ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" IAŞI
Tomul LI, s.I, Matematică, 2005, f.1

A VITALI TYPE THEOREM FOR THE SET-VALUED GOULD INTEGRAL

BY

B. SATCO

Abstract. We obtain a Vitali-type theorem of passing to the limit into the set-valued Gould integral (this kind of integral of a real function with respect to a finitely additive multimeasure was introduced and studied in [6] and [7]). As a consequence, a Lebesgue-type result is given.

Mathematics Subject Classification 2000: 28B20, 26E25.
Key words: set-valued Gould integral, multimeasure, $|M|$-totally measurable function.

1. Terminology and notations. Let Ω be a set, Σ an algebra of subsets of Ω and X a Banach space. Let X^* be the topological dual of X and B^* its unit closed ball. We denote by $\mathcal{P}_0(X)$, resp. $\mathcal{P}_{wkc}(X)$ the family of its nonempty, resp. nonempty weakly compact convex subsets. If D is the Hausdorff distance, we denote by $|A| = D(A, \{0\})$ for every $A \in \mathcal{P}_{wkc}(X)$.

Definition 1. *A multifunction $M : \Sigma \to \mathcal{P}_0(X)$ is said to be a finitely additive multimeasure if $M(\emptyset) = \{0\}$ and, for every $A, B \in \Sigma$ with $A \cap B = \emptyset$, we have:*

$$M(A \cup B) = M(A) + M(B). \tag{1}$$

Definition 2. *Let $M : \Sigma \to \mathcal{P}_0(X)$ be a finitely additive multimeasure.*
i) The set function $|M| : \Sigma \to \overline{\mathbb{R}_+}$ defined by

$$|M|(A) = v(M; A) = \sup_{\pi} \left\{ \sum_{i=1}^{n} |M(A_i)| \right\}, \forall A \in \Sigma,$$

where $\pi = \{A_i\}_{i=1}^n$ is a finite Σ-partition of A, is called the variation of M.
ii) If $|M|(\Omega) < \infty$, we say that M is of bounded variation.

It is known that $|M|$ is a $\overline{\mathbb{R}_+}$-valued finitely additive measure. In the sequel, $|M|^*$ will denote the outer measure associated to $|M|$: for every $E \subset \Omega$, $|M|^*(E) = \inf\{|M|(F) : F \in \Sigma, F \supseteq E\}$.

Now we recall some definitions which appear in the theory of Gould set-valued integral.

Definition 3. *A function $f : \Omega \to \mathbb{R}$ is said to be $|M|$-totally measurable if, for every $\varepsilon > 0$, there exists a Σ-partition of Ω, $\pi_\varepsilon = \{E_i\}_{i=0}^n$, such that $|M|(E_0) < \varepsilon$ and $osc(f, E_i) < \varepsilon, \forall i \in \{1, ..., n\}$, where $osc(f, E) = \sup\{|f(s) - f(t)| ; s, t \in E\}$ is the oscillation of f on the set E.*

Definition 4. *i) A finite Σ-partition $\pi_1 = \left\{A_j^{'}\right\}_{j\in J}$ is finer than another finite Σ-partition $\pi_2 = \{A_i\}_{i\in I}$ if every element of π_1 is contained into an element of π_2 ($\pi_1 \succcurlyeq \pi_2$).*

ii) If $\pi_1 = \left\{A_j^{'}\right\}_{j\in J}$ and $\pi_2 = \{A_i\}_{i\in I}$ are two finite Σ-partitions of Ω, the partition $\pi_0 = \pi_1 \cap \pi_2 = \left\{A_i \cap A_j^{'}\right\}_{i\in I, j\in J}$ is said to be the common refinement of the partitions π_1 and π_2. The common refinement of two partitions is finer than both of them.

