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GROWTH ESTIMATES ON CERTAIN DISCRETE INEQUALITIES INVOLVING ITERATED SUMS

BY

B.G. PACHPATTE

Abstract. In this paper growth estimates on certain discrete inequalities involving iterated sums are established. Applications are also given to illustrate the usefulness of one of our results.

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1. Introduction. In the book of Filatov and Sarova [4] (see, also [1,3]) the growth estimates on the following integral inequalities, due to Bykov and Salpagarov [2]:

$$u(t) \leq c + \int_{\alpha}^{t} k(t,\tau)\, u(\tau)\, d\tau + \int_{\alpha}^{t} \left(\int_{\alpha}^{s} h(t,s,\sigma)\, u(\sigma)\, d\sigma \right) ds, \tag{1.1}$$

$$\begin{aligned} u(t) \leq\; & c + \int_{\alpha}^{t} b(s)\, u(s)\, ds + \int_{\alpha}^{t} \left(\int_{\alpha}^{s} k(s,\tau)\, u(\tau)\, d\tau \right) ds + \\ & + \int_{\alpha}^{t} \left(\int_{\alpha}^{s} \left(\int_{\alpha}^{\tau} h(s,\tau,\sigma)\, u(\sigma)\, d\sigma \right) d\tau \right) ds, \end{aligned} \tag{1.2}$$

are given, under some suitable conditions on the functions involved in (1.1), (1.2). The estimates obtained on the above inequalities can be used as tools in certain applications for which the other inequalities do not apply directly. It is natural to expect that the growth estimates on the discrete versions of the above inequalities would be equally important in the study of certain finite difference equations. The main purpose of this paper is to establish the growth estimates on the discrete versions of the inequalities (1.1), (1.2) and their generalizations which can be used as handy tools in the study of certain classes of finite difference equations. Some applications of one of our results to study the behavior of solutions of a certain finite difference equation involving iterated sums are also given.

2. Statement of results. In what follows, R denotes the set of real numbers and $R_+ = [0, \infty)$, $N_0 = \{0, 1, 2, ...\}$ are the given subsets of R. We denote by $D = \{(n, s) \in N_0^2 : 0 \leq s \leq n < \infty\}$, $E = \{(n, s, \sigma) \in N_0^3 : 0 \leq \sigma \leq s \leq n < \infty\}$, and for any function $p(n)$, $n \in N_0$, $z(m, n)$, $m, n \in N_0$, denote $\Delta p(n) = p(n+1) - p(n)$, $\Delta_m z(m, n) = z(m+1, n) - z(m, n)$. We use the usual conventions that the empty sums and products are taken to be 0 and 1 respectively and assume that all the sums and products involved throughout the discussion exist on the respective domains of their definitions.

Our main results are established in the following theorems.

Theorem 1. *Let $u(n) : N_0 \to R_+$, $k(n, s), \Delta_n k(n, s) : D \to R_+$, $h(n, s, \sigma), \Delta_n h(n, s, \sigma) : E \to R_+$ and $c \geq 0$ is a constant.*

(a_1) If

$$u(n) \leq c + \sum_{s=0}^{n-1} k(n, s) u(s) + \sum_{s=0}^{n-1} \left(\sum_{\sigma=0}^{s-1} h(n, s, \sigma) u(\sigma) \right), \tag{2.1}$$

for $n \in N_0$, then

$$u(n) \leq c \prod_{s=0}^{n-1} [1 + A(s) + B(s)], \tag{2.2}$$

for $n \in N_0$, where

$$A(n) = k(n+1, n) + \sum_{\sigma=0}^{n-1} h(n+1, n, \sigma), \tag{2.3}$$

$$B(n)=\sum_{\tau=0}^{n-1}\Delta_n k(n,\tau)+\sum_{\tau=0}^{n-1}\left(\sum_{\sigma=0}^{\tau-1}\Delta_n h(n,\tau,\sigma)\right). \tag{2.4}$$

(a_2) Let $g(u)$ be a nondecreasing continuous function defined on R_+ with $g(u)>0$ for $u>0$. If

$$u(n)\le c+\sum_{s=0}^{n-1}k(n,s)\,g(u(s))+\sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1}h(n,s,\sigma)\,g(u(\sigma))\right), \tag{2.5}$$

for $n\in N_0$, then for $0\le n\le n_1,\ n,n_1\in N_0$,

$$u(n)\le G^{-1}\left[G(c)+\sum_{s=0}^{n-1}[A(s)+B(s)]\right], \tag{2.6}$$

where $A(n),B(n)$ are given by (2.3), (2.4),

$$G(r)=\int_{r_0}^{r}\frac{dt}{g(t)},\ \ r>0, \tag{2.7}$$

$r_0>0$ is arbitrary, G^{-1} is the inverse of G and $n_1\in N_0$, be chosen so that

$$G(c)+\sum_{s=0}^{n-1}[A(s)+B(s)]\in Dom\left(G^{-1}\right),$$

for all $n\in N_0$ such that $0\le n\le n_1$.

