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SOME REMARKS ON THE BIHARMONIC SUBMANIFOLDS OF $\mathbb{S}^3$ AND THEIR STABILITY*

BY

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Abstract. We give a geometric interpretation for the non-geodesic biharmonic curves on the unit euclidian sphere $\mathbb{S}^3$ and, using the stereographic projection, we study the second variation of the bienergy functional for some biharmonic surfaces in $\mathbb{S}^3$.
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1. Introduction. Let $\phi:(M,g)\to(N,h)$ be a smooth map between Riemannian manifolds. The *tension field* of ϕ is defined by $\tau(\phi)=\operatorname{trace}\nabla d\phi$, and when M is compact, the *bienergy* of ϕ is $E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2\, v_g$. The critical points of E_2 are called *biharmonic maps*. Jiang obtained in [15,16] the first and the second variation formula for the bienergy, showing that the Euler-Lagrange equation corresponding to E_2 is

$$\tau_2(\phi) = -J^{\phi}(\tau(\phi)) = 0,$$

where $J^{\phi} = \triangle^{\phi} + \operatorname{trace} R^N(d\phi\cdot,\ )d\phi\cdot$ is the Jacobi operator associated to ϕ. It is well known that J^{ϕ} gives the second variation for the energy functional $E(\phi) = \frac{1}{2}\int_M |d\phi|^2\, v_g$ at its critical points, which are the harmonic maps (see [9,10]).

As the Jacobi operator is linear, harmonic maps are trivially biharmonic, so we are interested in non-harmonic biharmonic maps.

There exist several results concerning the conditions under which a biharmonic map is harmonic. Chen and Ishikawa proved in [8] that there

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exist no non-minimal biharmonic submanifolds of the Euclidian space $\mathbb{R}^3$ and, similarly, in [4], the authors proved the non-existence of such submanifolds in 3-dimensional constant negatively curved manifolds.

In [5,6], Caddeo, Montaldo and Piu determined the non-geodesic biharmonic curves on revolution surfaces in $\mathbb{R}^3$. The non-minimal biharmonic submanifolds of the 3-dimensional euclidian sphere $\mathbb{S}^3$ were classified in [3]: the only non-geodesic biharmonic curves are certain helices, and $\mathbb{S}^2(\frac{1}{\sqrt{2}})$ is the unique non-minimal biharmonic surface. Two methods for constructing non-minimal biharmonic submanifolds in $\mathbb{S}^n$, $n > 3$ were given in [4].

In [18] it was given a partial classification result for the non-minimal biharmonic submanifolds M in $\mathbb{S}^n$ in terms of the mean curvature $|H|$: if $|H| = \text{constant} > 1$, then M can not be biharmonic; if $|H| = 1$, then M is biharmonic if and only if M is pseudo-umbilical and $\nabla^{\perp} H = 0$, i.e. M is minimal in the hypersphere $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$. In particular, if M is a hypersurface of $\mathbb{S}^n$ with $|H| = 1$, then it is biharmonic if and only if $M = \mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$.

The biharmonic submanifolds into a space of nonconstant sectional curvature were also investigated. In [7] the authors obtained the parametric equations of all non-geodesic biharmonic curves of the Heisenberg group, and Inoguchi classified the biharmonic Legendre curves and the Hopf cylinders in 3-dimensional Sasakian space forms (see [13]). Then Sasahara gave in [20] the explicit representation of the non-minimal biharmonic Legendre surfaces in 5-dimensional Sasakian space forms.

As biharmonicity derives from a variational principle, the study of the second variation of the bienergy comes as a natural problem. By taking into account the non-existence of non-harmonic biharmonic maps from compact to non-positively curved manifolds, the second author obtained the second variation formula for the bienergy of biharmonic maps into $\mathbb{S}^n$ ([17]) and computed the biharmonic nullity for the identity map of $\mathbb{S}^n$ and for the totally geodesic inclusion of $\mathbb{S}^m$ in $\mathbb{S}^n$. Other results concerning the stability of certain classes of biharmonic maps into spheres were obtained in [14].

In [21,22], the author showed that non-minimal biharmonic Legendre curves and surfaces in Sasakian space forms are unstable.

Further results concerning the biharmonic Legendre surfaces and 3-dimensional anti-invariant submanifolds in Sasakian space forms were obtained in [1].

The goal of the present paper is to give a geometric approach for the non-geodesic biharmonic curves on $\mathbb{S}^3$ and to study the stability properties of biharmonic surfaces of $\mathbb{S}^3$. Using the stereographic projection we obtain the geometric image in $\mathbb{R}^3$ of the biharmonic curves in $\mathbb{S}^3$. Then, we study the variation of the bienergy for the biharmonic inclusions of $\mathbb{S}^2$ and $\mathbb{S}^2(\frac{1}{\sqrt{2}})$ in $\mathbb{S}^3$. As $\mathbb{S}^2$ and $\mathbb{S}^2(\frac{1}{\sqrt{2}})$ are projected in $\mathbb{S}^2$, respectively $\mathbb{S}^2(1+\sqrt{2})$, their normal variation in $\mathbb{S}^3$ becomes the radial variation in $\mathbb{R}^3$. This enables us to give a nice geometric interpretation of the bienergy variation, recovering some results obtained in [3,17,14].

Conventions. We place ourselves in C^∞ category, i.e. manifolds, metrics, connections, maps are assumed to be smooth. By (M^m, g) we indicate a connected manifold of dimension m, without boundary, endowed with a Riemannian metric g, by ∇ its Levi-Civita connection and for the curvature tensor we adopt the following sign convention $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. For $\phi : (M,g) \to (N,h)$ we denote by ∇^ϕ the induced connection in the pull-back bundle $\phi^{-1}TN$.

2. Biharmonic curves on $\mathbb{S}^3$. In [3] the authors studied and classified the non-minimal biharmonic submanifolds of $\mathbb{S}^3$. First, they proved that an arc-length parameterized curve $\gamma : I \to \mathbb{S}^3$ is non-geodesic biharmonic if and only if it is a helix, i.e. its curvature k and its torsion τ are both constant, and $k^2 + \tau^2 = 1$.

We recall here that, as in $\mathbb{R}^3$, two helices of the same curvature and torsion in $\mathbb{S}^3$ are congruent, i.e. there exists an isometry of $\mathbb{S}^3$ which takes one onto another (see, for example, [12]).

Theorem 2.1. [3] *Let $\gamma : I \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an arc-length parameterized curve. Then it is biharmonic on $\mathbb{S}^3$ if and only if, as a curve in $\mathbb{R}^4$, it satisfies*

$$\gamma^{iv} + 2\ddot{\gamma} + (1-k^2)\gamma = 0,$$

where $k = |\nabla^{\mathbb{S}^3}_T T|$ is the curvature of γ, $\nabla^{\mathbb{S}^3}$ is the Levi-Civita connection on $\mathbb{S}^3$ and $T = \dot{\gamma}$.

By integrating this equation it results

Theorem 2.2. [3] *Let $\gamma : I \to \mathbb{S}^3 \subset \mathbb{R}^4$ be a non-geodesic biharmonic arc-length parameterized curve. Then k is constant, $k \in (0,1]$ and, up to an isometry of $\mathbb{S}^3$, we have two cases:*

(A) *$k = 1$, and γ is the circle of radius $\frac{1}{\sqrt{2}}$;*

(B) $0 < k < 1$, *and* γ *is geodesic on the Clifford torus* $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$.