Definition 5. *Let $M : \Sigma \to \mathcal{P}_{wkc}(X)$ be a finitely additive multimeasure with bounded variation and $f : \Omega \to \mathbb{R}$ a bounded function.*

i) We say that f is Gould integrable with respect to M (briefly, M-Gould-integrable) if there exists a nonempty weakly compact convex subset of X, denoted by $(G)\int_\Omega f dM$, which verifies the condition that, for every $\varepsilon > 0$, there exists a partition π_ε of Ω such that, for every other partition $\pi = \{E_i\}_{i=0}^n$ finer than π_ε and for every choice of $s_i \in E_i$, we have:

$$D\left(\sum_{i=0}^{n} f(s_i)M(E_i), (G)\int_\Omega f dM\right) < \varepsilon.$$

The set $(G)\int_\Omega f dM$ is called the M-Gould integral of f.

ii) If $E \in \Sigma$, f is said to be M-Gould-integrable over E if the restriction of f to E is Gould-integrable with respect to $(E, \Sigma E, M_E)$, where ΣE is the set of all Σ-measurable subsets of E.

To define the M-Gould-integral in the general, unbounded case, suppose that $M : \Sigma \to \mathcal{P}_{wkc}(X)$ is a finitely additive multimeasure for which $(\Omega, \Sigma, |M|)$ is a non-degenerate space (see [4], p. 210).

For a function $f : \Omega \to \mathbb{R}$ we define

$$E(f) = \{K \in \Sigma; |M|(K) < \infty \text{ and } f \text{ is bounded on } K\}. \tag{2}$$

Definition 6. *Under the above mentioned assumptions on M, a $|M|$-totally measurable function $f : \Omega \to \mathbb{R}$ is said to be generalized M-Gould-integrable (briefly, g-M-Gould-integrable) if there exists an element $A \in \mathcal{P}_{wkc}(X)$, denoted by $(G)\int_\Omega f dM$, such that, for every $\varepsilon > 0$, there exists $K_\varepsilon \in E(f)$ with the property that, for any $K \in E(f), K \supseteq K_\varepsilon$ we have*

$$D\left((G)\int_K f dM, A\right) < \varepsilon,$$

where $(G)\int_K f dM$ is the integral of f on K in the sense of Definition 5. $(G)\int_\Omega f dM$ is called the (generalized) M-Gould-integral of f with respect to M.

In ([7], Theorem 3.7), the following characterization for the g-M-Gould-integrability is given:

Theorem 7. *The $|M|$-totally measurable function $f : \Omega \to \mathbb{R}$ is g-M-Gould-integrable on Ω if and only if for every $\varepsilon > 0$, there exists $K_\varepsilon \in E(f)$ such that, for every $K \in E(f)$ with $K \cap K_\varepsilon = \emptyset$, we have $\left|(G)\int_K f dM\right| < \varepsilon$.*

Remark 8. All the results proved in [6] and [7] are still valid if the multimeasure M is $\mathcal{P}_{wkc}(X)$-valued. Moreover, excepting Theorem 2.14 in [6] and Theorem 3.11 in [7], the convexity assumption is not necessary.

In what follows, we shall shortly use the terms of M-Gould-integral and M-Gould-integrability for both situations bounded or unbounded, specifying, if it is the case, the particular situation.

2. A Vitali-type convergence theorem

Theorem 9. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real M-Gould-integrable functions convergent in $|M|$-measure to f, satisfying the following conditions:*

i) $\lim_{|M|(E)\to 0} \left|\int_E f_n dM\right| = 0$ uniformly in $n \in \mathbb{N}$;

ii) for every $\varepsilon > 0$, there exists $E_\varepsilon \in \Sigma$ such that $|M|(E_\varepsilon) < \infty$ and $\left|\int_E f_n dM\right| < \varepsilon$, for all measurable $E \subseteq \Omega \setminus E_\varepsilon$ and $n \in \mathbb{N}$.

Then f is M-Gould-integrable and $\lim_{n\to\infty}\int_E f_n dM = \int_E f dM$ uniformly with respect to $E \in \Sigma$.

