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A NONCOMMUTATIVE INTERPRETATION OF HOMOTOPY GROUPS OF A COMPACT SPACE

BY

OANA AGRIGOROAIEI

Abstract. In this paper we refer to Čerin homotopy groups $\pi_n(A;B)$ of a pair (A,B) of C^*-algebras [1]. We prove that for (X,x_0) a pointed compact Hausdorff space the homotopy group $\pi_n(X,x_0), n \geq 1$, is isomorphic with $\pi_n(A;\mathbb{C})$ where A is the C^*-algebra of all complex valued continuous functions vanishing at x_0 on X. Additional two theoretical results on Čerin's group are emphasised.

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Introduction. In 1993 Čerin [1] defined homotopy groups $\pi_n(A;B)$ of a pair (A,B) of C^*-algebras. These groups formally keep a series of properties of homotopy groups of topological spaces such as homotopy invariance, functoriality, commutativity for $n \geq 2$, existence of relative homotopy groups $\pi_n(A;(B,B'))$ and of the exact sequence of a triplet $(A;(B,B'))$ with A,B two C^*-algebras and B' a C^*-subalgebra of B. Even a "fibering property" is proved.

But for the moment this programme differs from other programmes whose aim is to extend some theories from classical algebraic topology to the so-called noncommutative algebraic topology. While in other cases the new theory applied to commutative C^*-algebras (via Gelfand duality) coincides with the classical "commutative" theory, in the case of homotopy groups in the sense of Čerin, things are no longer the same. One has not given for the moment a natural way of obtaining the homotopy groups $\pi_n(X,x_0)$ of a (pointed) locally compact Hausdorff space (X,x_0) starting from the C^*-algebra $C_0(X)$ of all complex valued continuous functions vanishing at infinity on X. The difficulty comes from the fact that ordinary homotopy groups depend on base points.

However, in the present paper the particular case of a compact Hausdorff space is considered and one proves the following results:

Theorem 2.11. For a pointed compact Hausdorff space (X, x_0) and $n \geq 1$, there exists an isomorphism between the homotopy group $\pi_n(X, x_0)$ and the Čerin homotopy group $\pi_n(A; \mathbb{C})$ where A is the C^*-algebra of all complex valued continuous functions vanishing at x_0 on X.

Theorem 2.13. For a commutative C^*-algebra there exists an isomorphism $\pi_n(A; \mathbb{C}) \simeq \pi_n(M(A^+), \Lambda)$, where A^+ is the unital C^*-algebra which contains A as a maximal ideal and $(M(A^+), \Lambda)$ is the (pointed) topological space of the characters of A.

These theorems are the main results of this paper.

In addition, using the theoretical results of [1], one proves that the group $\pi_n(A; (B, B))$ is trivial for any two C^*-algebras A and B, and if A and B are unitary C^*-algebras, then $\pi_n(A; (B, \mathbb{C})) \simeq \pi_n(A; B), \forall n \geq 2$.

1. Preliminaries. Every topological space considered from now on will be a Hausdorff space. We will also use the term "$*$-morphism" for homomorphisms between C^*-algebras.

For a compact topological space X and a C^*-algebra B, let $C(X, B)$ denote the C^*-algebra of all continuous functions from X into B. The norm on $C(X, B)$ is given by $\|f\| = \sup_{x \in X} \|f(x)\|$. If $x_0 \in X$ then $C_{x_0}(X, B)$ denotes the C^*-subalgebra of $C(X, B)$ consisting of all continuous functions $f : X \to B$ with $f(x_0) = 0$. If $B = \mathbb{C}$, we write $C(X)$ and $C_{x_0}(X)$ for $C(X, \mathbb{C})$ and $C_{x_0}(X, \mathbb{C})$ respectively.

The set of all characters of a Banach algebra A (nonzero homomorphisms from A into $\mathbb{C}$) will be denoted by $M(A)$. For a compact topological space X there is a homeomorphism $E_X : X \to M(C(X))$, given by $E_X(x) = \varepsilon_x$ where $\varepsilon_x(f) = f(x)$ is the evaluation at $x \in X$ (see [3], p.10).

