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NONLINEAR DIFFERENTIAL EQUATIONS IN FRÉCHET SPACES AND CONTINUUM CROSS-SECTIONS

BY

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Abstract. In this paper, we study a type of non linear differential equations in Fréchet spaces. Some basic results, like existence and asymptotic behavior, are established. The approach proposed is based on the realization of differential equations in Fréchet spaces as projective limits of appropriate sequences of equations in Banach spaces.

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1. Introduction. One of the main drawbacks in the study of infinite dimensional non-Banach locally convex topological vector spaces is the lack of a general solvability theory for differential equations, even for linear ones. This fact has also direct reflections to various applications which surpass the borders of Mathematical Analysis. For example, infinite dimensional differential manifolds and fiber bundles modelled on spaces of the aforementioned type, whose usage (especially in Theoretical Physics) is constantly increasing, appear to have very poor geometrical structure.

This situation led a number of authors to suggest alternative ways of studying concrete classes of differential equations in infinite dimension spaces and in particular in Fréchet spaces, (see for example: [3], [6], [11], [15]). In this frame also belong previous papers of the first author ([4], [5]) where a new method of solving linear differential equations in Fréchet spaces is proposed, based mainly on the fact that *every Fréchet space can be realized as a projective limit of Banach spaces.*

In the present paper we extend this work to nonlinear differential equations:

$$x' = \phi(t, x), \quad (t, x) \in \Omega = [t_0, \infty) \times \mathbb{F}, \tag{1}$$

where $\mathbb{F}$ is an arbitrarily chosen Fréchet space. Our methodology is based on the exploitation of the algebraic but, under appropriate assumptions, compatible with topological and differential tools, notion of projective limit, in order to *construct a system of countable many equations in Banach spaces which is equivalent to* (1). This approach does not only lead to the solvation of the equations in study, but provides us also with the *exact form of the solutions*, a fact that is used essentially in the applications.

In the second part of this note we present a detailed study of the cross-sections of the solutions obtained. More precisely, for the n-dimensional real case it is well known ([8], [13]) that the cross-section of all solutions emanating from a continuum, at any point in the common domain, is also a continuum, property known as Kneser's one. If now $\omega \subset \Omega$ is a certain set, it is important to get a solution of (1) which remains (i.e. having its graph) in ω. To this end, in [13] we used the consequent map $\mathcal{K}$, that is a mapping of ω into $\partial\omega$ under the action of solutions. Actually, if $\mathcal{K}$ is well defined, then the image of a continuum is also a continuum and there is no solution in ω. In the same paper, the behavior of the image for the *singular case* is also studied where any connected component of it approaches the boundary $\partial\Omega \cap \partial\omega$. Here all the above results are extended to the Fréchet case. As an application, we impose sufficient conditions and obtain an asymptotic in ω solution of (1) and further, by suitable choice of ω, we conclude the asymptotic behavior of this solution.

Concerning the differentiability of mappings in Fréchet spaces we adopt the definition of J.A.Leslie ([12]). However, the ideas of Kriegl-Michor ([9]) fit also to our approach.

2. Ordinary differential equations in Fréchet spaces. Let $\mathbb{F}$ be a Fréchet space, i.e. a linear topological vector space which is metrizable, locally convex and complete. The topology of such a space can be realized by an increasing countable family of seminorms

$$p_1 \leq p_2 \leq \dots \quad .$$

A really convenient realization of $\mathbb{F}$ is that of a projective limit of Banach spaces. More precisely: $\mathbb{F}$ coincides, up to isomorphism, with the limit of the

projective system $\{\mathbb{E}_i; f_{ji}\}_{i,j\in\mathbb{N}}$ where $\mathbb{E}_i$ is the completion of the quotient $\mathbb{F}/\ker(p_i)$ and the connecting morphisms f_{ji} $(j \geq i)$ are given by

$$f_{ji}([x + \ker p_j]_j) = [x + \ker p_i]_i.$$

Here the brackets denote the corresponding equivalence classes. In the sequel each point x of $\mathbb{F}$ will be considered identified with the family $(x_i)_{i\in\mathbb{N}}$ of its projections onto the Banach factors of the limit, with respect to the canonical mappings $f_i : \mathbb{F} \to \mathbb{E}_i$. The norms of $\mathbb{E}_i$'s will be denoted by $\|\cdot\|_i$ respectively. We recall here that f_{ji} and f_i always satisfy:

$$f_{ik} \circ f_{ji} = f_{jk}, \quad f_{ji} \circ f_j = f_i, \quad j \geq i \geq k.$$

