ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" IAŞI
Tomul LI, s.I, Matematică, 2005, f.1

ON THE GEOMETRY OF LIE ALGEBROIDS AND APPLICATIONS TO OPTIMAL CONTROL

BY

LIVIU POPESCU

Abstract. In this paper the geometry of Lie algebroids using the notions of semispray and nonlinear connection is studied. An application to distributional systems with homogeneous cost is provided.

Mathematics Subject Classification 2000: 17B66, 53C05, 93B05.

Key words: Lie algebroids, nonlinear connection, Euler- Lagrange equations, Hamilton -Jacobi equations, optimal control, homogeneous cost.

1. Introduction. The notion of Lie algebroid is a generalization of the concepts of Lie algebra and integrable distribution. In his paper [19] WEINSTEIN gives a generalized theory of Lagrangian on Lie algebroids and obtains the Euler-Lagrange equations using the structure of the dual of Lie algebroids and Legendre transformations associated with a regular Lagrangian. The same Euler-Lagrange equations were later obtained by MARTINEZ using the symplectic formalism for Lagrangian [10] and Hamiltonian [11] as it appears in KLEIN's paper [7]. In [9] LIBERMANN shows that such a formalism is not possible, in general, if we consider the tangent bundle TE to the Lie algebroid $\pi : E \to M$ as the space for developing the theory, one of the problems being the different dimensions of vertical and horizontal distributions. The difficulty is solved with the notion of prolongation of Lie algebra over a mapping introduced by HIGGINS and MACKENZIE [5]. Following the MIRON, ANASTASIEI [13] and DE LEON, RODRIQUES [8] for the case of tangent bundle, we will study in this paper the geometry of Lie algebroids using the notions of semispray and nonlinear connection

(see also GRIFONE [4] and CRAMPIN [3]) and we will give an application to distributional systems with homogeneous cost.

2. The geometry of lie algebroids. Let M be a differentiable, n-dimensional manifold and (TM, π_M, M) its tangent bundle. Let (E, π, M) be a vector bundle with the dimension of type fibre m. A Lie algebroid over a manifold M is a vector bundle (E, π, M) equipped with a Lie algebra structure $[,]$ on its space of sections and a map $\sigma : E \to TM$ (called the anchor) which induces a Lie algebra homomorphism (also denoted σ) from sections of E to vector fields on M, satisfying the compatibility conditions

$$[s_1, fs_2] = f[s_1, s_2] + (\sigma(s_1)f)s_2, \tag{1}$$

where f is a smooth function on M and s_1, s_2 are sections of E. Therefore, we also have the relations

$$[\sigma(s_1), \sigma(s_2)] = \sigma[s_1, s_2], \quad [s_1, [s_2, s_3]] + [s_2, [s_3, s_1]] + [s_3, [s_1, s_2]] = 0.$$

The properties of Lie algebroids permit us to define an exterior differential operator d on $\bigwedge E = Sec((E^*)^{\wedge p} \to M$. If f is a function on M then $df(x) \in E_x^*$ is defined by $< df(x), u >= \sigma(u)f$, for every $u \in E_x$. If ω is an element of $\bigwedge^p(E)$ with $p > 0$, then the element $d\omega \in \bigwedge^{p+1}(E)$ is given by the formula

$$\begin{aligned} d\omega(s_1, ..., s_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \sigma(s_i)\omega(s_1, ..., \hat{s_i}, ..., s_{p+1}) + \\ &+ \sum_{1 \le i < j \le p+1} (-1)^{i+j} \omega([s_i, s_j], s_1, ..., \hat{s_i}, ..., \hat{s_j}, ...s_{p+1}), \end{aligned}$$

where the hat over an argument means the absence of that argument.
If we take the local coordinates (x^i) on M and a local basis $\{s_\alpha\}$ of sections of the bundle, then we have the local coordinates (x^i, y^α) on E. These coordinates determine the local functions $\sigma_\alpha^i(x)$, $L_{\alpha\beta}^\gamma(x)$ on M given by

$$\sigma(s_\alpha) = \sigma_\alpha^i \frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta] = L_{\alpha\beta}^\gamma s_\gamma, \quad i, j = \overline{1, n}, \quad \alpha, \beta = \overline{1, m}, \tag{2}$$

and satisfying the relations

$$\sigma_\alpha^j \frac{\partial \sigma_\beta^i}{\partial x^j} - \sigma_\beta^j \frac{\partial \sigma_\alpha^i}{\partial x^j} = \sigma_\gamma^i L_{\alpha\beta}^\gamma, \quad \sum_{(\alpha,\beta,\gamma)} \left(\sigma_\alpha^i \frac{\partial L_{\beta\gamma}^\delta}{\partial x^i} + L_{\alpha\eta}^\delta L_{\beta\gamma}^\eta \right) = 0, \tag{3}$$

which are called the structure equations of Lie algebroid. In local coordinates the differential d is determined by

$$dx^i = \sigma^i_\alpha s^\alpha, \quad ds^\alpha = -\frac{1}{2} L^\alpha_{\beta\gamma} s^\beta \wedge s^\gamma, \tag{4}$$

where $\{s^\alpha\}$ is the dual basis of $\{s_\alpha\}$ and we have the relations $d^2x^i = 0$ and $d^2s^\alpha = 0$. The differential of a function f on M is given by $df = \frac{\partial f}{\partial x^i}\sigma^i_\alpha y^\alpha$ and in particular we have $\dot{x}^i = \sigma^i_\alpha y^\alpha$.

2.1. The induced vector bundle. Let (R, p, M) be a vector bundle. For the map $\sigma : E \to TM$ we can construct the induced vector bundle of (TR, T_p, TM) given by

$$\mathcal{T}E = \{(u, v) \in E \times_{TM} TR \mid \sigma(u) = T_p(v)\}, \tag{5}$$

but fibered over R by the projection $\pi_1(u, v) = \pi_R(v)$, where $\pi_R : TR \to R$ is the tangent projection. We have also the canonical projection $\pi_2 : \mathcal{T}E \to E$ given by $\pi_2(u, v) = u$. The projection onto the second factor $\sigma^1 : \mathcal{T}E \to TR$, $\sigma^1(u, v) = v$ will be the anchor of a new Lie algebroid over manifold R [17]. An element of $\mathcal{T}E$ is said to be vertical if it is in the kernel of the projection π_2. We will denote $(V\mathcal{T}E, \pi_{1|V\mathcal{T}E}, R)$ the vertical bundle of $(\mathcal{T}E, \pi_1, R)$.