We recall the concept of homotopy in noncommutative language. Let I denote the unit interval $[0, 1]$.

Definition 1.1. *If A, B are two C^*-algebras, two $*$-morphisms $\alpha, \beta : A \to B$ are said to be* homotopic *(written $\alpha \simeq_* \beta$) if there is a $*$-morphism $\varphi : A \to C(I, B)$ such that $\varphi(a)(0) = \alpha(a)$ and $\varphi(a)(1) = \beta(a)$, for all a in A.*

Following this, *the n-th homotopy group* of a pair of C^*-algebras is defined in [1].

Let $C_\partial(I^n, B)$ denote the set of all continuous functions from the n-th dimensional cube I^n into B which map the boundary ∂I^n of I^n into the zero element of the algebra B, and $F^n(A; B)$ the set of all $*$-morphisms from A into $C_\partial(I^n, B)$. These $*$-morphisms are divided into homotopy classes and $\pi_n(A, B)$ denotes the totality of these classes. Also, $[f]$ denotes the homotopy class which contains the $*$-morphism f, and 0 the homotopy class which contains the trivial $*$-morphism $\theta : A \to C_\partial(I^n, B)$.

For any $n \geq 1$ an addition is defined in $F^n(A; B)$ as follows: for any two $*$-morphisms $f, g : A \to C_\partial(I^n, B)$ their sum $f + g$ is the $*$-morphism from A into $C_\partial(I^n, B)$ given by

$$(f+g)(a)(t_1, t_2, \ldots, t_n) = \begin{cases} f(a)(2t_1, t_2, \ldots, t_n), & t_1 \in \left[0, \frac{1}{2}\right] \\ g(a)(2t_1 - 1, t_2, \ldots, t_n), & t_1 \in \left[\frac{1}{2}, 1\right] \end{cases}$$

for any a in $A, (t_1, t_2, \ldots, t_n)$ in I^n. Now there can be defined a binary operation of addition on the set $\pi_n(A; B)$ by the rule $[f] + [g] = [f + g]$; thus $(\pi_n(A; B), +)$ gets a group structure (the identity of the addition operation is $0 = [\theta]$).

Also in [1] *the n-th relative homotopy groups*, $\pi_n(A; (B, B'))$, are defined, for A, B two C^*-algebras and B' a C^*-subalgebra of B. Let $n \geq 1$ be an integer and consider again the n-th cube I^n. The initial (n-1)-face of I^n defined by $t_n = 0$ is denoted by J_n. The union of all remaining faces of I^n is denoted by K_n. Let $C_{K_n}(I^n, J_n; B, B')$ denote the C^*-algebra of all continuous functions from I^n into B which take J_n into B' and K_n into the zero element of B; let $F^n(A; (B, B'))$ denote the set of all $*$-morphisms from A into $C_{K_n}(I^n, J_n; B, B')$. The addition in $F^n(A; (B, B'))$, for $n \geq 2$, is the same as in $F^n(A; B)$, and so a binary operation of addition can be defined on the set $\pi_n(A; (B, B'))$ by the rule $[f] + [g] = [f + g]$. A group structure is thus obtained on $\pi_n(A; (B, B'))$, with $[\theta]$ as the identity of the addition operation.

We will also use the next results, that pertain to the transformations induced by $*$-morphisms. Let A, and C be C^*-algebras and (B, B'), (D, D') be pairs consisting of a C^*-algebra and its C^*-subalgebra. Then $g : (B, B') \to (D, D')$ is considered to be a $*$-morphism between C^*-algebra pairs if it is a $*$-morphism from B into D and $g(B') \subset D'$.