The previous construction allows us more to consider each f_i as an isometry in the sense that

$$p_i(x) = \|f_i(x)\|_i, x \in \mathbb{F}.$$

Our aim is to study ordinary differential equations in Fréchet spaces using the identification $\mathbb{F} \simeq \varprojlim \mathbb{E}_i$ by translating any possible problem or question to the corresponding ones on the factors. This methodology *apart from leading to an existence theorem*, also allows us to *obtain unique solutions* under appropriate conditions, a fact that in general is also false and, more important, to *clarify the exact form of these solutions.*

As a first step, we consider the following generalization of the notion of *Lipschitz condition* for functions in Fréchet spaces.

Definition 1. *A mapping $\phi : \mathbb{F} \to \mathbb{F}$ will be called k-Lipschitz, where k is a positive real number, if and only if*

$$p_i(\phi(x_2) - \phi(x_1)) \leq k \cdot p_i(x_2 - x_1)$$

for any x_1, $x_2 \in \mathbb{F}$ and $i \in \mathbb{N}$.

In case where ϕ is the projective limit of a system $\{\phi_i{:}\mathbb{E}_i \to \mathbb{E}_i\}_{i\in\mathbb{N}}$, i.e. ϕ and ϕ_i are related in the following way:

$$f_{ji} \circ \phi_j = \phi_i \circ f_{ji}, \quad f_i \circ \phi = \phi_i \circ f_i, \quad j \geq i,$$

then, we may check if ϕ satisfies the previous Definition or not by the behavior of its components:

Proposition 2. *$\phi = \varprojlim \phi_i$ is k-Lipschitz if and only if each ϕ_i is k-Lipschitz on $\mathbb{E}_i$.*

Proof. Let $x_i = f_i(x)$, $y_i = f_i(y)$ be two arbitrarily chosen points of $\mathbb{E}_i$. The relations between ϕ and ϕ_i's that we described above leads to

$$\|\phi_i(x_i) - \phi_i(y_i)\|_i = p_i(\phi(x) - \phi(y)).$$

Therefore, a k-Lipschitz condition for ϕ, in the generalized sense of Definition 1, leads directly to an analogous condition, in the classical sense, for each ϕ_i on $\mathbb{E}_i$:

$$\|\phi_i(x_i) - \phi_i(y_i)\|_i \ \leq \ k \cdot p_i(x-y) \ = \ k \cdot \|x_i - y_i\|_i.$$

On the other hand, k-Lipschitz conditions for all ϕ_i's (for the same constant k) concludes that:

$$p_i(\phi(x) - \phi(y)) \ \leq \ k \cdot \|x_i - y_i\|_i \ = \ k \cdot p_i(x-y).$$

□

The above result enable us to proceed with the proof of the first main Theorem of the paper.

Theorem 3. *Let* $\mathbb{F} = \varprojlim \mathbb{E}_i$ *be a Fréchet space and*

$$x' = \phi(t, x) \tag{2}$$

a differential equation on $\mathbb{F}$*, where* $\phi : \mathbb{R} \times \mathbb{F} \to \mathbb{F}$ *is a projective limit k-Lipschitz mapping. Assume that for an initial point* $(t_0, x_0) \in \mathbb{R} \times \mathbb{F}$ *there exists a constant* $\tau \in \mathbb{R}$ *such that*

$$M_1 := \sup\{p_i(\phi(t, x_0));\ i \in \mathbb{N},\ t \in [t_0 - \tau, t_0 + \tau]\} < +\infty.$$

Then (2) has a unique solution in the interval $I = [t_0 - a, t_0 + a]$*, where* $a = \inf\{\tau, \frac{1}{M_1 + k}\}$.