Proof. First, we prove that the limit function, f, is $|M|$-totally measurable. Indeed, for each positive ε, $\lim_{n\to\infty} |M|^* \left(\left\{s \in \Omega : |f_n(s) - f(s)| \geq \frac{\varepsilon}{3}\right\}\right) = 0$, whence, by the definition of the outer measure, there exists $n_\varepsilon \in \mathbb{N}$ such that, for a certain measurable set Ω_ε with $|M|(\Omega_\varepsilon) < \frac{2\varepsilon}{3}$, the following inclusion holds:

$$\Omega_\varepsilon \supseteq \left\{s \in \Omega; |f_{n_\varepsilon}(s) - f(s)| \geq \frac{\varepsilon}{3}\right\}.$$

Since by hypothesis f_{n_ε} is totally measurable, one can find a measurable partition $(\Omega_0, ..., \Omega_{m_\varepsilon})$ of Ω such that $|M|(\Omega_0) < \frac{\varepsilon}{3}$ and $osc(f_{n_\varepsilon}, \Omega_i) < \frac{\varepsilon}{3}$ for each $i \in \{1, ..., m_\varepsilon\}$.

Therefore, the partition of Ω defined by $(\Omega_0 \cup \Omega_\varepsilon, \Omega_1 \setminus \Omega_\varepsilon, ..., \Omega_{m_\varepsilon} \setminus \Omega_\varepsilon)$ satisfies that $|M|(\Omega_0 \cup \Omega_\varepsilon) < \varepsilon$ and $osc(f, \Omega_i \setminus \Omega_\varepsilon) < \varepsilon$, for each $i \in \{1, ..., m_\varepsilon\}$ and so the limit function is $|M|$-totally measurable.

We will verify the integrability of f and the convergence of the integrals of f_n to the integral of f in two steps.

Step 1. $|M|(\Omega) < \infty$.

Step 1.1. We prove that for every $\varepsilon > 0$, there exists $\overline{\delta}_\varepsilon > 0$ such that $\left|\int_E f dM\right| < \varepsilon$, for all measurable $E \in E(f)$ with $|M|(E) < \overline{\delta}_\varepsilon$ (here the integral is in the sense of Definition 5).

By hypothesis, there exists $\delta_\varepsilon > 0$ such that $\left|\int_E f_n dM\right| < \frac{\varepsilon}{5}$ whenever $|M|(E) \leq \delta_\varepsilon$ and $n \in \mathbb{N}$. Let us now consider a measurable set E where f is bounded with the variation $|M|(E) < \overline{\delta}_\varepsilon = \min(\delta_\varepsilon, 1)$.

Using the $|M|$-absolute continuity of the Gould integral in the bounded case (Theorem 2.12 in [6]), we can find a positive $\delta'_{\varepsilon,E}$ such that $\left|\int_{E'} f dM\right| < \frac{\varepsilon}{5}$, for every $E' \subseteq E$ satisfying $|M|(E') < \delta'_{\varepsilon,E}$.

Since the sequence is $|M|$-convergent to f, we are able to choose $n_{\varepsilon,E} \in \mathbb{N}$ and $F_{\varepsilon,E} \in \Sigma E$ with $|M|(F_{\varepsilon,E}) < \delta'_{\varepsilon,E}$, on whose complementary with respect to E the following inequality holds:

$$\left|f_{n_{\varepsilon,E}}(s) - f(s)\right| < \frac{\varepsilon}{5}.$$

We deduce that

$$\begin{aligned}\left|\int_E f dM\right| &\leq \left|\int_{F_{\varepsilon,E}} f dM\right| + \left|\int_{E\setminus F_{\varepsilon,E}} f dM\right| < \\ &< \frac{\varepsilon}{5} + \left|\int_{E\setminus F_{\varepsilon,E}} f_{n_\varepsilon} dM\right| + D\left(\int_{E\setminus F_{\varepsilon,E}} f_{n_\varepsilon} dM, \int_{E\setminus F_{\varepsilon,E}} f dM\right) < \\ &< \frac{2\varepsilon}{5} + D\left(\int_{E\setminus F_{\varepsilon,E}} f_{n_{\varepsilon,E}} dM, \int_{E\setminus F_{\varepsilon,E}} f dM\right).\end{aligned}$$