Theorem 1.2. *Let $f : C \to A$ and $g : (B, B') \to (D, D')$ be $*$-*

morphisms. There can be defined a function

$$(f,g)_n : F^n(A;\left(B,B'\right)) \to F^n(C;\left(D,D'\right))$$
$$(f,g)_n\,(\xi)\,(c)(t) = g\left(\xi\left(f(c)\right)(t)\right)$$

which induces a group homomorphism
$(f,g)_{*n} : \pi_n(A;(B,B')) \to \pi_n(C;(D,D'))$ *whenever* $n \geq 2$, *or* $n = 1$ *and* $C' = \{0_C\}\,, D' = \{0_D\}\,.$

Also $\left(id_A, id_{(B,B')}\right)_{*n} = id_{\pi_n(A;(B,B'))}$ *(where* id_A *denotes the identical application of* A *into* A*).*

Proposition 1.3. *Let* $f : C \to A, u : E \to C$ *and* $g : (B,B') \to (D,D')\,, v : (D,D') \to (F,F')$ *be* $*$*-morphisms. Then* $(u,v)_{*n} \circ (f,g)_{*n} = (f \circ u, v \circ g)_{*n}\,.$

Proposition 1.4. *There exists a group homomorphism*

$$\partial_n : \pi_{n+1}\left(A;\left(B,B'\right)\right) \to \pi_n\left(A;B'\right)$$

defined by $\partial_n([f]) = [f_{\partial_n}]$, *with* $f_{\partial_n}(a)(t_1,\ldots,t_n) = f(a)(t_1,\ldots,t_n,0)$.

Theorem 1.5. *Let* j *and* i *denote the inclusions of* C^**-algebra pairs* $(B,\{0\})$ *and* $(B',\{0\})$ *into* C^**-algebra pairs* (B,B') *and* $(B,\{0\})$, *respectively. To simplify the notation, let* $i_* := (id_A, i)_{*n}$ *and* $j_* := (id_A, j)_{*n}\,.$ *Then the group sequence*

$$\ldots \to \pi_n(A,B') \xrightarrow{i_*} \pi_n(A;B) \xrightarrow{j_*} \pi_n(A;(B,B')) \xrightarrow{\partial_n} \pi_{n-1}(A;B') \xrightarrow{i_*} \ldots$$
$$\ldots \xrightarrow{j_*} \pi_2(A;(B,B')) \xrightarrow{\partial_n} \pi_1(A;B') \xrightarrow{i_*} \pi_1(A;B) \tag{1}$$

is exact.

Other notations that will be encountered in the next section are:

$\Omega_n(X,x_0)$, the set of all continuous functions from I^n into a topological space X which take ∂I^n into x_0;

for any complex number z and a topological space X e_z is an element of $C\left(X\right)$ defined by $e_z\left(x\right) = z, \forall x \in X$.

2. Proof of results. Let $u : M(C_{x_0}(X)) \cup \{\theta\} \to M(C(X))$ be given by:

$$u(\mu) : C(X) \to \mathbb{C}$$
$$u(\mu)(f) = \mu(f - e_{f(x_0)}) + f(x_0)$$

for any homomorphism μ between the Banach algebras $C_{x_0}(X)$ and $\mathbb{C}$, and any f in $C(X)$.

Lemma 2.1. *The function u is a homeomorphism, with $u(\theta) = e_{x_0}$, that has the inverse $u^{-1} = v : M(C(X)) \to M(C_{x_0}(X)) \cup \{\theta\}$ given by $v(\eta) = \eta|_{C_{x_0}(X)}$.*

Proof. We first check that u and v are defined correctly. For v that is quite clear, and for u we will just show that for any $f, g \in C_{x_0}(X)$ and any $\mu \in M(C_{x_0}(X)) \cup \{\theta\}$ we have $u(\mu)(f) \cdot u(\mu)(g) = u(\mu)(fg)$; this is obtained using the fact that $\left(f - e_{f(x_0)}\right)\left(g - e_{g(x_0)}\right) = fg - g(x_0) f - f(x_0) g + e_{fg(x_0)}$.

We also have that $vu(\mu) = u(\mu)\Big|_{C_{x_0}(X)} = \mu$ and

$uv(\eta)(f) = u(\eta\Big|_{C_{x_0}(X)})(f) = \eta(f - e_{f(x_0)}) + f(x_0) = \eta(f)$ so $v = u^{-1}$.