Proof. Let $\{\phi_i : \mathbb{R} \times \mathbb{E}_i \to \mathbb{E}_i\}_{i \in \mathbb{N}}$ be a family of continuous functions realizing ϕ. Then, a system of ordinary differential equations on the Banach spaces $\mathbb{E}_i$ is defined:

$$x_i' = \phi_i(t, x_i) \tag{3}$$

According to Proposition 2, each of the mappings ϕ_i is also k-Lipschitz and, for any $i \in \mathbb{N}$ and $x \in \mathbb{E}_i$ with $\|x_i - f_i(x_0)\|_i \leq 1$ we see that:

$$\|\phi_i(t, x)\|_i \ \leq \ \|\phi_i(t, f_i(x_0))\|_i + \|\phi_i(t, x) - \phi_i(t, f_i(x_0))\|_i \ \leq$$

$$\leq \|f_i(\phi(t,x_0))\|_i + k\cdot\|x - f_i(x_0)\|_i \;=\; p_i(\phi(t,x_0)) + k \;\leq\; M_1 + k.$$

Therefore (see [2]; Corollary 1.7.2), a unique solution x_i can be defined for each equation (3) on the interval $I = [t_0 - a, t_0 + a]$.

On the other hand, these solutions are clearly related. Indeed, for any $j \geq i$ we observe that the mapping $f_{ji} \circ x_j$ is also a solution of (3) since:

$$(f_{ji}\circ x_j)'(t) = f_{ji}(x_j'(t)) = f_{ji}(\phi_j(t, x_j(t))) = \phi_i(t, (f_{ji}\circ x_j)(t)).$$

For more details concerning the calculations involving derivatives in Fréchet spaces, we refer to 4. Moreover, both $f_{ji} \circ x_j$ and x_i emanate from the same initial point $(t_0, f_i(x_0))$, therefore they coincide. As a result, the C^1 mapping

$$x = \varprojlim x_i$$

can be defined on I. This is the desired solution of (2) since:

$$x'(t) \;=\; (x_i'(t)) \;=\; (\phi_i(t, x_i(t))) \;=\; (\phi_i(t, f_i(x(t)))) \;=$$

$$= (f_i(\phi(t, x(t)))) \;=\; \phi(t, x(t)).$$

This solution is unique, with respect to the initial data (t_0, x_0). Indeed, if $y : J \subseteq \mathbb{R} \to \mathbb{F}$ is another solution of (2) satisfying $y(t_0) = x_0$, then y can be realized as the limit of the projective system $\{f_i \circ y_i : J \to \mathbb{E}_i\}_{i\in\mathbb{N}}$. Employing analogous calculations as above, we check that the mapping $f_i \circ y$ coincides with x_i (for each $i \in \mathbb{N}$) as a solution of (3) through the same initial condition $(t_0, f_i(x_0))$. Thus, $y = \varprojlim(f_i \circ y) = \varprojlim x_i = x$. □

Remark 4. It is now clear that we have succeeded not only to obtain a solution of the problem in study, but also to clarify the exact form of it. This is necessary for the applications that follow (see also relevant comments in the Introduction).

3. Asymptotic behavior of the solutions-applications. In this Section we grow up some topological results concerning the set of solutions of a differential equation in Fréchet spaces.

3.1. Cross-sections of solutions funnel. Our aim here is to study the cross-sections of the family of solutions of equation (2) which is assumed to be subjecting to the assumptions of Theorem 3. To this end let $\mathcal{X}(S)$ be the set of solutions of (2) with $x(t_0) \in S \subseteq \mathbb{F}$. If $\mathbb{F} = \mathbb{R}^n$ and S is a *continuum*

(i.e. compact and connected) subset of it, then there is an interval $[t_0, a)$ such that any solution $x \in \mathcal{X}(S)$ is defined on and further the cross-section of the solutions funnel over any point $\beta \in [t_0, a)$:

$$\mathcal{X}(\beta, S) = \{x(\beta);\ x \in \mathcal{X}(S)\}$$

is also a continuum of $\mathbb{R}^n$. Here we prove an analogous result on the Fréchet space $\mathbb{F}$, provided that its factors $\mathbb{E}_i$ are of finite dimension.