We can obtain Theorem 2.2 in a different manner, a geometric one: we shall study the conditions under which a curve on $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}})$ is biharmonic on $\mathbb{S}^3$.

Let

$$\begin{aligned} N &= \mathbb{S}^1(\tfrac{1}{\sqrt{2}}) \times \mathbb{S}^1(\tfrac{1}{\sqrt{2}}) = \\ &= \Big\{p = (x^1, x^2, y^1, y^2) : (x^1)^2 + (x^2)^2 = (y^1)^2 + (y^2)^2 = \frac{1}{2}\Big\}. \end{aligned}$$

The set of all vector fields tangent to N is given by

$$C(TN) = \big\{X = (X^1, X^2, Y^1, Y^2) : X^1x^1 + X^2x^2 = Y^1y^1 + Y^2y^2 = 0\big\}.$$

Let $p \in N$ and $\eta(p) = (x^1, x^2, -y^1, -y^2)$. Then η satisfies $\langle \eta, p\rangle = 0$, $\langle \eta, X\rangle = 0$ and $|\eta|^2 = 1$, i.e. η is a unit section in the normal bundle of N in $\mathbb{S}^3$.

For $X = (X^1, X^2, Y^1, Y^2) \in C(TN)$, we have $\nabla^{\perp}_X \eta = 0$ and $A_\eta X = (-X^1, -X^2, Y^1, Y^2)$, where A is the Weingarten operator, and $\nabla^{\perp}$ is the normal connection in the normal bundle of N in $\mathbb{S}^3$. We also recall that N is a flat, parallelized Riemannian manifold.

Indeed, $E_1(p) = (-x^2, x^1, 0, 0)$ and $E_2(p) = (0, 0, -y^2, y^1)$ are globally defined vector fields on N and they satisfy

$$|E_1|^2 = |E_2|^2 = \frac{1}{2}, \qquad \langle E_1, E_2\rangle = 0$$

and

$$\nabla^N_{E_i} E_j = 0, \quad \forall\, i, j = 1, 2.$$

The second fundamental form of N in $\mathbb{S}^3$ is given by

$$B(E_1, E_1) = -\frac{1}{2}\eta, \qquad B(E_2, E_2) = \frac{1}{2}\eta, \qquad B(E_1, E_2) = 0.$$

Let $\mathbf{i} : N \to \mathbb{S}^3$ be the inclusion map and for $\gamma : I \to N$, an arc-length parameterized curve on N, denote by $\widetilde{\gamma} = \mathbf{i} \circ \gamma : I \to \mathbb{S}^3$. We get

$$\tau(\widetilde{\gamma}) = \tau(\gamma) + c\eta,$$

where $T = \dot{\gamma} = f_1 E_1 + f_2 E_2$, $\tau(\gamma) = \nabla^N_T T = \dot{f}_1 E_1 + \dot{f}_2 E_2$ and $c = \frac{(f_2)^2 - (f_1)^2}{2}$.

In order to determine the bitension field of $\widetilde{\gamma}$, we have

$$\nabla^{\mathbb{S}^3}_T \tau(\widetilde{\gamma}) = \nabla^N_T \tau(\gamma) + c(f_1 E_1 - f_2 E_2) + \frac{3}{2}\dot{c}\eta,$$

$$\begin{aligned}\nabla^{\mathbb{S}^3}_T \nabla^{\mathbb{S}^3}_T \tau(\widetilde{\gamma}) &= \nabla^N_T \nabla^N_T \tau(\gamma) + ((c\dot{f_1}) + \frac{3}{2}\dot{c} f_1)E_1 - ((c\dot{f_2}) + \frac{3}{2}\dot{c} f_2)E_2 + \\ &+ \frac{1}{2}(-f_1 \ddot{f}_1 + f_2 \ddot{f}_2 + 3\ddot{c} - 2c)\eta\end{aligned}$$

and

$$R^{\mathbb{S}^3}(T, \tau(\widetilde{\gamma}))T = -\tau(\gamma) - c\eta,$$

so

$$\tau_2(\widetilde{\gamma})^{\perp} = \frac{1}{2}(-f_1 \ddot{f}_1 + f_2 \ddot{f}_2 + 3\ddot{c})\eta.$$

If $\widetilde{\gamma} : I \to \mathbb{S}^3$ is a biharmonic curve, then $\tau_2(\widetilde{\gamma})^{\perp} = 0$, i.e.

$$-f_1 \ddot{f}_1 + f_2 \ddot{f}_2 + 3\ddot{c} = 0. \tag{2.1}$$

From $2c = -(f_1)^2 + (f_2)^2$ and $(f_1)^2 + (f_2)^2 = 2$ we get $(f_1)^2 = 1 - c$, and $(f_2)^2 = 1 + c$.

If $\dot{c} = 0$, then f_1 and f_2 are constant, which implies that γ is a geodesic.

Assume now $\dot{c} \neq 0$.

As (2.1) is equivalent to

$$4\ddot{c} + \frac{c\,\dot{c}^2}{2(1 - c^2)} = 0,$$

dividing by $\dot{c}$ and integrating, we obtain

$$\frac{\dot{c}^{16}}{1 - c^2} = constant. \tag{2.2}$$

Since $\widetilde{\gamma}$ is biharmonic, its curvature is constant, so we have

$$k^2 = |\tau(\widetilde{\gamma})|^2 = \frac{(\dot{f}_1)^2 + (\dot{f}_2)^2}{2} + c^2 = \frac{\dot{c}^2}{4(1 - c^2)} + c^2 = constant.$$

Substituting $\dot{c}$ from (2.2) in the last equation we get a polinomial equation in c. So $c = constant$ and we have a contradiction.

Thus,

Proposition 2.3. *Let $\gamma : I \to N$ be an arc-length parameterized curve. If $\widetilde{\gamma} = \mathbf{i} \circ \gamma : I \to \mathbb{S}^3$ is non-geodesic biharmonic, then γ is a geodesic.*

Conversely, let $\gamma : I \to N$ be an arc-length parameterized geodesic on N. Writing

$$\gamma(t) = \frac{1}{\sqrt{2}}(\cos\alpha(t), \sin\alpha(t), \cos\beta(t), \sin\beta(t)),$$

we get $T = \dot{\gamma} = \dot{\alpha}E_1 + \dot{\beta}E_2$, and it results that $\dot{\alpha}$ and $\dot{\beta}$ are constant, i.e.

$$\gamma(t) = \frac{1}{\sqrt{2}}(\cos(a_1 t + b_1), \sin(a_1 t + b_1), \cos(a_2 t + b_2), \sin(a_2 t + b_2)),$$

where $(a_1)^2 + (a_2)^2 = 2$.

Further, we obtain

$$\tau(\widetilde{\gamma}) = \frac{(a_2)^2 - (a_1)^2}{2}\eta$$

and conclude that a geodesic on N is geodesic on $\mathbb{S}^3$ if and only if its slope is equal to ± 1.

Moreover, for $(a_1)^2 \neq (a_2)^2$, a straightforward computation shows that $\tau_2(\widetilde{\gamma}) = 0$.