Since a sequence $(\mu_\lambda)_{\lambda\in\Lambda}$ converges to μ in $M(C_{x_0}(X)) \cup \{\theta\}$ with its Gelfand topology if and only if $\mu_\lambda\left(f - e_{f(x_0)}\right)$ converges to $\mu\left(f - e_{f(x_0)}\right)$ for any f in $C(X)$, which is equivalent with the fact that $u(\mu_\lambda)$ converges to $u(\mu)$ in $M(C(X))$, it follows that u is a homeomorphism. □

We consider $\Phi : \Omega_n(X, x_0) \to F^n(C_{x_0}(X); B)$ given by $\Phi(\alpha) : C_{x_0}(X) \to C_\partial(I^n, B)$, $\Phi(\alpha)(h)(t) = h \circ \alpha(t)$ for any $\alpha \in \Omega(X, x_0), h \in C_{x_0}(X), t \in I^n$, and the induced function $\hat{\Phi} : \pi_n(X, x_0) \to \pi_n(C_{x_0}(X); B), \hat{\Phi}[\alpha] = [\Phi(\alpha)]$.

Proposition 2.2. *The functions Φ and $\hat{\Phi}$ are correctly defined.*

Proof. For any α in $\Omega(X, x_0)$ and any h in $C_{x_0}(X)$ we have that $h \circ \alpha$ is continuous, $\Phi(\alpha)$ is a $*$-morphism and if $t \in \partial I^n$ then $h \circ a(t) = f(x_0) = 0$.

If $[\alpha] = [\beta] \in \pi_n(X, x_0)$ then there exists a homotopy $F : I^n \times I \to X$ between α and β relative to ∂I^n. Let $H : C_{x_0}(X) \to C(I, C_\partial(I^n, B))$ be the $*$-morphism given by $H(f)(s)(t) = f \circ F(t, s)$.Then $H : \Phi(\alpha) \simeq \Phi(\beta)$ so $\hat{\Phi}$ is correctly defined. □

Theorem 2.3. *$\hat{\Phi} : \pi_n(X, x_0) \to \pi_n(C_{x_0}(X); B)$ is a group homomorphism, for any $n \geq 1$.*

Proof. If $[\alpha], [\beta] \in \pi_n(X, x_0)$ and $\gamma = \alpha + \beta$ then

$$\Phi(\gamma)(h)(t) = h\circ\gamma(t_1, t_2, \ldots, t_n) = \begin{cases} h\circ\alpha(2t_1, t_2, \ldots, t_n) & , t_1 \in \left[0, \frac{1}{2}\right] \\ h\circ\beta(2t_1 - 1, t_2, \ldots, t_n) & , t_1 \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$= (\Phi(\alpha) * \Phi(\beta))(h)(t)$ hence $\hat{\Phi}([\alpha] + [\beta]) = \hat{\Phi}([\alpha + \beta])$. □

Proposition 2.4. *Let (X, x_0) and (Y, y_0) be pointed compact spaces. If $\varphi : (X, x_0) \to (Y, y_0)$ is a continuous function then the following diagram commutes:*

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{\hat{\Phi}} & \pi_n(C_{x_0}(X); B) \\ \varphi_* \downarrow & & \downarrow (C\varphi, id_B)_{*n} \\ \pi_n(Y, y_0) & \xrightarrow[\hat{\Phi}]{} & \pi_n(C_{y_0}(Y); B) \end{array}$$

where $C\varphi$ is the $*$-morphism from $C_{y_0}(Y)$ into $C_{x_0}(X)$ given by $C_\varphi(f) = f\circ\varphi$ and φ_* is the group homomorphism induced by the continuous function φ.