As a first step we characterize continuum subsets of $\mathbb{F}$:

Proposition 5. *A subset S of $\mathbb{F}$ is continuum if and only if it is the limit of continuum subsets of $\mathbb{E}_i$'s.*

Proof. Let S be continuum in $\mathbb{F}$. We define the sets:

$$S_i := f_i(S) \subset \mathbb{E}_i, \quad i \in \mathbb{N},$$

which are also continuum since the canonical projections f_i are continuous mappings. Moreover the limit $\varprojlim S_i$ can be defined, since

$$f_{ji}(S_j) = f_{ji}(f_j(S)) = f_i(S) = S_i,\ j \geq i.$$

Any point now $s \in S$ belongs, by definition, also to $\varprojlim S_i$.Conversely, if (s_i) is an arbitrarily chosen point of $\varprojlim S_i$, then $f_{ji}(s_j) = s_i$ $(j \geq i)$ and we may find elements $\{s^i \in S\}_{i\in\mathbb{N}}$ such that $s_i = f_i(s^i)$, $i \in \mathbb{N}$. We construct in this way a sequence of S which, necessarily, has a subsequence $\{s^n\}$ converging to a point $s^* \in S$, since S is compact. This limit equals to (s_i). Indeed, for a fixed index i, by choosing any $n \geq i$, we observe that $\lim\limits_{n\to+\infty} s^n = s^*$ implies that $\lim\limits_{n\to+\infty} f_i(s^n) = f_i(s^*)$. Taking also into account that

$$f_i(s^n) = f_{ni}(f_n(s^n)) = f_{ni}(s_n) = s_i,$$

we conclude $f_i(s^*) = s_i$, for every $i \in \mathbb{N}$, or, equivalently, $s^* = (s_i)$. Therefore, (s_i) belongs also to S.

The above thoughts ensure that every continuum subset of $\mathbb{F}$ can be realized as a projective limit of continuum subsets of $\mathbb{E}_i$'s. The converse is also true, since compactness and conectness are properties compatibles with the notion of projective limit. □

Remark 6. The previous result is the best that one would expect to have in the framework of projective limits since not only guarantees

that treating continuums via limits we remain in the same category but in addition, it clarifies that any of these sets is a projective limit, a fact which is not succeeded for any class of subsets of $\mathbb{F}$.

We are now in a position to prove the main Theorem of this Section which extends to the Fréchet case a basic result of the study of solution space of ordinary differential equations.

Theorem 7. *Let $\mathbb{F}$ be a Fréchet space which is obtained as a limit of finite dimensional Banach spaces $\{\mathbb{E}_i; i \in \mathbb{N}\}$. Then for every continuum subset S of $\mathbb{F}$, the cross-section $\mathcal{X}(\beta, S)$, over $\beta \in [t_0, a)$, is also continuum.*

Proof. By Theorem's 3 proof, a system of differential equations on the factors $\mathbb{E}_i$

$$x_i' = \phi_i(t, x_i),\ i \in \mathbb{N}, \tag{3,i}$$

corresponds to the equation (2), where $\phi = \varprojlim \phi_i$. On the other hand, according to Proposition 5, S can be thought of as the limit of a projective system of continuum subsets $S_i \subset \mathbb{E}_i$ $(i \in \mathbb{N})$. Therefore, the following continuum cross-sections are defined for equations (3,i):

$$\mathcal{X}_i(\beta, S_i) = \{x_i(\beta);\ x_i \in \mathcal{X}_i(S_i)\ \}, \quad i \in \mathbb{N},$$

where $\mathcal{X}_i(S_i)$ denotes the space of solutions of (3,i) through all initial points $P_{0i} \in S_i$.