In what follows, since the variation of M on $E \setminus F_{\varepsilon,E}$ is finite and the two functions f and $f_{n_{\varepsilon,E}}$ are bounded on it, by definition, one can find a measurable partition $(E_0, ..., E_m)$ of $E \setminus F_{\varepsilon,E}$ satisfying that, for every intermediary points $s_i \in E_i, i \in \{0, ..., m\}$,

$$D\left(\sum_{i=0}^{m} f(s_i) M(E_i), \int_{E\setminus F_{\varepsilon,E}} f dM\right) < \frac{\varepsilon}{5}$$

and

$$D\left(\sum_{i=0}^{m} f_{n_{\varepsilon,E}}(s_i) M(E_i), \int_{E\setminus F_{\varepsilon,E}} f_{n_{\varepsilon,E}} dM\right) < \frac{\varepsilon}{5}.$$

Therefore, by using the inequality

$$D(\alpha A, \beta A) \leq |\alpha - \beta| \cdot |A|$$

that is available for every real α, β and every $A \in P_{wkc}(X)$, we obtain that

$$\begin{aligned}\left|\int_E f dM\right| &< \frac{4\varepsilon}{5} + D\left(\sum_{i=0}^{m} f(s_i) M(E_i), \sum_{i=0}^{m} f_{n_{\varepsilon,E}}(s_i) M(E_i)\right) \leq \\ &\leq \frac{4\varepsilon}{5} + \sum_{i=0}^{m} \left|f(s_i) - f_{n_{\varepsilon,E}}(s_i)\right| |M(E_i)| < \varepsilon.\end{aligned}$$

Step 1.2. Let us now prove that f is Gould-integrable.

The sequence is $|M|$-convergent to f, so we can choose $n_\varepsilon \in \mathbb{N}$ and, for each $n \geq n_\varepsilon$, $F'_{n,\varepsilon} \in \Sigma$ with $|M|(F'_{n,\varepsilon}) < \overline{\delta}_\varepsilon$, on whose complementary $|f_n(s) - f(s)| < \varepsilon$. The function f_{n_ε} being Gould-integrable, according to

Theorem 7, there is $F''_{n_\varepsilon,\varepsilon} \in E(f_{n_\varepsilon})$ such that, for every $E \in E(f_{n_\varepsilon})$ disjoint of $F''_{n_\varepsilon,\varepsilon}$,

$$\left|\int_E f_{n_\varepsilon} dM\right| < \varepsilon.$$

Then $F_\varepsilon = F''_{n_\varepsilon,\varepsilon} \setminus F'_{n_\varepsilon,\varepsilon}$ is in $E(f)$ and, if we consider an arbitrary $E \in E(f)$ disjoint of F_ε, then

$$\begin{aligned}
&\left|\int_E f dM\right| \leq \left|\int_{E\cap F'_{n_\varepsilon,\varepsilon}} f dM\right| + \left|\int_{E\setminus F'_{n_\varepsilon,\varepsilon}\setminus F''_{n_\varepsilon,\varepsilon}} f dM\right| < \\
&<\varepsilon + \left|\int_{E\setminus F'_{n_\varepsilon,\varepsilon}\setminus F''_{n_\varepsilon,\varepsilon}} f_{n_\varepsilon} dM\right| + D\left(\int_{E\setminus F'_{n_\varepsilon,\varepsilon}\setminus F''_{n_\varepsilon,\varepsilon}} f_{n_\varepsilon} dM, \int_{E\setminus F'_{n_\varepsilon,\varepsilon}\setminus F''_{n_\varepsilon,\varepsilon}} f dM\right) < \\
&< 2\varepsilon + D\left(\int_{E\setminus F'_{n_\varepsilon,\varepsilon}\setminus F''_{n_\varepsilon,\varepsilon}} f_{n_\varepsilon} dM, \int_{E\setminus F'_{n_\varepsilon,\varepsilon}\setminus F''_{n_\varepsilon,\varepsilon}} f dM\right)
\end{aligned}$$

whence, by passing through the integral sums, as it was done at the Step 1.1, we obtain that

$$\left|\int_E f dM\right| < (4 + |M|(\Omega))\,\varepsilon$$

and so, according to Theorem 7, the integrability of the limit function is proved.