Proof. If $\alpha \in \Omega_n(X, x_0)$ then $\bar{\varphi}(\Phi(\alpha))(f)(t) = id_{\mathbb{C}}(\Phi(\alpha)(C\varphi(f))(t)) = \Phi(\alpha)(f\circ\varphi)(t) = f\circ\varphi\circ\alpha(t) = \Phi(\varphi\circ\alpha)(f)(t)$ thus $\bar{\varphi}_*\circ\hat{\Phi}[\alpha] = [\bar{\varphi}(\Phi(\alpha))] = [\Phi\circ\varphi(\alpha)] = \hat{\Phi}\circ\varphi_*[\alpha]$. □

Let A be a C^*-algebra. We consider $\Psi : F^n(A;\mathbb{C}) \to \Omega_n(M(A)\cup\{\theta\}, \theta)$ defined as follows: for any $*$-morphism $h : A \to C_\partial(I^n, \mathbb{C})$ take

$$\Psi(h) : I^n \to M(A)\cup\{\theta\}$$
$$\Psi(h)(t) : A \to \mathbb{C}, \Psi(h)(t)(a) = h(a)(t)$$

and the induced function $\hat{\Psi} : \pi_n(A;\mathbb{C}) \to \pi_n(M(A)\cup\{\theta\}, \theta)$, $\hat{\Psi}[h] = [\Psi(h)]$.

Lemma 2.5. *Both Ψ and $\hat{\Psi}$ are correctly defined.*

Proof. For any $*$-morphism h we have

$$\begin{cases} h(\lambda a + \mu b)(t) = \lambda h(a)(t) + \mu h(b)(t) \\ h(ab)(t) = h(a)(t)\cdot h(b)(t) \end{cases} , \forall a, b \in A, \lambda, \mu \in \mathbb{C}$$

so $\Psi(h)(t) \in M(A)\cup\{\theta\}$. For any sequence t_n that converges to t in I^n we have, using the continuity of $h(a), \forall a \in A$, that $\Psi(h)(t_n)(a) \xrightarrow{\mathbb{C}} \Psi(h)(t)(a)$, for all $a \in A \implies \Psi(h)(t_n) \stackrel{w^*}{\to} \Psi(h)(t)$, thus $\Psi(h)$ is continuous; also, $\Psi(h)(\partial I^n) = \{\theta\}$.

If $[h_1] = [h_2] \in \pi_n(A, \mathbb{C})$ there is a $*$-morphism $H : A \to C(I, C_\partial(I^n, \mathbb{C}))$ so that $H(a)(0) = h_1(a)$ and $H(a)(1) = h_2(a), \forall a \in A$. Let $G : I^n \times I \to M(A) \cup \{\theta\}$ be given by $G(t, s)(a) = H(a)(s)(t)$, for any $t \in I^n, s \in I, a \in A$. To show that G is continuous consider r_n a sequence that converges to r in $I^n \times I$; therefore $r_n = (t_n, s_n)$ and $r = (t, s)$ where $t_n \stackrel{I^n}{\to} t, s_n \stackrel{I}{\to} s$. Then $G(r_n)(a) = H(a)(s_n)(t_n) \stackrel{\mathbb{C}}{\to} H(a)(s)(t) = G(r)(a), \forall a \in A$, so $G(r_n) \stackrel{w^*}{\to} G(r)$ proving that G is continuous.

Since $\begin{cases} G(t,0)(a) = h_1(a)(t) = \Psi(h_1)(t)(a) \\ G(t,1)(a) = h_2(a)(t) = \Psi(h_2)(t)(a) \\ G(t,s)(a) = 0 = \theta(a), \forall t \in \partial I^n, s \in I \end{cases}$ it follows that G is a homotopy between $\Psi(h_1)$ and $\Psi(h_2)$. $\square$

Theorem 2.6. *The function $\hat{\Psi} : \pi_n(A; \mathbb{C}) \to \pi_n(M(A) \cup \{\theta\}, \theta)$ is a group homomorphism, $\forall n \geq 1$.*