Theorem 2. *Let $u(n),k(n,s),h(n,s,\sigma),c$ be as in Theorem 1 and $b(n):N_0\to R_+$.*

(b_1) If

$$\begin{aligned}u(n)\le{}& c+\sum_{s=0}^{n-1}b(s)\,u(s)+\sum_{s=0}^{n-1}\left(\sum_{\tau=0}^{s-1}k(s,\tau)\,u(\tau)\right)+\\&+\sum_{s=0}^{n-1}\left(\sum_{\tau=0}^{s-1}\left(\sum_{\sigma=0}^{\tau-1}h(s,\tau,\sigma)\,u(\sigma)\right)\right),\end{aligned} \tag{2.8}$$

for $n\in N_0$, then

$$u(n)\le c\prod_{s=0}^{n-1}\left[1+b(s)+\sum_{\tau=0}^{s-1}k(s,\tau)+\sum_{\tau=0}^{s-1}\left(\sum_{\sigma=0}^{\tau-1}h(s,\tau,\sigma)\right)\right], \tag{2.9}$$

for $n \in N_0$.

(b_2) Let $g(u)$ be as in Theorem 1 part (a_2). If

$$u(n) \leq c + \sum_{s=0}^{n-1} b(s)\, g(u(s)) + \sum_{s=0}^{n-1} \left(\sum_{\tau=0}^{s-1} k(s,\tau)\, g(u(\tau)) \right) + \tag{2.10}$$
$$+ \sum_{s=0}^{n-1} \left(\sum_{\tau=0}^{s-1} \left(\sum_{\sigma=0}^{\tau-1} h(s,\tau,\sigma)\, g(u(\sigma)) \right) \right),$$

for $n \in N_0$, then for $0 \leq n \leq n_2,\ n, n_2 \in N_0$,

(2.11)

$$u(n) \leq G^{-1} \left[G(c) + \sum_{s=0}^{n-1} \left[b(s) + \sum_{\tau=0}^{s-1} k(s,\tau) + \sum_{\tau=0}^{s-1} \left(\sum_{\sigma=0}^{\tau-1} h(s,\tau,\sigma) \right) \right] \right],$$

where G, G^{-1} are as in Theorem 1 part (a_2) and $n_2 \in N_0$ be chosen so that

$$G(c) + \sum_{s=0}^{n-1} \left[b(s) + \sum_{\tau=0}^{s-1} k(s,\tau) + \sum_{\tau=0}^{s-1} \left(\sum_{\sigma=0}^{\tau-1} h(s,\tau,\sigma) \right) \right] \in Dom\left(G^{-1}\right),$$

for all $n \in N_0$ such that $0 \leq n \leq n_2$.

Remark 1. We note that the inequalities established in $(a_1), (b_1)$ provides the growth estimates on the discrete versions of the inequalities given in (1.1), (1.2) while the inequalities in $(a_2), (b_2)$ provides the growth estimates on the general versions of the inequalities in $(a_1), (b_1)$ which can be used conveniently in certain applications.

3. Proofs of Theorems 1 and 2. (a_1) Define a function $z(n)$ by the right hand side of (2.1), then $z(0) = c$ and $u(n) \leq z(n)$. From the hypotheses we observe that $z(n)$ is nondecreasing in n and

$$\Delta z(n) = z(n+1) - z(n) =$$
$$= \sum_{s=0}^{n} k(n+1,s)\, u(s) + \sum_{s=0}^{n} \left(\sum_{\sigma=0}^{s-1} h(n+1,s,\sigma)\, u(\sigma) \right) -$$
$$- \sum_{s=0}^{n-1} k(n,s)\, u(s) - \sum_{s=0}^{n-1} \left(\sum_{\sigma=0}^{s-1} h(n,s,\sigma)\, u(\sigma) \right) =$$