In this paper we will restrict our attention to the case $R = E$ (also considered in [16], where $\mathcal{T}E$ is called the "relative self-tangent space"). We have also that $KerT_p$ is a vertical bundle in (TE, π_E, E) which is denoted $(VTE, \pi_{E|VTE}, E)$ and $\sigma^1\big|_{V\mathcal{T}E} : V\mathcal{T}E \to VTE$ is an isomorphism.
The local basis of sections of $\mathcal{T}E$ is given by $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$, where

$$\mathcal{X}_\alpha = \left(s_\alpha(\pi(u)), \sigma^i_\alpha \frac{\partial}{\partial x^i}\Big|_u\right), \quad \mathcal{V}_\alpha = \left(0, \frac{\partial}{\partial y^\alpha}\Big|_u\right), \tag{6}$$

$(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha})$ is the local basis on TE. If V is a section of $\mathcal{T}E$ then in terms of basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ it is $V = Z^\alpha \mathcal{X}_\alpha + V^\alpha \mathcal{V}_\alpha$, and the vector field $\sigma^1(V) \in \chi(E)$ has the expression $\sigma^1(V) = \sigma^i_\alpha Z^\alpha \frac{\partial}{\partial x^i} + V^\alpha \frac{\partial}{\partial y^\alpha}$. The vertical lift of a section $\rho = \rho^\alpha s_\alpha$ and the corresponding vector field are $\rho^v = \rho^\alpha \mathcal{V}_\alpha$ and $\sigma^1(\rho^v) = \rho^\alpha \frac{\partial}{\partial y^\alpha}$. The structure functions of $\mathcal{T}E$ are given by the following formulas

$$\sigma^1(\mathcal{X}_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \quad \sigma^1(\mathcal{V}_\alpha) = \frac{\partial}{\partial y^\alpha}, \tag{7}$$

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = L^\gamma_{\alpha\beta}\mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{V}_\beta] = 0, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0. \tag{8}$$

If $\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$ denotes the corresponding dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ then the local expression of the differential of a function L on $\mathcal{T}E$ is $dL = \sigma^i_\alpha \frac{\partial L}{\partial x^i}\mathcal{X}^\alpha + \frac{\partial L}{\partial y^\alpha}\mathcal{V}^\alpha$, and therefore, we have $dx^i = \sigma^i_\alpha \mathcal{X}^\alpha$ and $dy^\alpha = \mathcal{V}^\alpha$. The differential of sections of $(\mathcal{T}E)^*$ is determined by

$$d\mathcal{X}^\alpha = -\frac{1}{2}L^\alpha_{\beta\gamma}\mathcal{X}^\beta \wedge \mathcal{X}^\gamma, \quad d\mathcal{V}^\alpha = 0. \tag{9}$$

2.2. Semispray

Definition 1. *The vertical section given by $\mathcal{C} = y^\alpha \mathcal{V}_\alpha$ is called the Liouville section.*

The Liouville section measures the homogeneity of functions and sections. A function $f \in C^\infty(E)$ is said to be homogeneous of degree $r \in \mathbb{Z}$ if $L_\mathcal{C} f = rf$, where $L_\mathcal{C}$ is the Lie derivation with respect to Liouville section on Lie algebroid. A section Z of $\mathcal{T}E$ is said to be homogeneous of degree $r \in \mathbb{Z}$ if $L_\mathcal{C} Z = rZ$.

Definition 2. *The linear mapping given by $\mathcal{J} = \mathcal{V}^\alpha \otimes \mathcal{X}_\alpha$ is called the tangent structure.*

We have the following properties:

$$\mathcal{J}(\mathcal{X}_\alpha) = \mathcal{V}^\alpha, \ \mathcal{J}(\mathcal{V}^\alpha) = 0, \ \mathcal{J}^2 = 0, \ \mathcal{J}\mathcal{C} = 0, \ Im\mathcal{J} = Ker\mathcal{J} = V\mathcal{T}E,$$

Moreover, $\mathcal{J}$ is an integrable structure and is homogeneous of degree -1.

Definition 3. *A section ξ of $\mathcal{T}E$ satisfying the condition $\mathcal{J}\xi\mathcal{C}$ is called semispray.*

In local coordinates a semispray has the expression

$$\xi(x, y) = y^\alpha \mathcal{X}_\alpha + \xi^\alpha(x, y)\mathcal{V}_\alpha \tag{10}$$

and the associated vector field is $\sigma^1(\xi) = \sigma^i_\alpha y^\alpha \frac{\partial}{\partial x^i} + \xi^\alpha \frac{\partial}{\partial y^\alpha}$. The integral curves of $\sigma^1(\xi)$ satisfy the differential equations

$$\frac{dx^i}{dt} = \sigma^i_\alpha(x)y^\alpha, \quad \frac{dy^\alpha}{dt} = \xi^\alpha(x, y). \tag{11}$$

We shall express now the non-homogeneity of the semispray.

Definition 4. *Let ξ be a semispray on E. We call deviation of ξ the section $\xi^* = [\mathcal{C}, \xi] - \xi$.*

Proposition 1. *The deviation ξ^* is a vertical section.*

Proof. We have $\mathcal{J}\xi^* = \mathcal{J}[\mathcal{C}, \xi] - \mathcal{J}\xi = \mathcal{J}[y^\alpha \mathcal{V}_\alpha, y^\beta \mathcal{X}_\beta + \xi^\beta \mathcal{V}_\beta] - \mathcal{C} = \mathcal{J}\left(y^\beta \mathcal{X}_\beta + y^\alpha \frac{\partial \xi^\beta}{\partial y^\alpha}\mathcal{V}_\beta - \xi^\beta \mathcal{V}_\beta\right) - \mathcal{C} = y^\beta \mathcal{V}_\beta - \mathcal{C} = 0$, therefore ξ^* is a vertical section. □

Definition 5. *A semispray ξ is called spray if $\xi^* = 0$, that is ξ is 2-homogeneous and it is of class C^1 on the null section. If, moreover, ξ is C^2 on the null section, then ξ is called quadratic spray.*

Proposition 2. *If ξ is a semispray on E, then we have*

$$\mathcal{J}[\xi, \mathcal{J}Z] = -\mathcal{J}Z, \quad Z \in \chi(E) \tag{12}$$

Proof. Since $N_{\mathcal{J}} = 0$ (Nijenhuis tensor of tangent structure) we deduce $N_{\mathcal{J}}(\xi, Z) = [\mathcal{J}\xi, \mathcal{J}Z] - \mathcal{J}[\mathcal{J}\xi, Z] - \mathcal{J}[\xi, \mathcal{J}Z] = 0$ and $[\mathcal{C}, \mathcal{J}Z] - \mathcal{J}[\mathcal{C}, Z] - \mathcal{J}[\xi, \mathcal{J}Z] = 0 \Rightarrow [\mathcal{C}, \mathcal{J}Z] - \mathcal{J}[\mathcal{C}, Z] = \mathcal{J}[\xi, \mathcal{J}Z]$. But, on the other hand, $(L_{\mathcal{C}}\mathcal{J})(Z) = [\mathcal{C}, \mathcal{J}Z] - \mathcal{J}[\mathcal{C}, Z]$ (Lie derivative) and $L_{\mathcal{C}}\mathcal{J} = -\mathcal{J}$ (J is (-1)-homogeneous, so $\mathcal{J}[\xi, \mathcal{J}Z] = -\mathcal{J}Z$. □