Proof. If $f, g \in F^n(A; \mathbb{C})$ and $h = f + g$. then $\Psi(f * g)(t)(a) = (f * g)(a)(t) = h(a)(t) = \Psi(h)(t)(a)$, so $\hat{\Psi}([f] + [g]) = \hat{\Psi}([f + g])$. $\square$

Proposition 2.7. *If A_1 and A_2 are two C^*-algebras and $\gamma : A_2 \to A_1$ is a $*$-morphism then the following diagram commutes:*

$$\begin{array}{ccc} \pi_n(A_1; \mathbb{C}) & \xrightarrow{\hat{\Psi}} & \pi_n(M(A_1) \cup \{\theta\}, \theta) \\ {\scriptstyle (\gamma, id_{\mathbb{C}})_{*n}} \downarrow & & \downarrow {\scriptstyle M\gamma_*} \\ \pi_n(A_2; \mathbb{C}) & \xrightarrow{\hat{\Psi}} & \pi_n(M(A_2) \cup \{\theta\}, \theta) \end{array}$$

where $M_\gamma : (M(A_1) \cup \{\theta\}, \theta) \to (M(A_2) \cup \{\theta\}, \theta)$ is the continuous function given by $M_\gamma(\eta) = \eta \circ \gamma$.

Proof. $M_\gamma(\eta)$ is a homomorphism between Banach algebras, $\forall \eta \in M(A_1) \cup \{\theta\}$, because γ is a $*$-morphism and η is a homomorphism between Banach algebras.

M_γ is continuous because for any sequence $(\eta_\lambda)_{\lambda \in \Lambda}$ which converges to η in $M(A_1) \cup \{\theta\}$ we have that $\eta_\lambda(\gamma(a)) \stackrel{\mathbb{C}}{\to} \eta(\gamma(a)), \forall a \in A_2$ hence $\eta_\lambda \circ \gamma$ converges to $\eta \circ \gamma$ in $M(A_2) \cup \{\theta\}$.

If h is an element of $F^n(A; \mathbb{C})$ then
$(M\gamma \circ \Psi(h))(t)(a) = (\Psi(h)(t) \circ \gamma)(a) = \Psi(h)(t)(\gamma(a)) = h(\gamma(a))(t) =$

$((\gamma, id_{\mathbb{C}})_n(h))(a)\,(t) = \Psi((\gamma, id_{\mathbb{C}})_n(h))(t)(a)$, for all $t \in I^n, a \in A_2$, so $M\gamma_* \circ \hat{\Psi}\,[h] = \hat{\Psi} \circ (\gamma, id_{\mathbb{C}})_{*n}\,[h]$. $\square$

In order to be able to compose the group homomorphisms $\hat{\Phi}{:}\pi_n(X, x_0) \rightarrow \pi_n(C_{x_0}(X); B)$ and $\hat{\Psi} : \pi_n(A; \mathbb{C}) \rightarrow \pi_n(M(A) \cup \{\theta\}, \theta)$ we consider $B := \mathbb{C}$ and $A := C_{x_0}(X)$.

Lemma 2.8. *The following diagram commutes:*

$$\begin{array}{ccccc} \pi_n(X, x_0) & \xrightarrow{\hat{\Phi}} & \pi_n(C_{x_0}(X); \mathbb{C}) & \xrightarrow{\hat{\Psi}} & \pi_n(M(C(X)) \cup \theta, \theta) \\ & \searrow^{E_{X*}} & & & \uparrow v_* \\ & & & & \pi_n(M(C(X)), \varepsilon_{x_0}) \end{array}$$

Proof. Let α be an element of $\Omega_n(X, x_0)$. For any $t \in I^n, f \in C_{x_0}(X)$ we have $\Psi(\Phi\alpha)(t)(f) = \Phi\alpha(f)(t) = f(\alpha(t)) = \varepsilon_{\alpha(t)}(f) = E_X \circ \alpha(t)\Big|_{C_{x_0}(X)}(f) = (v \circ E_X \circ \alpha)(t)(f)$, hence $\hat{\Psi} \circ \hat{\Phi}\,[\alpha] = v_* \circ E_{X*}\,[\alpha]$. $\square$