$$
\begin{aligned}
&= k\left(n+1,n\right)u\left(n\right)+\sum_{s=0}^{n-1}k\left(n+1,s\right)u\left(s\right)-\sum_{s=0}^{n-1}k\left(n,s\right)u\left(s\right)+\\
&+\sum_{\sigma=0}^{n-1}h\left(n+1,n,\sigma\right)u\left(\sigma\right)+\sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1}h\left(n+1,s,\sigma\right)u\left(\sigma\right)\right)-\\
&-\sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1}h\left(n,s,\sigma\right)u\left(\sigma\right)\right)=\\
&= k\left(n+1,n\right)u\left(n\right)+\sum_{s=0}^{n-1}\Delta_n k\left(n,s\right)u\left(s\right)+\\
&+\sum_{\sigma=0}^{n-1}h\left(n+1,n,\sigma\right)u\left(\sigma\right)+\sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1}\Delta_n h\left(n,s,\sigma\right)u\left(\sigma\right)\right)\leq\\
&\leq k\left(n+1,n\right)z\left(n\right)+\sum_{s=0}^{n-1}\Delta_n k\left(n,s\right)z\left(s\right)+\\
&+\sum_{\sigma=0}^{n-1}h\left(n+1,n,\sigma\right)z\left(\sigma\right)+\sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1}\Delta_n h\left(n,s,\sigma\right)z\left(\sigma\right)\right)\leq\\
&\leq\left[A\left(n\right)+B\left(n\right)\right]z\left(n\right).
\end{aligned}
\tag{3.1}
$$

Now a suitable application of the Corollary 1.2.2 given in [5, p.12] to (3.1) yields

$$z\left(n\right)\leq c\prod_{s=0}^{n-1}\left[1+A\left(s\right)+B\left(s\right)\right]. \tag{3.2}$$

Using (3.2) in $u\left(n\right)\leq z\left(n\right)$ we get the required inequality in (2.2).

(a_2) First we assume that $c>0$ and define a function $z(n)$ by the right hand side of (2.5), then $z(n)>0, z(0)=c$ and $u\left(n\right)\leq z\left(n\right)$. By following the arguments as in the proof of (a_1) up to (3.1) with suitable modifications we get

$$\Delta z\left(n\right)\leq\left[A\left(n\right)+B\left(n\right)\right]g\left(z\left(n\right)\right). \tag{3.3}$$

From (2.7) and (3.3) we observe that

$$G\left(z\left(n+1\right)\right)-G\left(z\left(n\right)\right)=\int\limits_{z(n)}^{z(n+1)}\frac{dt}{g\left(t\right)}\leq\frac{\Delta z\left(n\right)}{g\left(z\left(n\right)\right)}\leq\left[A\left(n\right)+B\left(n\right)\right]. \tag{3.4}$$

By taking $n = s$ in (3.4) and summing up over s from 0 to $n-1$, it follows that

$$G(z(n)) \le G(c) + \sum_{s=0}^{n-1} [A(s) + B(s)]. \tag{3.5}$$

From (3.5) we have

$$z(n) \le G^{-1}\left[G(c) + \sum_{s=0}^{n-1} [A(s) + B(s)]\right]. \tag{3.6}$$

Using (3.6) in $u(n) \le z(n)$ we get the inequality in (2.6). The subdomain $0 \le n \le n_1$ is obvious.

If $c \ge 0$, we carry out the above procedure with $c+\varepsilon$ instead of c, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (2.6).

(b_1) Define a function $z(n)$ by the right hand side of (2.8). Then $z(0) = c$, $u(n) \le z(n)$, $z(n)$ is nondecreasing in n and

(3.7)
$$\begin{aligned}\Delta z(n) =& b(n)u(n) + \sum_{\tau=0}^{n-1} k(n,\tau)u(\tau) + \sum_{\tau=0}^{n-1}\left(\sum_{\sigma=0}^{\tau-1} h(n,\tau,\sigma)u(\sigma)\right) \le \\ &\le b(n)z(n) + \sum_{\tau=0}^{n-1} k(n,\tau)z(\tau) + \sum_{\tau=0}^{n-1}\left(\sum_{\sigma=0}^{\tau-1} h(n,\tau,\sigma)z(\sigma)\right) \le \\ &\le \left[b(n) + \sum_{\tau=0}^{n-1} k(n,\tau) + \sum_{\tau=0}^{n-1}\left(\sum_{\sigma=0}^{\tau-1} h(n,\tau,\sigma)\right)\right] z(n).\end{aligned}$$

Now a suitable application of Corollary 1.2.2 given in [5,p.12] to (3.7) yields

$$z(n) \le c\prod_{s=0}^{n-1}\left[1 + b(s) + \sum_{\tau=0}^{s-1} k(s,\tau) + \sum_{\tau=0}^{s-1}\left(\sum_{\sigma=0}^{\tau-1} h(s,\tau,\sigma)\right)\right]. \tag{3.8}$$

Using (3.8) in $u(n) \le z(n)$ we get the desired inequality in (2.9).