The above family defines a projective system, since

$$f_{ji}(\mathcal{X}_j(\beta, S_j)) \subseteq \mathcal{X}_i(\beta, S_i),\ \text{for any } j \geq i.$$

Indeed if a belongs to $\mathcal{X}_j(\beta, S_j)$, that is $a = x_j(\beta)$, where x_j is a solution of (3,i) such that $x_j(t_0) \in S_j$, then $f_{ji}(a) = (f_{ji} \circ x_j)(\beta)$, where, as we have already seen, $f_{ji} \circ x_j$ is a solution of (3,i) and $(f_{ji} \circ x_j)(t_0) \in f_{ji}(S_j) \subseteq S_i$. As a result, $f_{ji}(a) \in \mathcal{X}_i(\beta, S_i)$.

Considering now a point $a \in \mathcal{X}(\beta, S)$ and following analogous thoughts, we easily verify that $f_i(a) \in \mathcal{X}_i(\beta, S_i)$ $(i \in \mathbb{N})$. Thus

$$\mathcal{X}(\beta, S) \subseteq \varprojlim \mathcal{X}_i(\beta, S_i). \tag{4}$$

On the other hand, if $a = (a_i) \in \varprojlim \mathcal{X}_i(\beta, S_i)$, then there exists a sequence of solutions $\{x_i\}_{i \in \mathbb{N}}$ such that $a_i = x_i(\beta)$ and $x_i(t_0) \in S_i$. These

solutions form a projective system. Indeed, working as in the proof of Theorem 3, and since x_j is a solution of (3,j), the composition $f_{ji} \circ x_j$ is a solution of (3,i) satisfying further at least one common initial condition with x_i:

$$(f_{ji} \circ x_j)(\beta) = f_{ji}(a_j) = a_i = x_i(\beta).$$

Therefore, based on the uniqueness of solutions of each equation (3,i), we immediate conclude that $f_{ji} \circ x_j = x_i$, for every $j \geq i$, or equivalently, $x := \varprojlim x_i$ can be defined.

This mapping x is a solution of (2) and

$$x(t_0) = (x_i(t_0)) \in \varprojlim S_i = S, \quad x(\beta) = a.$$

As a result $a \in \mathcal{X}(\beta, S)$ and so

$$\varprojlim \mathcal{X}_i(\beta, S_i) \subseteq \mathcal{X}(\beta, S). \tag{5}$$

Relations (4) and (5) combined with Proposition 5 yield now the result. □

3.2. The consequent mapping and a topological method. Consider again the differential equation

$$x' = \phi(t, x), \quad (t, x) \in \Omega, \tag{1}$$

where $\Omega \subset [t_0, +\infty) \times \mathbb{F}$ is open, and let ω be an open subset of Ω with $\Omega - \bar{\omega} \neq \emptyset$. We shall say that a solution x of (1) *remains (right) asymptotic in* ω if its graph is a subset of ω:

$$G(x|Dom^+(x)) = \{(t, x(t)) : t \in Dom^+(x) = [t_0, +\infty) \cap Dom(x)\} \subset \omega.$$

A point $P = (\tau, \xi) \in \Omega \cap \partial\omega$ is a point of*semi-egress* of ω if there exists a solution $x \in \mathcal{X}(P)$, a point $t_1 \in Dom^-(x) = (-\infty, \tau] \cap Dom(x)$ and an $\epsilon > 0$ such that $G(x|[t_1 - \epsilon, t_1); P) \subset \omega^0$ and $G(x|[t_1, \tau); P) \subset \partial\omega$. If, moreover, for any solution $x \in \mathcal{X}(P)$ there exist $t_2 \in Dom^+(x)$, $\epsilon > 0$ such that $G(x|[\tau, t_2); P) \subset \partial\omega$ and $G(x|[t_2, t_2 + \epsilon); P) \subset \Omega - \omega$, then P will be called a point of *strict semi-egress* of ω. The family of all points of semi-egress or strict semi-egress will be denoted by ω^s or ω^{ss} respectively.