We resume,

Theorem 2.4. *Let $\gamma : I \to N$ be an arc-length parameterized curve. Then $\widetilde{\gamma} = \mathbf{i} \circ \gamma : I \to \mathbb{S}^3$ is non-geodesic biharmonic if and only if γ is a geodesic of slope different from ± 1.*

Remark 2.5. If $(a_2)^2 - (a_1)^2 \in \{\pm 2\}$, then γ is the circle of radius $\frac{1}{\sqrt{2}}$.

Remark 2.6. As the curvature of the non-geodesic biharmonic curves $\widetilde{\gamma} = \mathbf{i} \circ \gamma : I \to \mathbb{S}^3$, $k = |\tau(\widetilde{\gamma})| = \frac{|(a_2)^2 - (a_1)^2|}{2}$, completely overlaps the interval $(0, 1]$, we can conclude that they are, up to an isometry, the unique non-geodesic biharmonic curves of S^3.

Let $pr : \mathbb{S}^3 \setminus \{(0,0,0,1)\} \to \mathbb{R}^3$, $pr(x^1, x^2, x^3, x^4) = \left(\frac{x^1}{1-x^4}, \frac{x^2}{1-x^4}, \frac{x^3}{1-x^4}\right)$, be the stereographic projection from $(0,0,0,1) \in \mathbb{S}^3$.

It transforms the Clifford torus, $r : [0, 2\pi] \times [0, 2\pi] \to \mathbb{S}^3$

$$r(u, v) = \frac{1}{\sqrt{2}}(\cos u, \sin u, \cos v, \sin v),$$

onto the torus in $\mathbb{R}^3$, $pr \circ r : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$

$$(pr \circ r)(u, v) = \Big(\frac{\cos u}{\sqrt{2} - \sin v}, \frac{\sin u}{\sqrt{2} - \sin v}, \frac{\cos v}{\sqrt{2} - \sin v}\Big).$$

In the following we shall present the geometric images through stereographic projection of the three classes of geodesics on the Clifford torus.

We noted that when the slope of the geodesic on N is ± 1, it is also a geodesic on $\mathbb{S}^3$. Such an example of geodesic is the curve

$$\gamma_0(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, \cos t, \sin t),$$

which, through stereographic projection becomes

$$(pr \circ \gamma_0)(t) = \Big(\frac{\cos t}{\sqrt{2} - \sin t}, \frac{\sin t}{\sqrt{2} - \sin t}, \frac{\cos t}{\sqrt{2} - \sin t}\Big),$$

and whose geometric image is presented in Figure 1.

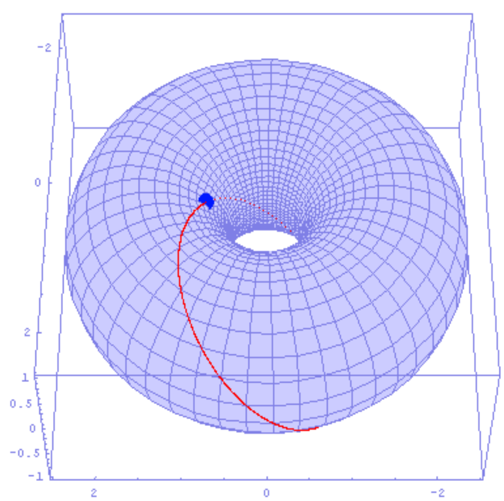

Figure 1: Geodesic on $\mathbb{S}^3$ (geodesic of slope 1 on the Clifford torus). Image through stereographic projection.

Considering the geodesic $\gamma_1 : I \to N$,

$$\gamma_1(t) = \frac{1}{\sqrt{2}}\Big(\cos 2\sqrt{\frac{2}{5}}t, \sin 2\sqrt{\frac{2}{5}}t, \cos\sqrt{\frac{2}{5}}t, \sin\sqrt{\frac{2}{5}}t\Big)$$

of rational, different from ± 1, slope we get a closed non-geodesic biharmonic curve on $\mathbb{S}^3$. We have

$$(pr \circ \gamma_1)(t) = \Big(\frac{\cos 2\sqrt{\frac{2}{5}}t}{\sqrt{2} - \sin \sqrt{\frac{2}{5}}}, \frac{\sin 2\sqrt{\frac{2}{5}}t}{\sqrt{2} - \sin \sqrt{\frac{2}{5}}}, \frac{\cos \sqrt{\frac{2}{5}}t}{\sqrt{2} - \sin \sqrt{\frac{2}{5}}} \Big)$$

and its image is presented in Figure 2.

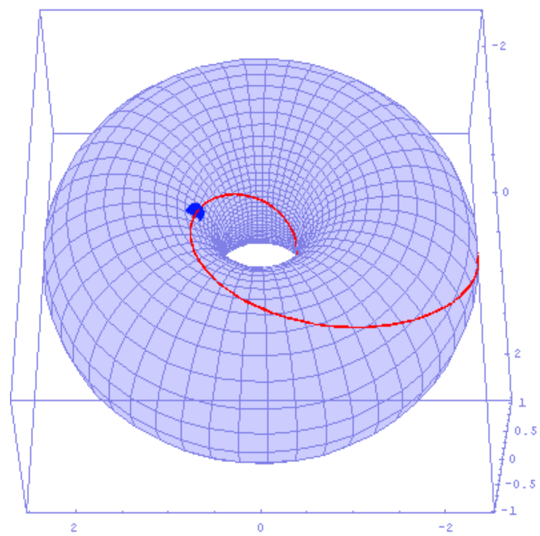

Figure 2: Closed non-geodesic biharmonic curve on $\mathbb{S}^3$ (geodesic of rational slope on the Clifford torus). Image through stereographic projection.

The curve

$$\gamma_2 : I \to N,\ \gamma_2 = \frac{1}{\sqrt{2}} \Big(\cos \sqrt{\frac{3}{2}}t, \sin \sqrt{\frac{3}{2}}t, \cos \sqrt{\frac{1}{2}}t, \sin \sqrt{\frac{1}{2}}t \Big)$$

is a geodesic of irrational slope, so dense, on the Clifford torus, and its image through stereographic projection,

$$(pr \circ \gamma_2)(t) = \Big(\frac{\cos \sqrt{\frac{3}{2}}t}{\sqrt{2} - \sin \sqrt{\frac{1}{2}}}, \frac{\sin \sqrt{\frac{3}{2}}t}{\sqrt{2} - \sin \sqrt{\frac{1}{2}}}, \frac{\cos \sqrt{\frac{1}{2}}t}{\sqrt{2} - \sin \sqrt{\frac{1}{2}}} \Big),$$

is presented in Figure 3.

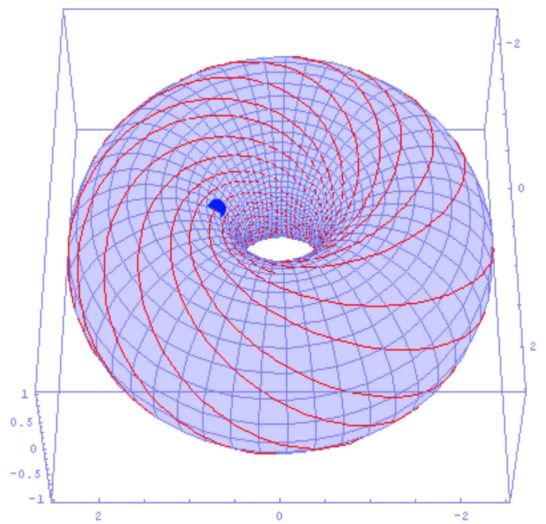

Figure 3: Non-geodesic biharmonic curve on $\mathbb{S}^3$ (geodesic of irrational slope, so dense, on the Clifford torus). Image through stereographic projection.