Step 1.3. It remains us to prove that $\lim\limits_{n\to\infty} \int_E f_n dM = \int_E f dM$ uniformly with respect to $E \in \Sigma$.

We use the $|M|$-absolute continuity of the Gould integral (Theorem 3.10 in [7]): for every $\varepsilon > 0$, there exists a positive $\widetilde{\delta}_\varepsilon \leq \delta_\varepsilon$ such that $\left|\int_E f dM\right| < \varepsilon$ for all measurable E with $|M|(E) < \widetilde{\delta}_\varepsilon$.

For every positive ε, we can choose, as at the Step 1.2, an $n_\varepsilon \in \mathbb{N}$ and, for every $n \geq n_\varepsilon$, $F'_{n,\varepsilon} \in \Sigma$ with $|M|(F'_{n,\varepsilon}) < \widetilde{\delta}_\varepsilon$, on whose complementary $|f_n(s) - f(s)| < \varepsilon$.

We obtain that, for every $n \geq n_\varepsilon$,

$$\begin{aligned} D\left(\int_E f_n dM, \int_E f dM\right) &\leq D\left(\int_{E\cap F'_{n,\varepsilon}} f_n dM, \int_{E\cap F'_{n,\varepsilon}} f dM\right) + \\ &+ D\left(\int_{E\setminus F'_{n,\varepsilon}} f_n dM, \int_{E\setminus F'_{n,\varepsilon}} f dM\right) \leq \left|\int_{E\cap F'_{n,\varepsilon}} f_n dM\right| + \\ &+ \left|\int_{E\cap F'_{n,\varepsilon}} f dM\right| + D\left(\int_{E\setminus F'_{n,\varepsilon}} f_n dM, \int_{E\setminus F'_{n,\varepsilon}} f dM\right) < \\ &< (4 + |M|(\Omega))\,\varepsilon, \end{aligned}$$

and so the statement is proved.

Step 2. The general, unbounded case.

By hypothesis, for each $\varepsilon > 0$, there exists $E_\varepsilon \in \Sigma$ with $|M|(E_\varepsilon) < \infty$ such that, for every $E \subseteq \Omega \setminus E_\varepsilon$ and every $n \in \mathbb{N}$, $\left|\int_E f_n dM\right| < \frac{\varepsilon}{5}$.
We have proved at the first step that f is Gould-integrable on E_ε, so one can find $\widetilde{E}_\varepsilon \subseteq E_\varepsilon$ on which f is bounded satisfying that, for every measurable set $E \subseteq E_\varepsilon \setminus \widetilde{E}_\varepsilon$ where f is bounded, $\left|\int_E f dM\right| < \frac{\varepsilon}{5}$.

We claim that every measurable set $\widetilde{E} \in E(f)$, $\widetilde{E} \subseteq \Omega \setminus \widetilde{E}_\varepsilon$ satisfies:

$$\left|\int_{\widetilde{E}} f dM\right| < \varepsilon.$$

In that purpose, let us take a positive $\delta_{\varepsilon,\widetilde{E}}$ such that $\left|\int_E f dM\right| < \frac{\varepsilon}{5}$ for every $E \subseteq \widetilde{E}$ with $|M|(E) < \delta_{\varepsilon,\widetilde{E}}$ and a natural n_ε satisfying (by the $|M|$-convergence of the sequence) that, for a certain measurable set F_ε with $|M|(F_\varepsilon) < \delta_{\varepsilon,\widetilde{E}}$, the following inclusion holds:

$$F_\varepsilon \supseteq \left\{ s \in \Omega; |f_{n_\varepsilon}(s) - f(s)| \geq \frac{\varepsilon}{5\,|M|(\widetilde{E})} \right\}.$$