A point $Q = (\sigma, \eta) \in \omega^s$ with $\sigma \geq \tau$ is a *consequent* of the point $P = (\tau, \xi) \in \omega$ if there exists a solution $x \in \mathcal{X}(P, Q) = \mathcal{X}(P) \cap \mathcal{X}(Q)$ and a

$t_1 \in [\tau, \sigma]$ such that $G(x|[t_1, \sigma]) \subset \partial\omega$ and $G(x|(\tau, t_1)) \subset \omega$, provided that $\tau < t_1$. The multipoint correspondence

$$\mathcal{K} : P \to \{Q : Q \text{ is a consequent point of P}\}$$

defines the so called *consequent mapping.*

Finally, we shall say that a set-valued mapping $\mathcal{F}$, which maps the points of a topological space X into compact subsets of another one Y is *upper semicontinuous* at a point $x_0 \in X$, if for any open subset V of Y with $\mathcal{F}(x_0) \subset V$ there exists a neighborhood U of x_0 such that $\mathcal{F}(x) \subset V$ for every $x \in U$.

Lemma 8. *The consequent mapping is an upper semicontinuous function and so if* $A \subset \omega(t_0) = \{(t, x) \in \omega;\ t = t_0\}$ *is a continuum, its image*

$$\mathcal{K}(A) = \cup\{\mathcal{K}(P) : P \in A\}$$

is also a continuum.

Proof. It follows directly by the analogous result in $\mathbb{R}^n$ (see [8]) and Proposition 5. □

Let $\mathcal{X}_\omega(A)$ be the family of all solutions $x \in \mathcal{X}(A)$ which remain asymptotic in ω, i.e. $G(x|Dom^+(x)) \subset \omega$. It is clear that whenever $\mathcal{X}(P_0) = \mathcal{X}_\omega(P_0)$, the consequent mapping cannot be defined at P_0. Further, by the previous Lemma, if A is a continuum and $\mathcal{X}_\omega(A) = \emptyset$, the image $\mathcal{K}(A)$ is also a continuum.

The set ω is said to be $\mathbb{R}$-*almost compact* in Ω if $\omega(I) = \{(t, x) \in \omega : t \in I\}$ is compact in Ω, for every compact subset I of $pr_1(\omega)$. In the sequel we assume that

$$t^* = \sup pr_1(\omega) = \sup pr_1(\Omega)\ .$$

We need also the following Lemma([10]; ch.V), which directly leads to Proposition 10 below (see also [13], [14]).

Lemma 9. *If* A *is an arbitrary proper subset of a continuum* C *and* S *a connected component of* A*, then* $\overline{S} \cap \partial A \neq \emptyset$.

Proposition 10. *If* $\omega^s = \omega^{ss}$ *and* Z *is a set such that* $\mathcal{X}_\omega(P_0) \neq \emptyset$*, for some* $P_0 \in Z$*, then either*

$$\mathcal{X}(Z) = \mathcal{X}_\omega(Z),$$

that is every solution $x \in \mathcal{X}(Z)$ remains (right) asymptotic in ω, or

i) any connected component S of $\mathcal{K}_\omega(Z)$ approaches the boundary $\partial\Omega$ of Ω, i.e. $\overline{S} \cap \partial\Omega \neq \emptyset$ and

ii) there is a $t_1 \in [t_0, t^*)$ such that for any $\widehat{t} \in [t_1, t^*)$ every connected component C of $\mathcal{X}_\omega(\widehat{t}; Z)$ approaches the boundary $\partial\omega$, i.e. $\overline{C} \cap \mathcal{K}_\omega(Z) \neq \emptyset$.

Proposition 11. *If the set ω is $\mathbb{R}$-almost compact in Ω, $t^* = \sup pr_1(\omega) = \sup pr_1(\Omega)$ and $\partial\omega(t_0, t^*) = \omega^s = \omega^{ss}$, then there exists a solution $x \in \mathcal{X}(P_0)$, $P_0 \in \omega(t_0)$, which remains (right) asymptotic in ω.*