(b_2) The proof can be completed by following the proof of (b_1) and closely looking at the proof of (a_2) given above. Here we omit the details.

4. Applications. In this section we present applications of the inequality established in Theorem 1, part (a_1) to study certain properties of solutions of the nonlinear sum-difference equation of the form

$$y(n) = f(n) + \sum_{s=0}^{n-1} F(n, s, y(s)) + \sum_{s=0}^{n-1} \left(\sum_{\sigma=0}^{s-1} H(n, s, \sigma, y(\sigma)) \right), \tag{4.1}$$

for $n \in N_0$, where $y(n)$ is an unknown function, $f : N_0 \to R$, $k : D \times R \to R$, $H : E \times R \to R$.

The following theorem deals with the estimate on the solution of equation (4.1).

Theorem 3. *Suppose that the functions f, F, H in equation (4.1) satisfy the conditions*

$$|f(n)| \leq c, \tag{4.2}$$

$$|F(n, s, y)| \leq k(n, s) |y|, \tag{4.3}$$

$$|H(n, s, \sigma, y)| \leq h(n, s, \sigma) |y|, \tag{4.4}$$

where $c, k(n, s)$, $h(n, s, \sigma)$ are as in Theorem 1. If $y(n)$ is any solution of (4.1) on N_0, then

$$|y(n)| \leq c \prod_{s=0}^{n-1} [1 + A(s) + B(s)], \tag{4.5}$$

for $n \in N_0$, where $A(n), B(n)$ are given by (2.3), (2.4).

Proof. Let $y(n)$ be a solution of (4.1). Using (4.2)-(4.4) in (4.1) we have

$$|y(n)| \leq c + \sum_{s=0}^{n-1} k(n, s) |y(s)| + \sum_{s=0}^{n-1} \left(\sum_{\sigma=0}^{s-1} h(n, s, \sigma) |y(\sigma)| \right). \tag{4.6}$$

Now an application of Theorem 1, part(a_1) to (4.6) yields the required estimate in (4.5). □

Remark 2. We note that, if the right hand side of (4.5) is finite, then the solution y(n) of (4.1) is bounded on N_0.

The next result deals with the uniqueness of solutions of equation (4.1).

Theorem 4. *Suppose that the functions F, H in equation (4.1) satisfy the conditions*

$$|F(n,s,y) - F(n,s,\bar{y})| \leq k(n,s)\,|y-\bar{y}|, \tag{4.7}$$

$$|H(n,s,\sigma,y) - H(n,s,\sigma,\bar{y})| \leq h(n,s,\sigma)\,|y-\bar{y}|, \tag{4.8}$$

where $k(n,s)$, $h(n,s,\sigma)$ are as in Theorem 1. Let $A(n), B(n)$ be as in Theorem 1, part (a_1). Then the equation (4.1) has at most one solution on N_0.

Proof. Let $u(n)$ and $v(n)$ be two solutions of equation (4.1) on N_0. Using this fact and the conditions (4.7), (4.8) we have

$$\begin{aligned} |u(n) - v(n)| \leq & \sum_{s=0}^{n-1} k(n,s)\,|u(s)-v(s)| + \\ & + \sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1} h(n,s,\sigma)\,|u(\sigma)-v(\sigma)|\right). \end{aligned} \tag{4.9}$$

Now a suitable application of Theorem 1, part (a_1) (when c=0) to (4.9) yields $u(n) = v(n)$ i.e. there is at most one solution of equation (4.1) on N_0.

Remark 3. We note that the inequality given in Theorem 2, part (b_1) can be used to study the similar properties as in Theorems 3 and 4 of the following sum-difference equation

$$\Delta y(n) = L\left(n, y(n), \sum_{\tau=0}^{n-1} F(n,\tau,y(\tau)), \sum_{\tau=0}^{n-1}\left(\sum_{\sigma=0}^{\tau-1} H(n,\tau,\sigma,u(\sigma))\right)\right), \tag{4.10}$$

with the given initial condition

$$y(0) = y_0, \tag{4.11}$$

under some suitable conditions on the functions involved in (4.10), (4.11).

In conclusion, we note that the results given here can be extended to obtain explicit bounds on discrete inequalities involving functions of two independent variables (see[5]). We also note that the inequalities in Theorems 1 and 2 can be extended very easily, involving several iterated sums.

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57 Shri Niketan Colony,
Near Abhinay Talkies,
Aurangabad 431 001,
Maharashtra
INDIA,
bgpachpatte@hotmail.com