2.3. Nonlinear connection

Definition 6. *A nonlinear connection in $(\mathcal{T}E, \pi_1, E)$ is a subbundle $H\mathcal{T}E$ (or a distribution H) such that $\mathcal{T}E = V\mathcal{T}E \oplus H\mathcal{T}E$.*

If H is a nonlinear connection, then a section of E may be decomposed as follows $s = vs + hs$ (vs is vertical component and hs is horizontal component of section s). Then v and h are vertical and horizontal projectors and we have the relations $h^2 = h,\ v^2 = v,\ hv = vh = 0$. But we know that $V\mathcal{T}E = Ker\mathcal{J} = Im\mathcal{J}$, where $\mathcal{J}$ is the tangent structure and we can easily prove that $\mathcal{J}h = \mathcal{J},\ h\mathcal{J} = 0,\ \mathcal{J}v = 0,\ v\mathcal{J} = \mathcal{J}$. We put $N = 2h - Id$ and it results

$$\mathcal{J}\mathcal{N} = \mathcal{J},\ \mathcal{N}\mathcal{J} = -\mathcal{J} \tag{13}$$

Conversely, if N satisfies (13) then we get $\mathcal{N}^2 = Id$. In fact, $\mathcal{N}(\mathcal{V}_\alpha) = \mathcal{N}(\mathcal{J}(\mathcal{X}_\alpha)) = -\mathcal{J}(\mathcal{X}_\alpha) = -\mathcal{V}_\alpha$ and if $\mathcal{N}(\mathcal{X}_\alpha) = A^\beta_\alpha \mathcal{X}_\beta + B^\beta_\alpha \mathcal{V}_\beta$ then $\mathcal{J}\mathcal{N}(\mathcal{X}_\alpha) =$

$\mathcal{J}(A^\beta_\alpha \mathcal{X}_\beta + B^\beta_\alpha \mathcal{V}_\beta) = A^\beta_\alpha \mathcal{V}_\beta$. But $\mathcal{J}\mathcal{N}(\mathcal{X}_\alpha) = \mathcal{J}(\mathcal{X}_\alpha) = \mathcal{V}_\alpha$ so $A^\beta_\alpha = \delta^\beta_\alpha$ and we obtain

$$\mathcal{N}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha + B^\beta_\alpha \mathcal{V}_\beta, \quad \mathcal{N}(\mathcal{V}_\alpha) = -\mathcal{V}_\alpha. \tag{14}$$

From (14) we get $\mathcal{N}^2 = Id$. Now, if we put $h = \frac{1}{2}(Id + \mathcal{N})$ and $v = \frac{1}{2}(Id - \mathcal{N})$ we obtain $Imv = V\mathcal{T}E$, $Imh = H\mathcal{T}E$, and h and v define a nonlinear connection in $\mathcal{T}E$. From previous results, we get:

Theorem 3. *A nonlinear connection is a section on $\mathcal{T}E$, C^∞ on $\mathcal{T}E \setminus \{0\}$ such that*

$$\mathcal{J}\mathcal{N} = \mathcal{J}, \ \mathcal{N}\mathcal{J} = -\mathcal{J}. \tag{15}$$

From (14) we have that $\mathcal{N}$ is locally given by

$$\mathcal{N}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha - 2\mathcal{N}^\beta_\alpha(x,y)\mathcal{V}_\beta, \quad \mathcal{N}(\mathcal{V}_\alpha) = -\mathcal{V}_\alpha. \tag{16}$$

where $B^\beta_\alpha = -2\mathcal{N}^\beta_\alpha$ and $\mathcal{N}^\beta_\alpha$ are the components of nonlinear connection. If ρ is a section on E, $\rho = \rho^\alpha s_\alpha$, then we define the horizontal lift $\rho^H = \rho^\alpha \mathcal{X}_\alpha - \rho^\alpha \mathcal{N}^\beta_\alpha \mathcal{V}_\beta$, and the corresponding vector field is $\sigma^1(\rho^H) = \rho^\alpha \sigma^i_\alpha \frac{\partial}{\partial x^i} - \rho^\alpha \mathcal{N}^\beta_\alpha \frac{\partial}{\partial y^\beta}$. From (16) we deduce that h is locally given by

$$h(\mathcal{X}_\alpha) = \mathcal{X}_\alpha - \mathcal{N}^\beta_\alpha \mathcal{V}_\beta, \quad h(\mathcal{V}_\alpha) = 0. \tag{17}$$

Next, we express the non-homogeneity of a nonlinear connection.

Definition 7. *The tension of a nonlinear connection $\mathcal{N}$ is a section $\mathcal{H}$ on $\mathcal{T}E$ given by $\mathcal{H} = \frac{1}{2} L_{\mathcal{C}} \mathcal{N}$, where L is the Lie derivative on Lie algebroid $(L_{\mathcal{C}}\mathcal{N})(Z) = [\mathcal{C}, \mathcal{N}Z] - \mathcal{N}[\mathcal{C}, Z]$.*

In local coordinates we have the expression

$$\mathcal{H} = \left(\mathcal{N}^\beta_\alpha - \frac{\partial \mathcal{N}^\beta_\alpha}{\partial y^\gamma} y^\gamma \right) \mathcal{X}^\alpha \otimes \mathcal{V}_\beta. \tag{18}$$

Definition 8. *A nonlinear connection is said to be homogeneous if $\mathcal{N}$ is homogeneous of degree 1, that is, the tension $\mathcal{H}$ vanishes.*

Definition 9. *A homogeneous nonlinear connection $\mathcal{N}$ is said to be linear connection if $\mathcal{N}$ is C^1 on the null section.*

If $\mathcal{N}$ is a linear connection, then the function $\mathcal{N}_\alpha^\beta$ are linear on y, then we may write $\mathcal{N}_\alpha^\beta(x,y) = \Gamma_{\alpha\gamma}^\beta(x)y^\gamma$.

2.4. Covariant derivative associated to a nonlinear connection. A tangent vector v to E at a point u is called admissible if $T_u\pi(v) = \sigma(u)$ [19]. A curve in E is admissible if its tangent vectors are admissible.

Definition 10. *A law of covariant derivative associated to a nonlinear connection $\mathcal{N}$ is an application $\mathcal{D} : Sec(E)\times Sec(E) \to Sec(E)$ $(\rho,\eta) \to \mathcal{D}_\rho\eta$ such that*

$$\mathcal{D}_{f\rho}\eta = f\mathcal{D}_\rho\eta, \quad \mathcal{D}_\rho(f\eta) = (\sigma(\rho)f)\eta + f\mathcal{D}_\rho\eta,$$

for any function $f \in C^\infty(M)$ and $\rho,\eta \in Sec(E)$.