Proof. Consider a sequence $\{P_n\}, P_n = (t_n, \xi_n) \in \omega^s$, such that $t_n < t_{n+1}$, $\lim t_n = t^*$ and a sequence of solutions $x_n \in X(P_n)$, $n = 1, 2, ...$ Since $\partial\omega(t_0, t^*) = \omega^s = \omega^{ss}$, we get, by the extensibility of solutions, that $[t_0, t_n] \subseteq Dom x_n$ and $G(x_n|[t_0, t_n]) \subseteq \bar{\omega}$. Moreover, $x_n(t_0) \in \omega(t_0)$, the last set $\omega(t_0)$ being compact, and, therefore, we may assume that

$$\lim(t_0, x_n(t_0)) \;=\; P_0.$$

As a result, by the well known Kamke's theorem ([1]), which is valid for (1) (since the solutions are projective limits of corresponding solutions on finite dimensional spaces), there is a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ and a solution $x \in X(P_0)$ such that $\lim x_{k_n} = x$ uniformly on every compact subset of $Dom^+ x$.

Further, $Dom^+ x = [t_0, t^*)$ and $G(x|(t_0, t^*), P_0) \subseteq \omega$. If not, by the compactness of $\omega[t_0, \tau]$ for any $\tau < t^*$, there would have been a $\widehat{t} \in (t_0, t^*)$ such that

$$(\widehat{t}, x(\widehat{t})) \in \Omega - \omega.$$

But $(\widehat{t}, x_{k_n}(\widehat{t})) \in \overline{\omega}$ for sufficiently large $n \in N$ and this contradicts the above convergence of $\{x_{k_n}\}$. □

Combining the outcomes of the previous Propositions we obtain the next general existence result:

Theorem 12. *If the set ω is $\mathbb{R}$-almost compact in Ω, $t^* = \sup pr_1(\omega) = \sup pr_1(\Omega)$, $\partial\omega(t_0, t^*) = \omega^s = \omega^{ss}$ and in addition, the cross-section $\omega(t_0)$ is a continuum, then there exists a solution $x \in \mathcal{X}(\omega(t_0))$ which remains asymptotic in ω. Furthermore, $S \cap \partial\omega(t_0) \neq \emptyset$ for every connected component S of the set $\mathcal{X}_\omega(\widehat{t}; \omega(t_0))$, provided that $\mathcal{K}_\omega(\widehat{t}; \omega(t_0)) \neq \emptyset$.*

3.3. Applications. In this subsection, we will give an application which concerns the asymptotic behavior of solutions of a 1-order differential system. The key of our approach is the fact that the basic open sets of $\mathbb{F}$, which is assumed to be the limit of finite dimensional Banach spaces, are of the form

$$\{p_i^{-1}(U); i \in \mathbb{N}, U \subseteq \mathbb{R} \text{ open}\}.$$

Consider the $\mathbb{R}$-almost compact set

$$\omega \ := \ \{(t,x) \in \mathbb{R} \times \mathbb{F} : t \geq t_0, \ p_i(x) \leq \psi(t)\}$$

where $\psi : [t_0, +\infty) \to \mathbb{R}$ is a positive differentiable map. On the spaces $\mathbb{R}^n$ we use the norms

$$\|(x_1, x_2, ..., x_n)\| = \max\{|x_i|, \ i = 1, 2, ..., n\}$$

and let

$$p_n(\phi(t,x)) = \phi_n(t,x) = (\phi_{n1}(t,x), ..., \phi_{nn}(t,x)), \ (t,x) \in \mathbb{R} \times \mathbb{F}.$$

Theorem 13. *Suppose that for any $x \in \mathbb{F}$ with $p_n(x) = \psi(t)$, we have*

$$(6) \qquad \begin{cases} \min\{\phi_{ni}(t,x), \ 1 \leq i \leq n\} \ > \ \psi'(t), \ t \geq t_0, \ x \geq 0 \text{ and} \\ \phi_n(t,-x) = -\phi_n(t,x), \quad (t,x) \in [t_0, +\infty) \times \mathbb{F}, \quad n = 1, 2, ... \end{cases}$$

Then, there exists a solution $x \in \mathcal{X}(\omega(t_0))$ of (1) which remains (right) asymptotic in ω.