Corollary 2.9. *The function $\hat{\Psi}$ is surjective and $\hat{\Phi}$ is injective.*

Proof. This happens because $\hat{\Psi} \circ \hat{\Phi} = v_* \circ E_{X*}$ and both v_* and E_{X*} are group isomorphisms. $\square$

Proposition 2.10. *The function $\hat{\Psi}$ is injective.*

Proof. Let h be an element of $F^n(C_{x_0}(X); \mathbb{C})$ with $[\Psi h] = [\theta]$. Then there is a continuous function

$$H : I^{n+1} \rightarrow M(C_{x_0}(X)) \cup \{\theta\} \text{ with } \begin{cases} H(t, 0) = \theta & , \forall t \in I^n \\ H(t, 1) = \Psi h(t) & , \forall t \in I^n \\ H(t, s) = \theta & , \forall t \in \partial I^n, s \in I \end{cases}.$$

We define $F : C_{x_0}(X) \rightarrow C(I, C_\partial(I^n, \mathbb{C}))$ by $F(f)(s)(t) = H(t, s)(f)$.

Because H and$f$ are continuous and $F(f)(s)(\partial I^n) = \{0\}$ we have that $F(f)(s) \in C_\partial(I^n, \mathbb{C})$, for all f in $C_{x_0}(x)$ and s in I.

Let $(s_n)_{n \in \mathbb{N}}$ be a sequence which converges to s in I. Then we have that $\|F(f)(s_n) - F(f)(s)\| = \sup_{t \in I^n} \|H(t, s_n)(f) - H(t, s)(f)\|$. Because I^n is a compact topological space there is a $t_0 \in I^n$ verifying

$$\sup_{t \in I^n} \|H(t, s_n)(f) - H(t, s)(f)\| = \|H(t_0, s_n)(f) - H(t_0, s)(f)\|$$

As $H(t_0, s) : I \to M(C_{x_0}(X)) \cup \{\theta\}$ is a continuous function, we have that $\|F(f)(s_n) - F(f)(s)\| \overset{n\to\infty}{\to} 0$. Therefore $F(f) \in C(I, C_\partial(I^n, \mathbb{C}))$.

As $H(t, s)$:$C_{x_0}(X) \to \mathbb{C}$ is a Banach algebra homomorphism for all $t \in I^n$ and $s \in I$, we have that F is a Banach algebra homomorphism too.We now prove the condition pertaining to the involution.

Consider the pair (t, s) set in $I^n \times I$. Because $u \circ H(t, s) \in M(C(X))$ (u is the homeomorphism from Lemma 2.1) and $E_X : X \to M(C(X))$ is a surjection we have that there is y in X (related to t and s) with $u \circ H(t, s) = \varepsilon_y$. So $H(t, s) = v \circ \varepsilon_y$, and we get that $H(t, s)(f) = \varepsilon_y(f) = f(y)$. We now have $H(t, s)(f^*) = f^*(y) = \overline{f(y)} = \overline{H(t, s)(f)}$, so $F(f^*) = F(f)^*$. Therefore F is a $*$-morphism.

Because $\left.\begin{array}{r} F(f)(0) = \theta \\ F(f)(1)(t) = \Psi h(t)(f) = h(f)(t) \end{array}\right\}$ we have that $F : \theta \simeq_* h$, thus $\hat{\Psi}$ is injective. □

Theorem 2.11. *If (X, x_0) is a compact topological space with a base point then for $A = C_{x_0}(X)$ and $B = \mathbb{C}$ we have $\pi_n(X, x_0) \simeq \pi_n(A; B)$.*

Proof. Using Proposition 2.10.we now have that $\hat{\Psi}$ is a group isomorphism. Then $\hat{\Phi} = \hat{\Psi}^{-1} \circ v_* \circ E_{X*}$ is the isomorphism from $\pi_n(X, x_0)$ onto $\pi_n(A; B)$. □