The covariant derivative has the local expression

$$\mathcal{D}_\rho\eta = \rho^\alpha\left(\sigma_\alpha^i\frac{\partial\eta^\beta}{\partial x^i} + \mathcal{N}_\alpha^\beta\right)s_\beta, \tag{19}$$

where $\rho = \rho^\alpha s_\alpha$ and $\eta = \eta^\alpha s_\alpha$. Let $a : I \to E$ be an admissible curve and let $b : I \to E$ be a curve in E, both of them projecting by π onto the same curve γ in M, $\pi(a(t)) = \pi(b(t)) = \gamma(t)$. We can define the derivative of $b(t)$ along the $a(t)$, denoted $\mathcal{D}_a b$. In local coordinates we have

$$\mathcal{D}_{a(t)}b(t) = \left(\frac{db^\beta}{dt} + \mathcal{N}_\alpha^\beta a^\alpha\right)s_\beta(\gamma(t)). \tag{20}$$

Definition 11. *An admissible curve $c(t)$ is a path (autoparallel) for nonlinear connection $\mathcal{N}$ if and only if $\mathcal{D}_{c(t)}c(t) = 0$.*

In local coordinates we get

$$\frac{dc^\beta}{dt} + \mathcal{N}_\alpha^\beta(x,y)c^\alpha = 0. \tag{21}$$

Proposition 4. *A curve in E is autoparallel for the nonlinear connection if and only if*

$$\frac{dx^i}{dt} = \sigma_\alpha^i y^\alpha, \quad \frac{dy^\beta}{dt} + \mathcal{N}_\alpha^\beta y^\alpha = 0. \tag{22}$$

2.5. Adapted basis of nonlinear connection. Let $\mathcal{N}$ be the nonlinear connection with components $\mathcal{N}_\alpha^\beta(x,y)$. We set

$$\delta_\alpha = h(\mathcal{X}_\alpha) = \mathcal{X}_\alpha - \mathcal{N}_\alpha^\beta\mathcal{V}_\beta, \tag{23}$$

then $\{\delta_\alpha, \mathcal{V}_\alpha\}$ is a local basis which is called the adapted basis with respect to $\mathcal{N}$. We have

$$\sigma^1(\delta_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i} - \mathcal{N}^\beta_\alpha \frac{\partial}{\partial y^\beta}, \quad \sigma^1(\mathcal{V}_\alpha) = \frac{\partial}{\partial y^\alpha}. \tag{24}$$

The dual adapted basis is given by $\{\mathcal{X}^\alpha, \delta\mathcal{V}^\alpha\}$ where

$$\delta\mathcal{V}^\alpha = \mathcal{V}^\alpha + \mathcal{N}^\alpha_\beta \mathcal{X}^\beta, \tag{25}$$

and $\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$ is the dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$. On the vector bundle (TE, π_E, E) we give also a nonlinear connection (see MIRON and ANASTASIEI [13]), such that $TE = HTE \oplus VTE$. The nonlinear connection on TE has the coefficients denoted by $N^\alpha_i(x, y)$ and the adapted basis is $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^\alpha}\}$ where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^\alpha_i \frac{\partial}{\partial y^\alpha}$. The dual adapted basis is $\{dx^i, \delta y^\alpha\}$ where $\delta y^\alpha = dy^\alpha + N^\alpha_i dx^i$ and $\{dx^i, dy^\alpha\}$ is the dual basis of $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha}\}$.

Proposition 5. *The relations between the coefficients of nonlinear connections and adapted basis on $\mathcal{T}E$ and TE are given by*

$$\mathcal{N}^\beta_\alpha = N^\beta_i \sigma^i_\alpha, \quad \sigma^1(\delta_\alpha) = \sigma^i_\alpha \frac{\delta}{\delta x^i}. \tag{26}$$

Proof. We know that $\sigma^{*1}(dx^i) = \sigma^i_\alpha \mathcal{X}^\alpha$ and $\sigma^{*1}(dy^\alpha) = \mathcal{V}^\alpha$, where σ^{*1} is the dual application of σ^1. Then we get $\sigma^{*1}(\delta y^\alpha) = \sigma^{*1}(dy^\alpha + N^\alpha_i dx^i) = \mathcal{V}^\alpha + N^\alpha_i \sigma^i_\beta \mathcal{X}^\beta$. But on $(\mathcal{T}E)^*$ we have the dual adapted basis $\delta\mathcal{V}^\alpha = \mathcal{V}^\alpha + \mathcal{N}^\alpha_\beta \mathcal{X}^\beta$ and $\sigma^{*1}\big|_{(VTE)^*} : (VTE)^* \to (V\mathcal{T}E)^*$ is an isomorphism. Then we obtain $\sigma^{*1}(\delta y^\alpha) = \delta\mathcal{V}^\alpha$ and it results $\mathcal{V}^\alpha + N^\alpha_i \sigma^i_\beta \mathcal{X}^\beta = \mathcal{V}^\alpha + \mathcal{N}^\alpha_\beta \mathcal{X}^\beta \Rightarrow N^\alpha_i \sigma^i_\beta = \mathcal{N}^\alpha_\beta$. Finally we obtain $\sigma^1(\delta_\alpha) = \sigma^1(\mathcal{X}_\alpha - \mathcal{N}^\beta_\alpha \mathcal{V}_\beta) = \sigma^i_\alpha \frac{\partial}{\partial x^i} - \mathcal{N}^\beta_\alpha \frac{\partial}{\partial y^\beta} = \sigma^i_\alpha \left(\frac{\partial}{\partial x^i} - N^\beta_i \frac{\partial}{\partial y^\beta}\right) = \sigma^i_\alpha \frac{\delta}{\delta x^i}$. □

Lemma 6. *The Lie brackets of the sections from adapted basis $\{\delta_\alpha, \mathcal{V}_\alpha\}$ are*

$$[\delta_\alpha, \delta_\beta] = L^\gamma_{\alpha\beta}\delta_\gamma + \mathcal{R}^\gamma_{\alpha\beta}\mathcal{V}_\gamma, \quad [\delta_\alpha, \mathcal{V}_\beta] = \frac{\partial \mathcal{N}^\gamma_\alpha}{\partial y^\beta}\mathcal{V}_\gamma, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0, \tag{27}$$

$$\mathcal{R}^\gamma_{\alpha\beta} = \sigma^i_\beta \frac{\partial \mathcal{N}^\gamma_\alpha}{\partial x^i} - \sigma^i_\alpha \frac{\partial \mathcal{N}^\gamma_\beta}{\partial x^i} - \mathcal{N}^\varepsilon_\beta \frac{\partial \mathcal{N}^\gamma_\alpha}{\partial y^\varepsilon} + \mathcal{N}^\varepsilon_\alpha \frac{\partial \mathcal{N}^\gamma_\beta}{\partial y^\varepsilon} + L^\varepsilon_{\alpha\beta}\mathcal{N}^\gamma_\varepsilon. \tag{28}$$

Remark 1. The following formula holds $\sigma^1[\delta_\alpha, \delta_\beta] = [\sigma^1(\delta_\alpha), \sigma^1(\delta_\beta)]$.