3. The stability of biharmonic submanifolds in $\mathbb{S}^3$. In [17] the author studied the second variation for the bienergy of biharmonic maps in spheres. Let $\phi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds. Suppose M is compact and orientable. We remind that *the bienergy* of ϕ is given by

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \ v_g,$$

and the bitension field $\tau_2(\phi)$ has the expression

$$\tau_2(\phi) = -\Delta^{\phi}\tau(\phi) - \operatorname{trace}\ R^N(d\phi\cdot, \tau(\phi))d\phi\cdot . \tag{3.1}$$

Suppose now $\phi : (M, g) \to \mathbb{S}^n$ is a biharmonic map and consider a two-parameter smooth variation of ϕ, $\{\phi_{s,t}\}_{s,t\in\mathbb{R}}$, i.e. consider the map Φ given by

$$\Phi : \mathbb{R} \times \mathbb{R} \times M \to \mathbb{S}^n, \ \Phi(s, t, p) = \phi_{s,t}(p),$$

where $\Phi(0, 0, p) = \phi_{0,0}(p) = \phi(p)$, $\forall p \in M$.

The variational vector fields corresponding to this variation, V and W, are given by

$$V(p) = \frac{d}{ds}\Big|_{s=0}\phi_{s,0}(p) = d\Phi_{(0,0,p)}\Big(\frac{\partial}{\partial s}\Big) \in T_{\phi(p)}\mathbb{S}^n,$$

and

$$W(p) = \frac{d}{dt}\Big|_{t=0}\phi_{0,t}(p) = d\Phi_{(0,0,p)}\Big(\frac{\partial}{\partial t}\Big) \in T_{\phi(p)}\mathbb{S}^n.$$

V and W are sections in $\phi^{-1}T\mathbb{S}^n$, i.e. $V, W \in C(\phi^{-1}T\mathbb{S}^n)$.

The Hessian of E_2 in its critical point ϕ is defined by

$$H(E_2)_\phi(V,W) = \frac{\partial^2}{\partial s \partial t}\Big|_{(s,t)=(0,0)} E_2(\phi_{s,t}).$$

The next result gives the explicit form of the Hessian for the bienergy.

Theorem 3.1. [17] *Let $\phi : (M,g) \to \mathbb{S}^n$ be a biharmonic map. Then the Hessian of the bienergy E_2 in ϕ is given by*

$$H(E_2)_\phi(V,W) = \int_M \langle I(V), W\rangle \ v_g,$$

where

(3.2)

$$\begin{aligned} I(V) &= \Delta^\phi(\Delta^\phi V) + \Delta^\phi\{\operatorname{trace}\langle V, d\phi\cdot\rangle d\phi\cdot - |d\phi|^2 V\} + \\ &\quad + 2\langle d\tau(\phi), d\phi\rangle V + |\tau(\phi)|^2 V - 2\operatorname{trace}\langle V, d\tau(\phi)\cdot\rangle d\phi\cdot - \\ &\quad - 2\operatorname{trace}\langle \tau(\phi), dV\cdot\rangle d\phi\cdot - \langle \tau(\phi), V\rangle \tau(\phi) + \\ &\quad + \operatorname{trace}\langle d\phi\cdot, \Delta^\phi V\rangle d\phi\cdot + \operatorname{trace}\langle d\phi\cdot, \operatorname{trace}\langle V, d\phi\cdot\rangle d\phi\cdot\rangle d\phi\cdot - \\ &\quad - 2|d\phi|^2 \operatorname{trace}\langle d\phi\cdot, V\rangle d\phi\cdot + 2\langle dV, d\phi\rangle\tau(\phi) - |d\phi|^2\Delta^\phi V + |d\phi|^4 V. \end{aligned}$$

Although the expression of the operator I is complicated, at least for some particular cases it becomes rather easy to study.

Theorem 3.2. [17] *The totally geodesic inclusion $\mathbf{i} : \mathbb{S}^m \to \mathbb{S}^n$ is biharmonic weakly-stable and we have*

a) *if $m = 2$, then* $\operatorname{nullity}(\mathbf{i}) = 3n$ *and*

$$\begin{aligned} \ker I &= \{X \in C(T\mathbb{S}^2) : X \text{ is Killing}\} \oplus \\ &\quad \oplus \{\operatorname{grad} f : f \in C^\infty(\mathbb{S}^2) \text{ and } \Delta f = \lambda_1 f\} \\ &\quad \bigoplus_{j=3}^{n} \{f\eta_j : f \in C^\infty(\mathbb{S}^2) \text{ and } \Delta f = \lambda_1 f\}. \end{aligned}$$

b) *if* $m>2$, *then* $\operatorname{nullity}(\mathbf{i})=(n-m)(m+1)+\frac{m(m+1)}{2}$ *and*

$$\begin{aligned}\ker I &= \{X\in C(T\mathbb{S}^m): X \textit{ is Killing}\}\\ &\quad \bigoplus_{j=m+1}^{n}\{f\eta_j : f\in C^\infty(\mathbb{S}^m) \textit{ and } \Delta f=\lambda_1 f\}.\end{aligned}$$

Here $\{\eta_{m+1},...,\eta_n\}$ *denotes a global parallel orthonormal frame in the normal bundle of* $\mathbb{S}^m$ *in* $\mathbb{S}^n$, *and* $\{\lambda_j\}_{j=0}^{\infty}$ *are the eigenvalues of the Laplacian on* $\mathbb{S}^m$.

Theorem 3.3. [14] *The biharmonic index of the canonical inclusion* $\mathbf{i}:\mathbb{S}^n(\frac{1}{\sqrt{2}})\to\mathbb{S}^{n+1}$ *is exactly* 1, *and its nullity is* $\frac{n(n+1)}{2}+n+1$. *Moreover, the subspace which gives the index is*

$$\{c\eta : c\in\mathbb{R}\}$$

and the subspace which gives the nullity is

$$\begin{aligned}\ker I &= \{X\in C(T\mathbb{S}^n(\tfrac{1}{\sqrt{2}})): X \textit{ is Killing}\}\oplus\\ &\quad \oplus\{2f\eta+\operatorname{grad} f : f\in C^\infty(\mathbb{S}^n(\tfrac{1}{\sqrt{2}})),\Delta f=\lambda_1 f\}.\end{aligned}$$

Here $\eta(p)=(p,-\frac{1}{\sqrt{2}})$ *is a unit section in the normal bundle of* $\mathbb{S}^n(\frac{1}{\sqrt{2}})$ *in* $\mathbb{S}^{n+1}$.

Proof. We shall present here a sketch of a proof somewhat different from that presented in [14].