Therefore

$$\begin{aligned} \left|\int_{\widetilde{E}} f dM\right| &\leq \left|\int_{\widetilde{E}\setminus E_\varepsilon} f dM\right| + \left|\int_{\widetilde{E}\cap\left(E_\varepsilon\setminus\widetilde{E}_\varepsilon\right)} f dM\right| < \\ &< \left|\int_{\left(\widetilde{E}\setminus E_\varepsilon\right)\cap F_\varepsilon} f dM\right| + \left|\int_{\widetilde{E}\setminus E_\varepsilon\setminus F_\varepsilon} f dM\right| + \frac{\varepsilon}{5} < \end{aligned}$$

$$< \left| \int_{\widetilde{E}\setminus E_\varepsilon \setminus F_\varepsilon} f_{n_\varepsilon} dM \right| + D\left(\int_{\widetilde{E}\setminus E_\varepsilon \setminus F_\varepsilon} f_{n_\varepsilon} dM, \int_{\widetilde{E}\setminus E_\varepsilon \setminus F_\varepsilon} f dM \right) + \frac{2\varepsilon}{5} <$$

$$< D\left(\int_{\widetilde{E}\setminus E_\varepsilon \setminus F_\varepsilon} f_{n_\varepsilon} dM, \int_{\widetilde{E}\setminus E_\varepsilon \setminus F_\varepsilon} f dM \right) + \frac{3\varepsilon}{5} \leq$$

$$\leq \int_{\widetilde{E}\setminus E_\varepsilon \setminus F_\varepsilon} |f_{n_\varepsilon} - f| \, d|M| + \frac{3\varepsilon}{5} < \varepsilon,$$

whence the limit function, f, is Gould-integrable.

In order to prove the second assertion, we use the first step and obtain that $\lim\limits_{n\to\infty} \int_E f_n dM = \int_E f dM$ uniformly in $E \in \Sigma E_\varepsilon$, so, for every $\varepsilon > 0$, we are able to find $n_\varepsilon \in \mathbb{N}$ such that, for every $n \geq n_\varepsilon$, $D\left(\int_E f_n dM, \int_E f dM\right) < \frac{2\varepsilon}{5}$ for each $E \in \Sigma E_\varepsilon$.

We can write, for every measurable E:

$$D\left(\int_E f_n dM, \int_E f dM \right) \leq D\left(\int_{E\cap E_\varepsilon} f_n dM, \int_{E\cap E_\varepsilon} f dM \right) +$$

$$+ D\left(\int_{E\setminus E_\varepsilon} f_n dM, \int_{E\setminus E_\varepsilon} f dM \right) < \frac{2\varepsilon}{5} + \left| \int_{E\setminus E_\varepsilon} f_n dM \right| +$$

$$+ \left| \int_{E\setminus E_\varepsilon} f dM \right| < \frac{3\varepsilon}{5} + \left| \int_{E\setminus E_\varepsilon} f dM \right| < \frac{9\varepsilon}{5}, \forall n \geq n_\varepsilon$$

since, by the integrability of f, there exists $E^\varepsilon \in E(f)$, $E^\varepsilon \subseteq E \setminus E_\varepsilon$, such that $D\left(\int_{E\setminus E_\varepsilon} f dM, \int_{E^\varepsilon} f dM\right) < \frac{\varepsilon}{5}$; moreover, E^ε is disjoint of $\widetilde{E}_\varepsilon$, so $\left|\int_{E^\varepsilon} f dM\right| < \varepsilon$. □

Remark 10. It is easy to see, using the definition of the Gould-integral, that the preceding Vitali-type theorem is valid if we replace the condition ii) by ii') $\forall \varepsilon > 0, \exists E_\varepsilon \in \Sigma$ with $|M|(E_\varepsilon) < \infty$ such that $\left|\int_E f_n dM\right| < \varepsilon$, for every $n \in \mathbb{N}$ and each $E \subseteq \Omega \setminus E_\varepsilon$ with $|M|(E) < \infty$ where f_n is bounded.

As a consequence, we can obtain a Lebesgue-type result, that was already directly obtained in [7].