2.6. Semispray and nonlinear connection. Let $\mathcal{N}$ be a nonlinear connection on $\mathcal{T}E$, ξ' an arbitrary semispray on $\mathcal{T}E$ and $h = \frac{1}{2}(Id + \mathcal{N})$ the horizontal projector of $\mathcal{N}$. We consider $\xi = h\xi'$ and for any other semispray ξ'' on $\mathcal{T}E$ we have $h(\xi' - \xi'') = h((\xi'^\alpha - \xi''^\alpha)\mathcal{V}_\alpha) = 0$ and it results that ξ doesn't depends on the choose of ξ'. We have $\mathcal{J}\xi\mathcal{J}h\xi' = \mathcal{J}\xi' = \mathcal{C}$ so ξ is a semispray, which is called the associated semispray to $\mathcal{N}$. If $\xi' = y^\alpha\mathcal{X}_\alpha + \xi'^\alpha\mathcal{V}_\alpha$ then $h\xi' = y^\alpha(\mathcal{X}_\alpha - \mathcal{N}^\beta_\alpha\mathcal{V}_\beta)$ so $\xi = y^\alpha\mathcal{X}_\alpha - \mathcal{N}^\beta_\alpha y^\alpha\mathcal{V}_\beta$. If $\mathcal{N}$ is a linear connection, then we have $\xi = y^\alpha\mathcal{X}_\alpha - \Gamma^\beta_{\alpha\gamma}(x)y^\alpha y^\gamma\mathcal{V}_\beta$.

Remark 2. If $\mathcal{N}$ is homogeneous (linear), then ξ is a spray (quadratic spray).

Now, let ξ be an arbitrary semispray on $\mathcal{T}E$. We consider

$$\mathcal{N} = -L_\xi\mathcal{J} \tag{29}$$

that is $\mathcal{N}(Z) = -[\xi, \mathcal{J}Z] + \mathcal{J}[\xi, Z]$, where $Z \in Sec(\mathcal{T}E)$.

Theorem 7. *N is a nonlinear connection in $\mathcal{T}E$ with associated semispray given by $\xi + \frac{1}{2}\xi^*$.*

Proof. Using (12) we get: $\mathcal{J}\mathcal{N}(Z)=\mathcal{J}(-[\xi, \mathcal{J}Z]+\mathcal{J}[\xi, Z])=-\mathcal{J}[\xi, \mathcal{J}Z] = \mathcal{J}(Z)$ and $\mathcal{N}\mathcal{J}(Z) = -[\xi, \mathcal{J}^2Z] + \mathcal{J}[\xi, \mathcal{J}Z] = \mathcal{J}[\xi, \mathcal{J}Z] = -\mathcal{J}(Z)$. If ξ' is the associated semispray to $\mathcal{N}$ we obtain $\xi' = h\xi = \frac{1}{2}\xi + \frac{1}{2}\mathcal{N}(\xi) = \frac{1}{2}\xi + \frac{1}{2}(-[\xi, \mathcal{J}\xi] + \mathcal{J}[\xi, \xi]) = \frac{1}{2}\xi + \frac{1}{2}[\mathcal{C}, \xi] = \xi + \frac{1}{2}\xi^*$. □

Proposition 8. *For $\xi = y^\alpha\mathcal{X}_\alpha + \xi^\alpha\mathcal{V}_\alpha$ the nonlinear connection $\mathcal{N} = -L_\xi\mathcal{J}$ is given by*

$$\mathcal{N}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha + \left(\frac{\partial\xi^\beta}{\partial y^\alpha} + y^\gamma L^\beta_{\gamma\alpha}\right)\mathcal{V}_\beta, \quad \mathcal{N}(\mathcal{V}_\alpha) = -\mathcal{V}_\alpha, \tag{30}$$

$$\mathcal{N}^\beta_\alpha = \frac{1}{2}(-\frac{\partial\xi^\beta}{\partial y^\alpha} + y^\gamma L^\beta_{\alpha\gamma}) \tag{31}$$

Proof. We have $\mathcal{N}(\mathcal{X}_\alpha) = -[\xi, \mathcal{J}(\mathcal{X}_\alpha)] + \mathcal{J}[\xi, \mathcal{X}_\alpha] = \mathcal{X}_\alpha + \frac{\partial\xi^\beta}{\partial y^\alpha}\mathcal{V}_\beta + \mathcal{J}(y^\beta L^\gamma_{\beta\alpha}\mathcal{X}_\gamma - \sigma^i_\alpha\frac{\partial\xi^\beta}{\partial x^i}\mathcal{V}_\beta)\mathcal{X}_\alpha + \left(\frac{\partial\xi^\beta}{\partial y^\alpha} + y^\gamma L^\beta_{\gamma\alpha}\right)\mathcal{V}_\beta$ and using the relation $\mathcal{N}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha - 2\mathcal{N}^\beta_\alpha(x, y)\mathcal{V}_\beta$ we obtain (30) and (31). □

Corollary 9. *If ξ is a spray then $\mathcal{N} = -L_\xi \mathcal{J}$ is a homogeneous nonlinear connection, whose associated spray is ξ. If ξ is a quadratic spray then $\mathcal{N} = -L_\xi \mathcal{J}$ is a linear connection.*

Proof. If $\mathcal{H}$ is the tension of $\mathcal{N}$ then $\mathcal{H}(\mathcal{X}_\alpha) = \frac{1}{2}\left(-\frac{\partial \xi^\gamma}{\partial y^\alpha} + y^\beta \frac{\partial^2 \xi^\gamma}{\partial y^\alpha \partial y^\beta}\right)\mathcal{V}_\gamma$. But ξ is a spray and it results that ξ^γ is homogeneous of degree 2, so $2\xi^\gamma = y^\beta \frac{\partial \xi^\gamma}{\partial y^\beta}$ and $\frac{\partial \xi^\gamma}{\partial y^\alpha} = y^\beta \frac{\partial^2 \xi^\gamma}{\partial y^\alpha \partial y^\beta}$ and therefore $\mathcal{H} = 0$. □

Theorem 10. *A nonlinear connection $\mathcal{N}$ and its associated semispray have the same path.*

Proof. The associated semispray is $\xi = y^\alpha \mathcal{X}_\alpha - \mathcal{N}_\alpha^\beta y^\alpha \mathcal{V}_\beta$ so $\xi^\beta = -\mathcal{N}_\alpha^\beta y^\alpha$ and it results from (11) and (22). □

Definition 12. *The curvature of nonlinear connection $\mathcal{N}$ is given by $\Omega = -N_h$, where h is the associated horizontal projector of $\mathcal{N}$, and N_h is the Nijenhuis tensor associated to h.*

In the local coordinates we get

$$\Omega(\delta_\alpha, \delta_\beta) = -\mathcal{R}_{\alpha\beta}^\gamma \mathcal{V}_\gamma, \quad \Omega(\delta_\alpha, \mathcal{V}_\beta) = 0, \quad \Omega(\mathcal{V}_\alpha, \mathcal{V}_\beta) = 0. \tag{32}$$

where $\mathcal{R}_{\alpha\beta}^\gamma$ is given by (28) and are called the curvature of nonlinear connection $\mathcal{N}$ on $\mathcal{T}E$.