The idea is to split $C(\mathbf{i}^{-1}T\mathbb{S}^{n+1})$ into a sum of (finite dimensional) subspaces, mutually orthogonal and I-invariant. Then we shall compute the index and nullity of I restricted to these subspaces. We have

$$\begin{aligned}C(\mathbf{i}^{-1}T\mathbb{S}^{n+1}) &= \{f\eta : f\in C^\infty(\mathbb{S}^n(\tfrac{1}{\sqrt{2}}))\}\oplus\{X : X\in C(T\mathbb{S}^n(\tfrac{1}{\sqrt{2}}))\}=\\ &= \{f\eta : f\in C^\infty(\mathbb{S}^n(\tfrac{1}{\sqrt{2}}))\}\oplus\{\operatorname{grad} f : f\in C^\infty(\mathbb{S}^n(\tfrac{1}{\sqrt{2}}))\}\oplus\\ &\quad \oplus\{X\in C(T\mathbb{S}^n(\tfrac{1}{\sqrt{2}})) : \operatorname{div} X=0\}=\\ &= \bigoplus_{j=0}^{\infty}\{f\eta : \Delta f=\lambda_j f\}\oplus\bigoplus_{j=1}^{\infty}\{\operatorname{grad} f : \Delta f=\lambda_j f\}\oplus\\ &\quad \oplus\{X : \operatorname{div} X=0\}=\\ &= S_0\oplus\bigoplus_{j=1}^{\infty}S_j\oplus S,\end{aligned}$$

where $S_0 = \{c\eta : c \in \mathbb{R}\}$, $S_j = \{f\eta : \Delta f = \lambda_j f\} \oplus \{\operatorname{grad} f : \Delta f = \lambda_j f\}$ and $S = \{X : \operatorname{div} X = 0\}$.

By a straightforward computation we obtain

$$I(S_0) \subset S_0, \qquad I(S_j) \subset S_j, \forall j \geq 1, \qquad I(S) \subset S,$$

and these spaces are mutually orthogonal.

Moreover,

- $(I(\eta), \eta) = -4n^2 \int_{\mathbb{S}^n(\frac{1}{\sqrt{2}})} v_g < 0$
- $(I(X), X) \geq 0$, $\forall X \in S$, with equality if and only if X is Killing
- $(I(V), V) > 0$, $\forall V \in S_j, j \geq 2$
- $(I(V), V) \geq 0$, $\forall V \in S_1$, with equality if and only if $V = 2f\eta + \operatorname{grad} f$, $\Delta f = \lambda_1 f$.

In order to prove the last two statements, let $\{f_1, ..., f_s\}$ be an L^2-orthonormal basis in $\{f : \Delta f = \lambda_j f\}$, $j \geq 1$. Then

$$B_j = \left\{ f_1\eta, \dots, f_s\eta, \frac{1}{\sqrt{\lambda_j}} \operatorname{grad} f_s, \dots, \frac{1}{\sqrt{\lambda_j}} \operatorname{grad} f_1 \right\}$$

is an L^2-orthonormal basis in S_j. With respect to B_j, I restricted to S_j has the associated matrix

$$\left(\begin{array}{ccccc|ccccc}
a & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & d \\
0 & a & \dots & 0 & 0 & 0 & 0 & \dots & d & 0 \\
\vdots & & & & & \vdots & & & & \\
0 & 0 & \dots & a & 0 & 0 & d & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & a & d & 0 & \dots & 0 & 0 \\
\hline
0 & 0 & \dots & 0 & d & b & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & d & 0 & 0 & b & \dots & 0 & 0 \\
\vdots & & & & & \vdots & & & & \\
0 & d & \dots & 0 & 0 & 0 & 0 & \dots & b & 0 \\
d & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & b
\end{array}\right),$$

where

$$a = (I(f_l\eta), f_l\eta) = \lambda_j^2 + 4\lambda_j - 4n^2, \forall\, l = \overline{1, s}$$

$$b=\Big(I\big(\frac{1}{\sqrt{\lambda_j}}\operatorname{grad} f_l\big),\frac{1}{\sqrt{\lambda_j}}\operatorname{grad} f_l\Big)=\lambda_j^2+2(n+2-2e)\lambda_j+4e^2-4en, \forall\, l=\overline{1,s}$$

$$d=\Big(I(f_l\eta),\frac{1}{\sqrt{\lambda_j}}\operatorname{grad} f_l\Big)=4\sqrt{\lambda_j}(e-\lambda_j), \forall\, l=\overline{1,s}$$

and $e=2(n-1)$.

This matrix has two real eigenvalues of multiplicity s, x_1 and x_2, which satisfy

$$\begin{cases} x_1x_2=ab-d^2\\ x_1+x_2=a+b>0.\end{cases}$$

If $j>1$, then $ab-d^2>0$, thus $x_1>0$ and $x_2>0$, i.e. I restricted to S_j is positive definite.

If $j=1$, then $ab-d^2=0$, thus $x_1=0$ and $x_2>0$, i.e. I restricted to S_1 is non-negative definite and its nullity is $s=n+1$. □

The stability of the inclusions $\mathbb{S}^2\to\mathbb{S}^3$ and $\mathbb{S}^2(\frac{1}{\sqrt{2}})\to\mathbb{S}^3$ can be studied by using the stereographic projection and by analyzing the problem in $\mathbb{R}^3$. The normal variation in $\mathbb{S}^3$ corresponds, in this context, to the radial variation in $\mathbb{R}^3$. In the following we shall only study the variation along the normal direction.

For the study of the first case, let $\phi:\mathbb{S}^2\to\mathbb{S}^3\setminus\{(0,0,0,1)\}$ be a smooth map. Through the stereographic projection we can identify ϕ with

$$\phi:(\mathbb{S}^2,e^{2\rho\circ\mathbf{i}}\langle\,,\rangle)\to(\mathbb{R}^3,e^{2\rho}\langle\,,\rangle),$$

where $\langle\,,\rangle$ is the canonic metric on $\mathbb{R}^3$, $\mathbf{i}:\mathbb{S}^2\to\mathbb{S}^3$ is the inclusion map and $\rho:\mathbb{R}^3\to\mathbb{R}$, $\rho(p)=\ln\frac{2}{|p|^2+1}$.

Let $\widetilde{\phi}:(\mathbb{S}^2,\langle\,,\rangle)\to(\mathbb{R}^3,\langle\,,\rangle)$ such that $\phi=\mathbf{1}\circ\widetilde{\phi}\circ\mathbf{1}$, where $\mathbf{1}$ denotes the identities $\mathbf{1}:(\mathbb{R}^3,\langle\,,\rangle)\to(\mathbb{R}^3,e^{2\rho}\langle\,,\rangle)$ and $\mathbf{1}:(\mathbb{S}^2,e^{2\rho\circ\mathbf{i}}\langle\,,\rangle)\to(\mathbb{S}^2,\langle\,,\rangle)$.

