \tag{9}$$

for all t in $[t_0, t_0+\tau]$. We denote by $z(t)$ the right-hand member of inequality (9). Thus,

$$z'(t) \le \psi(z(t))$$

and, by integration over $[t_0, t]$, we have

$$\int_0^{z(t)} \frac{du}{\psi(u)} \le t - t_0 \le \tau \tag{10}$$

for all t in $[t_0, t_0 + \tau]$. In a similar way, the same inequality is established for $z(t) = \int_t^{t_0} \psi(\|u(s)\|)ds$, where $t \in [t_0 - \tau, t_0]$. We shall confine ourselves to the case $t_0 \le t \le t_0 + \tau$.

Inequality (10) reads as

$$G(z(t)) \le t - t_0 \le \tau.$$

Since we have (8), we get

$$z(t) \le G^{-1}(\tau) \le G^{-1}(G(b)). \tag{11}$$

We have obtained in the end that

$$\|u(t)\| \le z(t) \le b, \qquad |t - t_0| \le \tau.$$

The validity of our claim being established, the conclusion of Peano's theorem follows from the Alternative. $\square$

The example below shows that *the estimate of* τ *given in Theorem 6 is the best possible one.*

Indeed, consider the problem

$$(12)\qquad \begin{cases} x' = e^{|x|} \\ |x(t)| \le b, \quad |t| \le c \\ x(0) = 0, \end{cases}$$

where b, $c > 0$ are fixed. Here, $F(r) = e^r$, $M = e^b$, $G(r) = 1 - e^{-r}$ for all $r \ge 0$. Its solution $x(t)$ verifies the identities

$$1 - e^{-x(t)} = t, t \ge 0, \qquad e^{x(t)} - 1 = t, t \le 0.$$

According to Theorem 6, $x(t)$ exists in $[-\tau, \tau]$, where $\tau = \min\{c, 1-e^{-b}\}$. Since $0 \le t \le 1-e^{-|x(t)|} \le 1-e^{-b}$ and $0 \ge t \ge -1+e^{-|x(t)|} \ge -1+e^{-b}$, it is obvious that the interval $[-\tau, \tau]$ is the maximal interval of existence of $x(t)$ as solution of problem (12). Simultaneously, we deduce from Theorem 1 that $x(t)$ exists in $[-\delta, \delta]$, where $\delta = \min\{c, be^{-b}\}$. The elementary inequality $e^b - 1 > b$ for all $b > 0$ yields $1 - e^{-b} > be^{-b}$. In other words, $\tau(b) > \delta(b)$ for all $b > 0$.

Remark 1. A similar "splitting" in the statement of Peano's existence theorem can be found in [13, Theorem I-2-5, p. 12].

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