Corollary 2.12. *Let $\omega : I \to X$ be a path in X from x_0 to x_1. Then $\pi_n(C_{x_0}(X); \mathbb{C}) \simeq \pi_n(C_{x_1}(X); \mathbb{C})$.*

Proof. This happens because $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic.□

Theorem 2.13. *If A is a non-unital commutative C^*-algebra then $\pi_n(A; \mathbb{C}) \simeq \pi_n(M(A^+), \Lambda)$ where $\Lambda(a, \lambda) = \lambda$ is a character for A^+.*

Proof. Because A is a non-unital commutative C^*-algebra, using the Gelfand-Naĭmark theorem, A is isomorphic with $C_0(M(A))$. But $M(A) \cup \{\theta\}$ and $M(A^+)$ are homeomorphic and the homeomorphism takes θ into Λ, it follows that $C_0(M(A))$ and $C_\Lambda(M(A^+))$ are isomorphic. Therefore $\hat{\Psi}$ is the isomorphism from $\pi_n(A; \mathbb{C})$ onto $\pi_n(M(A^+), \Lambda)$. □

Theorem 2.14. *If A is an unital C^*-algebra, B is a C^*-subalgebra of $C(X)$ and X is a connected compact topological space then $\pi_n(A; B) = 0$. In particular, if A is unital then $\pi_n(A; \mathbb{C}) = 0$.*

Proof. If $[h] \in \pi_n(A;B)$ let $f_t := h(1)(t)$ (where 1 denotes the unit for the C^*-algebra A). Then $f_t^2 = f_t$ so we have a continuous function $f_t : X \to \{0,1\}$. Since X is connected we get that f_t can only be e_0 or e_1. Then $h(1_A) : I^n \to \{e_0, e_1\}$ is continuous and $h(1_A)(\partial I^n) = \{e_0\}$. Since I^n is connected $h(1_A)(I^n) = \{e_0\}$. Therefore $h(a)(t) = h(a)(t) \cdot h(1_A)(t) = e_0$.□

Corollary 2.15. *If A and B are two unital C^*-algebras then $\pi_n(A;(B,\mathbb{C})) \simeq \pi_n(A;B), \forall n \geq 2$.*

Proof. We consider the exact group sequence given by (1), for $B' = \mathbb{C}$:$\ldots \to \pi_n(A;\mathbb{C}) \overset{i_*}{\to} \pi_n(A;B) \overset{j_*}{\to} \pi_n(A;(B,\mathbb{C})) \overset{\partial_n}{\to} \pi_{n-1}(A;\mathbb{C}) \overset{i_*}{\to} \ldots$

As A is unital we have $\pi_n(A;\mathbb{C}) = 0, \forall n \geq 1$ so $\mathrm{Im} i_* = 0 = \ker j_*$ and $\ker \partial_n = \pi_n(A;(B,\mathbb{C})) = \mathrm{Im}\ j_*$ thus j_* is an isomorphism. □

Theorem 2.16. *For any two C^*-algebras A and B and for all $n \geq 2$ we have $\pi_n(A;(B,B)) = 0$.*

Proof. We consider the exact group sequence given by (1), for $B' = B$:$\ldots \to \pi_n(A;B) \overset{i_*}{\to} \pi_n(A;B) \overset{j_*}{\to} \pi_n(A;(B,B)) \overset{\partial_n}{\to} \pi_{n-1}(A;B) \overset{i_*}{\to} \ldots$ Then $i_* = (id_A, id_B)_* = id_{\pi_n(A;B)}$. We have $\ker j_* = \mathrm{Im}\ i_* = \pi_n(A,B)$ which tells us that $\mathrm{Im}\ j_* = 0 = \ker \partial_n$, hence ∂_n is injective. Since $\mathrm{Im}\ \partial_n = \ker i_* = 0$ we have $\pi_n(A;(B,B)) = 0$. □

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Faculty of Mathematics,
University "Al.I.Cuza" Iaşi,
Bd. Carol 1, no 11, 700506-Iaşi
ROMÂNIA,
oanaag@yahoo.com