Denoting $\sigma=\rho\circ\mathbf{i}$ and $\widetilde{\sigma}=\rho\circ\phi$, we can describe this situation in the following diagram

$$\begin{tikzcd}
(\mathbb{S}^2,\langle\,,\,\rangle) \arrow[r, "\phi"] \arrow[d, "pr"'] & (\mathbb{S}^3\setminus\{(0,0,0,1)\},\langle\,,\,\rangle) \arrow[d, "pr"] \\
(\mathbb{S}^2,e^{2\sigma}\langle\,,\,\rangle) \arrow[r, "\phi"] \arrow[d, "\mathbf{1}"'] & (\mathbb{R}^3,e^{2\rho}\langle\,,\,\rangle) \\
(\mathbb{S}^2,\langle\,,\,\rangle) \arrow[r, "\widetilde{\phi}"] & (\mathbb{R}^3,\langle\,,\,\rangle) \arrow[u, "\mathbf{1}"']
\end{tikzcd}$$

The tension field of ϕ has the expression

$$\tau(\phi) = e^{-2\sigma}\big\{\tau(\widetilde{\phi}) + 2d\widetilde{\phi}(\operatorname{grad}\widetilde{\sigma}) - |d\widetilde{\phi}|^2(\operatorname{grad}\rho)\circ\widetilde{\phi}\big\}$$

and the bienergy

$$E_2(\phi) = \frac{1}{2}\int_{\mathbb{S}^2} e^{2\widetilde{\sigma}}|\tau(\phi)|^2 \ v_{e^{2\sigma}\langle\,,\rangle},$$

can be written as

$$\begin{aligned} E_2(\phi) \ &= \frac{1}{2}\int_{\mathbb{S}^2} e^{2(\widetilde{\sigma}-\sigma)}\Big|\tau(\widetilde{\phi}) + 2d\widetilde{\phi}(\operatorname{grad}\widetilde{\sigma}) - \\ &\quad -|d\widetilde{\phi}|^2(\operatorname{grad}\rho)\circ\widetilde{\phi}\Big|^2 v_{\langle\,,\rangle}. \end{aligned} \tag{3.3}$$

Consider now ϕ to be

$$\phi_t(p) = (1+tf)\mathbf{i}(p),$$

i.e. a normal variation of the map $\mathbf{i}$, with $f\in C^\infty(\mathbb{S}^2)$ and $\Delta f = \lambda f$.

In the following we shall determine the bienergy of ϕ_t. We have

$$\tau(\widetilde{\phi}_t) = -t\lambda f\mathbf{i} - 2(1+tf)\mathbf{i} + 2t\operatorname{grad} f,$$

$$\widetilde{\sigma}_t = \rho\circ\phi_t = \ln\frac{2}{(1+tf)^2+1},$$

$$\operatorname{grad}(\widetilde{\sigma}_t) = -\frac{2t(1+tf)}{(1+tf)^2+1}\operatorname{grad} f,$$

$$d\widetilde{\phi}_t(\operatorname{grad}\widetilde{\sigma}) = -\frac{2t(1+tf)}{(1+tf)^2+1}\big(t|\operatorname{grad} f|^2\mathbf{i} + (1+tf\operatorname{grad} f)\big),$$

$$|d\widetilde{\phi}_t|^2 = t^2|\operatorname{grad} f|^2 + 2(1+tf)^2$$

and

$$(\operatorname{grad}\rho)\circ\widetilde{\phi}_t = -\frac{2(1+tf)}{(1+tf)^2+1}\mathbf{i}.$$

By replacing these expressions in (3.3) we obtain

$$\begin{aligned} E_2(\phi_t) \ &= \int_{\mathbb{S}^2} 2t^2\Big\{\frac{\big(-2(1+ft)(2f+f^2t-\alpha^2t)+f(2+ft(2+ft))\lambda\big)^2}{\big(2+ft(2+ft)\big)^4} + \\ &\quad + \frac{4f^2\alpha^2t^2(2+f\ \ t)^2}{\big(2+ft(2+ft)\big)^4}\Big\}\ v_{\langle\,,\rangle} \end{aligned} \tag{3.4}$$

where $\alpha^2 = |\operatorname{grad} f|^2$.

From (3.4) we get

$$E_2(\phi_0) = 0, \quad \frac{d}{dt}\Big|_{t=0} E_2(\phi_t) = 0$$

and

$$\frac{d^2}{dt^2}\Big|_{t=0} E_2(\phi_t) = (I(f\mathbf{i}), f\mathbf{i}) = (-2+\lambda)^2 \int_{\mathbb{S}^2} f^2 v_{\langle , \rangle} \geq 0.$$

As $\mathbf{i} : \mathbb{S}^2 \to \mathbb{S}^3$ is totally geodesic, it is an absolute minimum for the bienergy, so weakly-stable as a biharmonic map. We also note here that the first non-zero eigenvalue of the Laplacian on $\mathbb{S}^2$, $\lambda_1 = 2$, contributes to the nullity of this map and the nullity provided by the normal variations is 3 (the dimension of the eigenspace corresponding to λ_1), remark conformal to Theorem 3.2.

We shall now analyze the case of $\lambda_0 = 0$ and of the corresponding eigenfunction $f = 1$. In this case

$$E_2(\phi_t) = \frac{32\pi t^2(1+t)^2(2+t)^2}{\big(2+t(2+t)\big)^4}$$

and its graphic is presented in Figure 4.

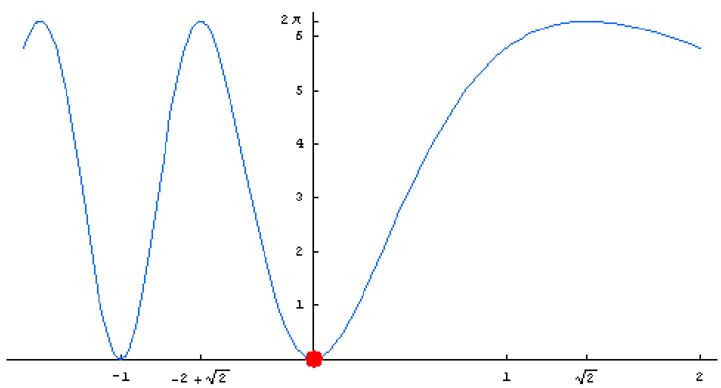

Figure 4: The bienergy $E_2(\phi_t)$ corresponding to $\lambda_0 = 0$, for $\mathbf{i} : \mathbb{S}^2 \to \mathbb{S}^3$

The critical points of the bienergy, i.e. the solutions of

$$\frac{d}{dt} E_2(\phi_t) = -\frac{64\pi t(-8 - 28t - 28t^2 + 14t^4 + 7t^5 + t^6)}{\big(2+t(2+t)\big)^5} = 0,$$

in $[-1,\infty)$ are $-1, 0, \sqrt{2}, -2+\sqrt{2}$. As we can notice, among these -1 and 0 are minimum points: -1 corresponds to the constant map, which is trivially biharmonic, and 0 corresponds to the inclusion $\mathbf{i}:\mathbb{S}^2\to\mathbb{S}^3$. Among the maximum points of the bienergy for this variation, only $\sqrt{2}$ corresponds, up to a homothetic transformation, to the biharmonic submanifold $\mathbb{S}^2(\frac{1}{\sqrt{2}})$ in $\mathbb{S}^3$.