Proof. By Theorem 12, it is enough to show that $\partial\omega[t_0, +\infty) = \omega^s = \omega^{ss}$ since obviously $t^* = \sup pr_1(\omega) = +\infty$. Let $P = (\tau, \xi) \in \partial\omega(\tau)$ and a solution $x \in \mathcal{X}(P)$. Since P is a boundary point, any open neighborhood U of P intersects both ω° and $(\Omega - \omega)^\circ$. Especially, we may choose as U any basic open ball

$$B_n(P, \delta) = \{x \in \mathbb{F} : p_n(x - \xi) < \delta\}; \ \delta > 0.$$

Considering the ordinary vector initial value problem

$$(7) \qquad x_n'(t) \ = \ \phi_n(t, x_n), \quad x_n(\tau) = \xi_n,$$

where $x_n = f_n(x)$, $\xi_n = f_n(\xi)$, $n \in \mathbb{N}$, and using the existence theorem in $\mathbb{R}^n$, we immediate conclude that there is an $\epsilon \in (0, \delta)$ such that the solution x_n of (7) satisfies:

$$(\tau - \epsilon, \tau + \epsilon] \subseteq Dom(x_n) \text{ and } G(x_n \mid [\tau - \epsilon, \tau + \epsilon]) \subseteq B_n(P, \delta).$$

However, there is an $i \in \{1, 2, ..., n\}$ such that

$$\| x_n(\tau) \| = \max\{|pr_j(x_n(\tau))| \, ; 1 \leq j \leq n\} = | x_{ni}(\tau) | = p_i(x(\tau)) = \psi(\tau)$$

and so we have to examine the following two cases:

If $x_{ni}(\tau) > 0$, then $x'_{ni}(\tau) = \phi_{ni}(\tau, x_i(\tau)) > \psi'(\tau)$. Consequently,

$$\begin{cases} x_{ni}(t) > \psi(t), \ t \in (\tau, \tau + \epsilon^*) \ \text{ and} \\ x_{ni}(t) < \psi(t), \ t \in (\tau - \epsilon^*, \tau) \end{cases}$$

for some $\epsilon^* \leq \epsilon$.

Similarly by (6), if $x_{ni}(\tau) < 0$, then

$$x'_{ni}(\tau) = \phi_{ni}(\tau, x_i(\tau)) = -\phi_{ni}(\tau, -x_i(\tau)) < -\psi'(\tau).$$

Consequently,

$$\begin{cases} x_{ni}(t) < -\psi(t), \ t \in (\tau, \tau + \epsilon^{**}) \ \text{ and} \\ x_{ni}(t) > -\psi(t), \ t \in (\tau - \epsilon^{**}, \tau). \end{cases}$$

Thus, by the definition of the topology of $\mathbb{F}$, we have

$$\begin{cases} ||x_n(\tau)|| \geq \psi(\tau) \ \text{ and} \\ ||x_n(\tau)|| \leq \psi(\tau) \end{cases}$$

i.e. $P \in \omega^s = \omega^{ss}$. □

Remark 14. If further we assume that $\lim_{t \to +\infty} \psi(t) = 0$, then by the previous Theorem, we get a solution x of (1) such that $\lim_{t \to +\infty} x(t) = 0$.

Example 15. Consider the system

$$x' = x^3 \cdot (1 + \exp(-t^2 - 5x^2)), \ \ (t, x) \in [0, +\infty) \times \mathbb{C}^\infty,$$

where exp denotes the generalized exponential mapping on $\mathbb{C}^\infty$ defined in [5]. We shall prove that the above system has a solution such that $\lim x(t) = 0$ as $t \to +\infty$.

Proof. Using the notations of previous Theorem and setting $\psi(t) = \exp(-t)$, we may check that for any $x = (x_n) \in \mathbb{C}^\infty,\ x_n \geq 0$ and $t \geq 0$ we have

$$\min_{1\leq i\leq n} \phi_{ni}(t, x_n) \ > \ \psi'(t)$$
$$\phi_n(t, -x_n) = -\phi_n(t, x_n) \ \ (n = 1, 2, ...)$$

Since $p_n(x) = \exp(-t)$, the result follows by the previous Remark. □

Notation 16. The above applications make once again obvious the importance of the exact clarification of the form of the solutions in study, since the analogous classical proofs on finite dimensional spaces collapses within the Fréchet framework.

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