Lemma 11. *The relation between the curvatures of nonlinear connections on $\mathcal{T}E$ and TE is given by*

$$\mathcal{R}_{\alpha\beta}^\gamma = \sigma_\alpha^i \sigma_\beta^j R_{ij}^\gamma, \tag{33}$$

where $R_{ij}^\gamma = \frac{\delta N_i^\gamma}{\delta x^j} - \frac{\delta N_j^\gamma}{\delta x^i}$.

Proof. Using the relations $\mathcal{N}_\alpha^\beta = N_i^\beta \sigma_\alpha^i$ and (28). □

Theorem 12. *The horizontal distribution $H\mathcal{T}E$ of a nonlinear connection $\mathcal{N}$ is integrable iff the curvature vanishes.*

Proof. We have $[\delta_\alpha, \delta_\beta] = L_{\alpha\beta}^\gamma \delta_\gamma + \mathcal{R}_{\alpha\beta}^\gamma \mathcal{V}_\gamma$ therefore $H\mathcal{T}E$ is integrable iff $\mathcal{R}_{\alpha\beta}^\gamma = 0$. □

Definition 13. *The weak torsion of $\mathcal{N}$ is defined by $t = [\mathcal{J}, h]$, that is*

$$[\mathcal{J}, h](Z, W) = [\mathcal{J}Z, hW] + [hZ, \mathcal{J}W] + \mathcal{J}[Z, W] - \mathcal{J}[Z, hW] - \\ -\mathcal{J}[hZ, W] - h[Z, \mathcal{J}W] - h[\mathcal{J}Z, W]$$

for any sections Z, W on $\mathcal{T}E$.

In the local coordinates we get

$$t(\delta_\alpha, \delta_\beta) = \left(\frac{\partial N^\gamma_\alpha}{\partial y^\beta} - \frac{\partial N^\gamma_\beta}{\partial y^\alpha} - L^\gamma_{\alpha\beta} \right) \mathcal{V}_\gamma, t(\delta_\alpha, \mathcal{V}_\beta) = 0, t(\mathcal{V}_\alpha, \mathcal{V}_\beta) = 0 \tag{34}$$

Definition 14. *The strong torsion $\mathcal{T}$ on $\mathcal{N}$ is given by $\mathcal{T} = i_\xi t - \mathcal{H}$, where $\mathcal{H}$ is the tension of $\mathcal{N}$, and i_ξ is the contraction with ξ.*

Locally we obtain

$$\mathcal{T}(\delta_\alpha) = \left(\frac{\partial \mathcal{N}^\gamma_\beta}{\partial y^\alpha} y^\beta - \mathcal{N}^\gamma_\alpha + y^\beta L^\gamma_{\alpha\beta} \right) \mathcal{V}_\gamma, \quad T(\mathcal{V}_\alpha) = 0. \tag{35}$$

Proposition 13. *The strong torsion $\mathcal{T}$ of a nonlinear connection $\mathcal{N}$ vanishes if and only if the weak torsion and the tension of $\mathcal{N}$ vanish.*

Proof. If $\mathcal{T} = 0$ then $\mathcal{N}^\gamma_\alpha = \frac{\partial \mathcal{N}^\gamma_\beta}{\partial y^\alpha} y^\beta + y^\beta L^\gamma_{\alpha\beta}$ and $t(\delta_\alpha, \delta_\beta) \frac{\partial \mathcal{N}^\gamma_\beta}{\partial y^\alpha} + \frac{\partial^2 \mathcal{N}^\gamma_\varepsilon}{\partial y^\alpha \partial y^\beta} y^\varepsilon - \frac{\partial \mathcal{N}^\gamma_\alpha}{\partial y^\beta} - \frac{\partial^2 \mathcal{N}^\gamma_\varepsilon}{\partial y^\alpha \partial y^\beta} y^\varepsilon + L^\gamma_{\alpha\beta} - L^\gamma_{\beta\alpha} - L^\gamma_{\alpha\beta} = \frac{\partial \mathcal{N}^\gamma_\beta}{\partial y^\alpha} - \frac{\partial \mathcal{N}^\gamma_\alpha}{\partial y^\beta} - L^\gamma_{\beta\alpha} = -t(\delta_\alpha, \delta_\beta)$ so, we obtain $t(\delta_\alpha, \delta_\beta) = 0$ and $\mathcal{H} = 0$. □

Proposition 14. *Let $\mathcal{N}$ and $\mathcal{N}'$ be two nonlinear connections in $\mathcal{T}E$ with the same strong torsion and the same associated semispray. Then $\mathcal{N} = \mathcal{N}'$.*

The proof follows the case of tangent bundle (see [8]).

3. Lagrangian systems. Let (E, π, M) be a Lie algebroid with the anchor $\sigma : E \to TM$ and $\mathcal{L} : E \to \mathbb{R}$ a differentiable function on E called the Lagrange function. When $\mathcal{L} \in C^\infty(E)$ we can define a dynamical system on E. The Euler-Lagrange equations on Lie algebroids [19] are

$$\frac{dx^i}{dt} = \sigma^i_\alpha y^\alpha, \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial y^\alpha} \right) = \sigma^i_\alpha \frac{\partial \mathcal{L}}{\partial x^i} - L^\gamma_{\alpha\beta} y^\beta \frac{\partial \mathcal{L}}{\partial y^\gamma}, \tag{36}$$

and from [11] we get the Hamilton-Jacobi equations

$$\frac{dx^i}{dt}=\sigma^i_\alpha\frac{\partial\mathcal{H}}{\partial\mu_\alpha},\quad \frac{d\mu_\alpha}{dt}=-\sigma^i_\alpha\frac{\partial\mathcal{H}}{\partial x^i}-\mu_\gamma L^\gamma_{\beta\alpha}\frac{\partial\mathcal{H}}{\partial\mu_\beta}. \tag{37}$$