For the case of $\mathbb{S}^2(\frac{1}{\sqrt{2}})$, if $\phi:\mathbb{S}^2(\frac{1}{\sqrt{2}})\to\mathbb{S}^3\setminus\{(0,0,0,1)\}$ is a smooth map, through stereographic projection we can identify it with

$$\phi:(\mathbb{S}^2(1+\sqrt{2}), e^{2\rho\circ\mathbf{i}}\langle\,,\rangle)\to(\mathbb{R}^3, e^{2\rho}\langle\,,\rangle),$$

where $\langle\,,\rangle$ is the canonic metric on $\mathbb{R}^3$, $\mathbf{i}:\mathbb{S}^2(\frac{1}{\sqrt{2}})\to\mathbb{S}^3$ denotes the inclusion map and $\rho:\mathbb{R}^3\to\mathbb{R}$, $\rho(p)=\ln\frac{2}{|p|^2+1}$.

Let $\widetilde{\phi}:(\mathbb{S}^2(1+\sqrt{2}),\langle\,,\rangle)\to(\mathbb{R}^3,\langle\,,\rangle)$ be such that $\phi=\mathbf{1}\circ\widetilde{\phi}\circ\mathbf{1}$, where $\mathbf{1}$ denotes the identities $\mathbf{1}:(\mathbb{R}^3,\langle\,,\rangle)\to(\mathbb{R}^3,e^{2\rho}\langle\,,\rangle)$ and
$\mathbf{1}:(\mathbb{S}^2(1+\sqrt{2}),e^{2\rho\circ\mathbf{i}}\langle\,,\rangle)\to(\mathbb{S}^2(1+\sqrt{2}),\langle\,,\rangle)$.
Denote by $\sigma=\rho\circ\mathbf{i}$ and $\widetilde{\sigma}=\rho\circ\phi$. We get the diagram

$$\begin{array}{ccc}
(\mathbb{S}^2(\frac{1}{\sqrt{2}}),\langle\,,\,\rangle) & \xrightarrow{\phi} & (\mathbb{S}^3\setminus\{(0,0,0,1)\},\langle\,,\,\rangle)\\
{\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle pr}\\
(\mathbb{S}^2(1+\sqrt{2}),e^{2\sigma}\langle\,,\,\rangle) & \xrightarrow{\phi} & (\mathbb{R}^3,e^{2\rho}\langle,\,\rangle)\\
{\scriptstyle \mathbf{1}}\downarrow & & \uparrow{\scriptstyle \mathbf{1}}\\
(\mathbb{S}^2(1+\sqrt{2}),\langle\,,\,\rangle) & \xrightarrow{\widetilde{\phi}} & (\mathbb{R}^3,\langle\,,\,\rangle)
\end{array}$$

and

$$E_2(\phi)=\frac{1}{2}\int_{\mathbb{S}^2(1+\sqrt{2})} e^{2\widetilde{\sigma}}|\tau(\phi)|^2\ v_{e^{2\sigma}\langle\,,\rangle}$$

becomes
(3.5)
$$E_2(\phi)=\frac{1}{2}\int_{\mathbb{S}^2(1+\sqrt{2})} e^{2(\widetilde{\sigma}-\sigma)}\Big|\tau(\widetilde{\phi})+2d\widetilde{\phi}(\operatorname{grad}\widetilde{\sigma})-|d\widetilde{\phi}|^2(\operatorname{grad}\rho)\circ\widetilde{\phi}\Big|^2 v_{\langle\,,\rangle}.$$

Let now ϕ be a normal variation of the inclusion map $\mathbf{i}:\mathbb{S}^2(1+\sqrt{2})\to\mathbb{R}^3$,

$$\phi_t(p)=(1+\sqrt{2}tf)\mathbf{i}(p),$$

where $f \in C^\infty(\mathbb{S}^2(1+\sqrt{2}))$ with $\Delta f = \lambda f$, Δ being the usual Laplacian on $\mathbb{S}^2(1+\sqrt{2})$.

In this case

$$\tau(\widetilde{\phi}_t) = -t\sqrt{2}\lambda f \mathbf{i} - \frac{2(1+\sqrt{2}tf)}{(1+\sqrt{2})^2}\mathbf{i} + 2\sqrt{2}t \operatorname{grad} f,$$

$$\widetilde{\sigma}_t = \rho \circ \phi_t = \ln \frac{2}{(1+\sqrt{2}tf)^2(1+\sqrt{2})^2+1},$$

$$\operatorname{grad}(\widetilde{\sigma}_t) = -\frac{2\sqrt{2}t(1+\sqrt{2})^2(1+\sqrt{2}tf)}{(1+\sqrt{2}tf)^2(1+\sqrt{2})^2+1} \operatorname{grad} f,$$

$$d\widetilde{\phi}_t(\operatorname{grad}\widetilde{\sigma}_t) = -\tfrac{2\sqrt{2}t(1+\sqrt{2})^2(1+\sqrt{2}tf)}{(1+\sqrt{2}tf)^2(1+\sqrt{2})^2+1}\{(1+\sqrt{2}tf)\operatorname{grad} f + \sqrt{2}\,t|\operatorname{grad} f|^2\mathbf{i}\},$$

$$|d\widetilde{\phi}_t|^2 = 2t^2(1+\sqrt{2})^2|\operatorname{grad} f|^2 + 2(1+\sqrt{2}tf)^2,$$

$$(\operatorname{grad}\rho)\circ\widetilde{\phi}_t = -\frac{2(1+\sqrt{2}tf)}{(1+\sqrt{2}\,tf)^2(1+\sqrt{2})^2+1}\mathbf{i},$$

and by replacing these in (3.5) we get the expression for the bienergy of ϕ_t,

$$\begin{aligned} E_2(\phi_t) = \int_{\mathbb{S}^2(1+\sqrt{2})} \Big\{ \Big(2(6+4\sqrt{2}) \Big(\alpha^2 \Big(2\sqrt{2}t - \tfrac{4\sqrt{2}(1+\sqrt{2})^2 t(1+\sqrt{2}ft)^2}{1+(1+\sqrt{2})^2(1+\sqrt{2}ft)^2} \Big)^2 + \\ + (3+2\sqrt{2}) \Big(\tfrac{2(1+\sqrt{2}ft)}{(1+\sqrt{2})^2} - \tfrac{4(1+\sqrt{2}ft)((3+2\sqrt{2})\alpha^2t^2+(1+\sqrt{2}ft)^2)}{1+(3+2\sqrt{2})(1+\sqrt{2}ft)^2} + \\ + \sqrt{2}ft\lambda + \tfrac{8(3+2\sqrt{2})\alpha^2t^2(1+\sqrt{2}ft)}{1+(3+2\sqrt{2})(1+\sqrt{2}ft)^2} \Big)^2 \Big) \Big) \Big/ (1+(3+2\sqrt{2})(1+\sqrt{2}ft)^2)^2 \Big\}\, v_{\langle\,,\rangle}, \end{aligned}$$

where $\alpha^2 = |\operatorname{grad} f|^2$.
From this expression we note that $t=0$ gives a critical point for the bienergy,

$$\frac{d}{dt}\Big|_{t=0} E_2(\phi_t) = -2\int_{\mathbb{S}^2(1+\sqrt{2})} \lambda f\ v_{\langle\,,\rangle} = -2\int_{\mathbb{S}^2(1+\sqrt{2})} \Delta f\ v_{\langle\,,\rangle} = 0.$$

Furthermore,

$$\frac{d^2}{dt^2}\Big|_{t=0} E_2(\phi_t) = (I(f\sqrt{2}\,\mathbf{i}), f\sqrt{2}\,\mathbf{i}) = (\mu^2+4\mu-16)\int_{\mathbb{S}^2(1+\sqrt{2})} f^2 v_{e^{2\sigma}\langle\,,\rangle},$$

with $\mu = e^{-2\sigma}\lambda$, result conformal to Proposition 2.1 in [14].