Corollary 11. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real $|M|$-totally measurable functions $|M|$-convergent to the function f such that there exists a positive M-Gould-integrable function, g, satisfying the inequalities*

$$|f_n| \leq g, \forall n \in \mathbb{N}.$$

Then $f_n, n \in \mathbb{N}$ and f are M-Gould-integrable and $\lim_{n\to\infty} \int_E f_n dM = \int_E f dM$ uniformly with respect to $E \in \Sigma$.

Proof. By the Gould-integrability hypothesis on g, for every positive ε, there exists $E_\varepsilon \in E(g)$ such that, for each $E \in \Sigma(\Omega \setminus E_\varepsilon)$ where g is bounded:

$$\left| \int_E g dM \right| < \frac{\varepsilon}{2}.$$

Theorem 3.12 in [7] yields that $\left|\int_E f_n dM\right| < \varepsilon$, for every $n \in \mathbb{N}$ and each $E \subseteq \Omega \setminus E_\varepsilon$ with $|M|(E) < \infty$, on which f_n is bounded.

So the functions f_n are Gould-integrable and the condition *ii'*) of the preceding remark is satisfied.

Let us now prove that $\lim_{|M|(E)\to 0} \left|\int_E f_n dM\right| = 0$ uniformly in $n \in \mathbb{N}$.

For every positive ε, there exists $\delta_\varepsilon > 0$ such that, for each $E' \in \Sigma E_\varepsilon$, $|M|(E') < \delta_\varepsilon$: $\left|\int_{E'} g dM\right| < \varepsilon$, so $\left|\int_{E'} f_n dM\right| < 2\varepsilon, \forall n \in \mathbb{N}$.

We claim that every $E \in \Sigma$ with $|M|(E) < \delta_\varepsilon$ satisfies $\left|\int_E f_n dM\right| < 4\varepsilon, \forall n \in \mathbb{N}$.

Indeed, $\left|\int_E f_n dM\right| < \varepsilon$ on every $E \in \Sigma(\Omega \setminus E_\varepsilon)$ of finite variation where f_n is bounded; therefore, by using the definition of the Gould integral, $\left|\int_E f_n dM\right| < 2\varepsilon$ on every E disjoint of E_ε.

Then

$$\left| \int_E f_n dM \right| \leq \left| \int_{E \cap E_\varepsilon} f_n dM \right| + \left| \int_{E \setminus E_\varepsilon} f_n dM \right| < 4\varepsilon,$$

and so the hypothesis of the Vitali-type theorem are satisfied. □

Acknowledgement. The author is very grateful to Prof. A.M. Precupanu for her careful reading of the paper and for the valuable help.

REFERENCES

1. Brooks J.K. – *An integration theory for set-valued measures*, I, II, Bull. Soc. Roy. Sciences de Liège, no. 37, 1968, p. 312-319, pp. 375-380.
2. Diestel J.; Uhl J.J. Jr. – *Vector measures,* Mathematical surveys and monographs, No 15.

3. Dunford N.; Schwartz L. – *Linear operators,* Part I, Wiley Interscience, New-York, 1958.

4. Gould G.G. – *Integration over vector-valued measures,* Proc. London Math. Soc. 15, 1965, pp. 193-225.

5. Hu S.; Papageorgiou N.S. – *Handbook of Multivalued Analysis,* Vol. I, Kluwer Acad. Publ., Dordrecht, 1997.

6. Precupanu A.M.; Croitoru A. – *A Gould type integral with respect to a multi-measure I,* Analele Universitatii "Al. I. Cuza", Iasi, 2002.

7. Precupanu A.M.; Croitoru A. – *A Gould type integral with respect to a multi-measure II,* Analele Universitatii "Al. I. Cuza", Iasi, 2003.

8. Precupanu A.M. – *Analiza matematica - Functii reale,* Ed. Did. si Pedag., Bucuresti, 1976.

Received: 12.IV.2005

"Al. I. Cuza" University,
Faculty of Mathematics,
Bd. Carol I, No. 11,
Iasi,
ROMÂNIA,
bisatco@eed.usv.ro