If $\mathcal{L}$ is a regular and 2-homogeneous Lagrangian and we denote $g_{\alpha\beta}=\frac{\partial^2\mathcal{L}}{\partial y^\alpha\partial y^\beta}$ then we get:

Theorem 15. *The corresponding homogeneous nonlinear connection $\mathcal{N}=-L_\xi\mathcal{J}$ has the coefficients given by*

$$\mathcal{N}^\delta_\varepsilon=\frac{1}{2}\left(-\frac{\partial\xi^\delta}{\partial y^\varepsilon}+y^\gamma L^\delta_{\varepsilon\gamma}\right), \tag{38}$$

$$\xi^\delta=\frac{1}{2}g^{\delta\beta}\left(\sigma^i_\alpha\frac{\partial g_{\beta\gamma}}{\partial x^i}+\sigma^i_\gamma\frac{\partial g_{\alpha\beta}}{\partial x^i}-\sigma^i_\beta\frac{\partial g_{\alpha\gamma}}{\partial x^i}+\right.$$
$$\left.+g_{\varepsilon\alpha}L^\varepsilon_{\beta\gamma}+g_{\varepsilon\gamma}L^\varepsilon_{\beta\alpha}-g_{\varepsilon\beta}L^\varepsilon_{\gamma\alpha}\right)y^\alpha y^\gamma,$$

and is called the canonical nonlinear connection associated to the homogeneous Lagrangian $\mathcal{L}$.

Proof. Using [11] we get the semispray associated to the Lagrangian $\mathcal{L}$

$$\xi^\varepsilon=g^{\varepsilon\beta}\left(\sigma^i_\beta\frac{\partial\mathcal{L}}{\partial x^i}-\sigma^i_\alpha\frac{\partial^2\mathcal{L}}{\partial x^i\partial y^\beta}y^\alpha-L^\gamma_{\beta\alpha}y^\alpha\frac{\partial\mathcal{L}}{\partial y^\gamma}\right),$$

and from $\mathcal{L}=\frac{1}{2}g_{\alpha\beta}y^\alpha y^\beta$ and (31) we get (38). □

Definition 15. *A function $\mathcal{F}:E\to[0,\infty]$ which satisfies the following properties*
1) $\mathcal{F}$ is C^∞ on $E\backslash\{0\}$
2) $\mathcal{F}(\lambda u)=\lambda\mathcal{F}(u)$ for $\lambda>0$ and $u\in E_x$. $x\in M$.
3) For each $y\in E_x\backslash\{0\}$ the quadratic form

$$g_{\alpha\beta}(x,y)=\frac{1}{2}\frac{\partial^2\mathcal{F}^2}{\partial y^\alpha\partial y^\beta},$$

is positive definite, will be called the Finsler function on Lie algebroid.

We can associate to any Finsler function $\mathcal{F}$ on Lie algebroid a Finsler function F on $Im\sigma \subset TM$ defined by

$$F(v) = \{\mathcal{F}(u) \,|u \in E_x, \quad \sigma(u) = v\}, \tag{39}$$

for each $v \in (Im\sigma)_x \subset T_xM$, $x \in M$. If we consider the Lagrangians $L = \frac{1}{2}F^2$ and $\mathcal{L} = \frac{1}{2}\mathcal{F}^2$ we obtain $\mathcal{L} = L \circ \sigma$ and the Hamiltonian is

$$\begin{aligned} H(p) &= \sup_v \{\langle p, v\rangle - L(v)\} = \sup_v \{\langle p, v\rangle - \mathcal{L}(u); \sigma(u) = v\} = \\ &= \sup_u \{\langle p, \sigma(u)\rangle - \mathcal{L}(u)\} = \sup_u \{\langle \sigma^\star(p), u\rangle - \mathcal{L}(u)\} = \mathcal{H}(\sigma^\star(p)), \end{aligned}$$

and we get

$$H(p) = \mathcal{H}(\mu), \quad \mu = \sigma^\star(p), \tag{40}$$

$p \in T_x^*M$, $\mu \in E_x^*$.

3.1. Applications to optimal control. Let M be a smooth n-dimensional manifold. We consider on M the distributional system

$$\dot{x} = \sum_{i=1}^{m} u_i(t) X_i(x), \tag{41}$$

where $x \in M$ represents the state of system and $u \in \mathbb{R}^m$ represents the controls, such that the distribution $D = \langle X_1, X_2, ..., X_m\rangle$ has constant rank m and is integrable. The cost is given by

$$\min_{u(.)} \int_I \mathcal{F}(u(t))dt, \tag{42}$$

where $\mathcal{F}$ is a Minkowski norm on D. A piecewise smooth curve $c : I \subset \mathbb{R} \to M$ is called horizontal if the tangent vectors are in D, i.e. $\dot{c}(t) \in D_{c(t)} \subset TM$ for almost every $t \in I$. In order to apply the theory of Lie algebroids we consider $E = M \times D$ and the anchor $\sigma : E \to TM$ given by

$$\sigma(x, u) = \sum_{i=1}^{m} u_i(t) X_i(x). \tag{43}$$

Let $u : I \to E$ be an admissible curve projecting by π onto the horizontal curve $c : I \to M$ (i.e. $\sigma(u(t)) = \dot{c}(t)$ and $\pi(u(t)) = c(t)$). The length of the horizontal curve c is defined by

$$length(c) = \int_I \mathcal{F}(u(t))dt = \int_I F(\dot{c}(t))dt,$$

and the distance is given by $d(a, b) = \inf length(c)$ where the infimum is taken over all horizontal curves connecting a and b. The distance is infinite if there is no admissible curve that connects these two points. The distribution D is integrable and it determines a foliation on M and two points can be joint if and only if are situated in the same leaf. Considering $\mathcal{L} = \frac{1}{2}\mathcal{F}^2$ we obtain a regular and 2-homogeneous Lagrangian on Lie algebroid.