For $\lambda_0 = 0$ and $f = 1$ we get

$$E_2(\phi_t) = \frac{8\pi(1+t)^2\big(4+\sqrt{2}(1+2t)(3+5t)+2t(8+7t)\big)^2}{(2+4t+3t^2+\sqrt{2}(1+t)(1+2t))^4},$$

so

$$E_2(\phi_0) = 4\pi > 0, \quad \frac{d}{dt}\Big|_{t=0} E_2(\phi_t) = 0$$

and

$$\frac{d^2}{dt^2}\Big|_{t=0} E_2(\phi_t) = (I(\sqrt{2}\,\mathbf{i}), \sqrt{2}\,\mathbf{i}) = -32\pi < 0.$$

The graphic of the bienergy shows that in the critical points obtained for $t = -2+\sqrt{2}$ and $t = 0$, the bienergy assumes the same value. These are maximum points for the bienergy, but only $\phi_0 = \mathbf{i}$ is a biharmonic map. Also, the critical point obtained for $t = 1-\sqrt{2}$ is an absolute minimum for the bienergy, the corresponding map being, up to a homothetic transformation, the inclusion of $\mathbb{S}^2$ into $\mathbb{S}^3$.

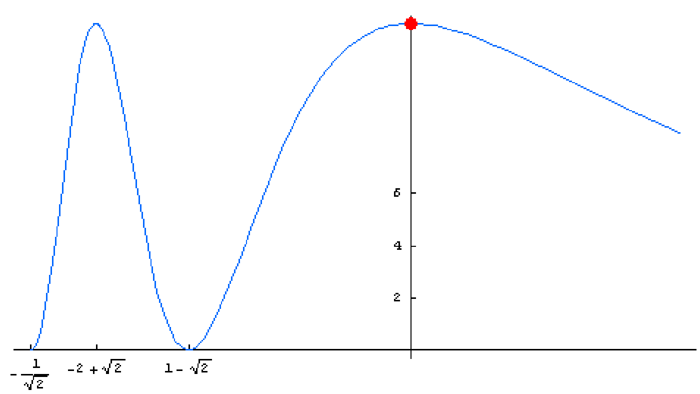

Figure 5: The bienergy $E_2(\phi_t)$ corresponding to $\lambda_0 = 0$, for the inclusion $\mathbf{i} : \mathbb{S}^2(\frac{1}{\sqrt{2}}) \to \mathbb{S}^3$

For the first non-zero eigenvalue λ_1,

$$\frac{d}{dt}\Big|_{t=0} E_2(\phi_t) = 0,$$

but

$$\frac{d^2}{dt^2}\Big|_{t=0} E_2(\phi_t) = (24-16\sqrt{2}) \int_{\mathbb{S}^2(1+\sqrt{2})} f^2\, v_{\langle\,,\rangle} > 0.$$

So the variations corresponding to this eigenvalue do not contribute to the index of the biharmonic map in question, remark conformal to Theorem 3.3.

We mention that the figures and some of the computations from the last part of this section are made with Mathematica 4.0.

REFERENCES

1. Arslan, K.; Ezentas, R.; Murathan, C.; Sasahara, T. – 3-*dimensional biharmonic anti-invariant submanifolds in Sasakian space forms*, preprint.
2. Arslan, K.; Ezentas, R.; Murathan, C.; Sasahara, T. – *Biharmonic submanifolds in generalized* (k, μ)*-manifolds*, preprint.
3. Caddeo, R.; Montaldo, S.; Oniciuc, C. – *Biharmonic submanifolds of* $\mathbb{S}^3$, Internat. J. Math., 12 (2001), no. 8, pp. 867-876.
4. Caddeo, R.; Montaldo, S.; Oniciuc, C. – *Biharmonic submanifolds in spheres*, Israel J. Math., 130 (2002), pp. 109-123.
5. Caddeo, R.; Montaldo, S.; Piu, P. – *Biharmonic curves on a surface*, Rendiconti di Matematica e delle sue Applicazioni, Serie VII, Volume 21, Roma (2001), pp. 143-157.
6. Caddeo, R.; Montaldo, S.; Piu, P. – *On Biharmonic Maps*, Contemporary Mathematics, Volume 288(2001), pp. 286-290.
7. Caddeo, R.; Oniciuc, C.; Piu, P. – *Explicit formulas for non-geodesic biharmonic curves of the Heisenberg group*, Rend. Sem. Mat. Univ. Politec. Torino, 62 (2004), no. 3, pp. 265-277.
8. Chen, B.Y.; Ishikawa, S. – *Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces*, Kyushu J. Math. 52 (1998), pp. 167-185.
9. Eells, J.; Lemaire, L. – *Selected Topics in Harmonic Maps*, Conf. Board Math. Sci 50 (1983).
10. Eells, J.; Lemaire, L. – *A Report on harmonic maps*, Bull. London Math. Soc., 10 (1978), pp. 1-68.
11. Gheorghiev, Gh.; Oproiu, V. – *Varietăţi diferenţiabile finit şi infinit dimensionale*, Vol. 2, Editura Academiei Române, 1978.
12. Gluck, H. – *Geodesics in the unit tangent bundle of a round sphere*, L'Enseignement Mathématique, 34(1988), pp. 233-246.
13. Inoguchi, J. – *Submanifolds with harmonic mean curvature in contact 3-manifolds*, Colloq. Math. 100 (2004), pp. 163-179.
14. Loubeau, E.; Oniciuc, C. – *The index of biharmonic maps in sferes*, Compositio Math., 141 (2005), 3, pp. 729-745.

15. JIANG, G.Y. – *2-harmonic isometric immersions between Riemannian manifolds*, Chinese Ann. Math. Ser. A 7 (1986), no 2, pp. 130-144.

16. JIANG, G.Y. – *2-harmonic maps and their first and second variation formulas*, Chinese Ann. Math. Ser. A 7 (1986), no 4, pp. 389-402.

17. ONICIUC, C. – *On the second variation formula for biharmonic maps to a sphere*, Publ. Math. Debrecen, 61/3-4 (2002), pp. 613-622.

18. ONICIUC, C. – *Tangency and Harmonicity Properties* (in romanian), PhD Thesis, Geometry Balkan Press 2003, http://vectron.mathem.pub.ro.

19. OPROIU, V. – *Geometrie diferenţială*, Editura Universităţii "Al.I. Cuza" Iaşi, 2002.

20. SASAHARA, T. – *Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors*, Publ. Math. Debrecen, to appear.

21. SASAHARA, T. – *Instability of biharmonic Legendre surfaces in Sasakian space forms*, preprint.

22. SASAHARA, T. – *Stability of biharmonic Legendre submanifolds in Sasakian space forms*, preprint.

23. URAKAWA, H. – *Calculus of Variations and Harmonic Maps*, Translations of Mathematical Monographs, 1993, Vol. 132.

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