3.2. Example. We consider the following distributional system with positive homogeneous cost (Randers metric):

$$\dot{x} = u_1 X_1 + u_2 X_2, \quad x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3, \; X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; X_2 = \begin{pmatrix} x \\ 1 \\ 1 \end{pmatrix}$$

$$\min_{u(.)} \int_0^T \mathcal{F}(u(t))dt, \quad \mathcal{F} = \|u\| + \langle b, u \rangle, \; b = (\varepsilon, 0)^t, \; u = (u_1, u_2)^t, \; 0 \leq \varepsilon < 1.$$

We are looking for the geodesics starting from the origin and parametrized by arclength. The associated distribution $D =< X_1, X_2 >$ is integrable, because $X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, and therefore $[X_1, X_2] = X_1$. In the case of Lie algebroid we consider $E = \mathbb{R}^3 \times D$ and the anchor $\sigma : E \to T\mathbb{R}^3$ has the components

$$\sigma_\alpha^i = \begin{pmatrix} 1 & x \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{44}$$

and we get the Lagrangian $\mathcal{L} = \frac{1}{2}\left(\sqrt{(u_1)^2 + (u_2)^2} + \varepsilon u_1\right)^2$. Using a result from [6] we can find the Hamiltonian on E^* given by

$$\mathcal{H} = \frac{1}{2}\left(\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}} - \frac{\varepsilon\mu_1}{1-\varepsilon^2}\right)^2 \tag{45}$$

From (3) and the relation $[X_\alpha, X_\beta] = L^\gamma_{\alpha\beta} X_\gamma$ we obtain the non-zero components $L^1_{12} = 1$, $L^1_{21} = -1$ and from (37) we have

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial \mu_1} + x\frac{\partial \mathcal{H}}{\partial \mu_2}, \quad \dot{y} = \dot{z} = \frac{\partial \mathcal{H}}{\partial \mu_2}, \quad \dot{\mu}_1 = -\mu_1 \frac{\partial \mathcal{H}}{\partial \mu_2}, \quad \dot{\mu}_2 = -\mu_1 \frac{\partial \mathcal{H}}{\partial \mu_1} \tag{46}$$

where

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \mu_1} &= \frac{\left(1+\varepsilon^2\right)\mu_1}{(1-\varepsilon^2)^2} - \frac{\varepsilon\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}}}{1-\varepsilon^2} - \frac{\varepsilon\mu_1^2}{(1-\varepsilon^2)^3\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}}} \\ \frac{\partial \mathcal{H}}{\partial \mu_2} &= \frac{\mu_2}{1-\varepsilon^2} - \frac{\varepsilon\mu_1\mu_2}{(1-\varepsilon^2)^2\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}}} \end{aligned} \tag{47}$$

The form of the last relations leads to the following change of variables $\mu_1(t) = (1-\varepsilon^2) r(t) \sec h\theta(t)$, $\mu_2(t) = \sqrt{1-\varepsilon^2} r(t) \tanh\theta(t)$, and from (46) we obtain

$$\begin{aligned} \sqrt{1-\varepsilon^2}\,\dot{r} &= r^2 \varepsilon \sec h\theta \tanh\theta(\varepsilon \sec h\theta - 1), \\ \sqrt{1-\varepsilon^2}\,\dot{\theta} &= r(\varepsilon \sec h\theta - 1)^2 \end{aligned} \tag{48}$$

and we have $|r| = \frac{1}{c(\varepsilon \sec h\theta - 1)}$. But the geodesics are parameterized by arclength, that corresponds to fix the level $1/2$ of the Hamiltonian and we have $\mathcal{H} = \frac{1}{2c^2}$ so $c = \pm 1$. From (48) we get $t = \sqrt{1-\varepsilon^2}\int \frac{1}{1-\varepsilon \sec h\theta} d\theta$. Finally, from (46) by straightforward computation, we get the following result $x(\theta) = \frac{\sinh\theta}{\sqrt{1-\varepsilon^2}} \pm \frac{b(1-\cosh\theta)}{(1-\varepsilon^2)}$, $\quad y(\theta) = z(\theta) = \ln\frac{\cosh\theta - \varepsilon}{1-\varepsilon}$.

Remark 3. If $\varepsilon = 0$ we obtain the case of distributional systems with quadratic cost with the solution $x(t) = \sinh t \pm b(1-\cosh t)$, $\quad y(t) = z(t) = \ln\cosh t$.

REFERENCES

1. Anastasiei, M. – *Finsler vector bundle-metrizable connections*, Periodica Mathem atica Hungarica, vol. 48 (1-2), (2004) pp. 83-91.
2. Bao, D.; Chern, S.S.; Shen, Z. – *An introduction to Riemann-Finsler Geometry*, Springer-Verlag, New York, 2000.
3. Crampin, M. – *Tangent bundle geometry for Lagrangian dynamics*, J. Phys. A: Math. Gen. 16 (1983), pp. 3755-3722.

4. Grifone, J. – *Structure presque tangente et connections I,* Ann. Inst. Fourier 22 (1) (1972), pp. 287-334.
5. Higgins, P.J.; Mackenzie, K. – *Algebraic constructions in the category of Lie algebroids*, Journal of Algebra 129 (1990), pp. 194-230.
6. Hrimiuc, D.; Shimada, H. – *On the L-duality between Lagrange and Hamilton Manifold*, Nonlinear World, 3 (1996), pp. 613-641.
7. Klein, J. – *Espaces Variationnels et mecanique*, Ann. Inst. Fourie 12 (1962), pp. 1-124.
8. de Leon, M.; Rodriques, P. – *Methods of differential geometry in analytical mechanics*, North-Holland 158, 1989.
9. Libermann, P. – *Lie Algebroids and Mechanics*, Arch. Math. (Brno), 32 (1996), pp. 147-162.
10. Martinez, E. – *Lagrangian mechanics on Lie algebroids*, Acta Applicadae Mathematicae, 67, (2001), pp. 295-320.
11. Martinez, E. – *Geometric formulation of mechanics on Lie algebroids*, Proc. of the VIII Workshop on Geometry and Physics (Medina del Campo, 1999), vol. 2 of Publ. R. Soc. Mat. Esp. (2001), pp. 209-222.
12. Mackenzie, K. – *Lie grupoids and Lie algebroids in differential geometry*, 124 London Mathematical Society Lecture Note Series, Cambridge, 1987.
13. Miron, R.; Anastasiei, M. – *The Geometry of Lagrange spaces: Theory and Applications*, Kluwer Academic Publishers, no. 59, 1994.
14. Miron, R.; Hrimiuc, D.; Shimada, H.; Sabau, S. – *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Academic Publishers, no. 118, 2001.
15. Oproiu, V. – *Regular vector fields and connections on cotangent bundle*, An. Stiint. Univ. Al. I. Cuza Iasi, Ser. Noua, Mat. 37 no. 1 (1991), pp. 87-104.
16. Popescu, L. – *Geometrical structures on relative self-tangent spaces*, Proceeding of the 3^{th} annual meeting of the Romanian Society of Mathematical Sciences, Craiova, May 26-29, 1999, (2000) pp. 184-192.
17. Popescu, P. – *On the geometry of relative tangent spaces*, Rev. Roumain Math. Pures and Applications, 37, (1992), pp. 779-789.
18. Vranceanu, Gh. – *Sur quelques tenseurs dans les variétés non holonomes*, C.R. Acad. Sci. Paris, 186, (1928) 995.
19. Weinstein, A. – *Lagrangian mechanics and grupoids*, Fields Institute Communications, 7, (1996), pp. 206-231.

Received: 06.X.2004

University of Craiova,
Faculty of Economic Sciences,
13, A.I. Cuza st.,
Craiova, 1100,
ROMÂNIA,
liviupopescu@central.ucv.ro