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SCHOUTEN-VAN KAMPEN AND VRĂNCEANU CONNECTIONS ON FOLIATED MANIFOLDS

BY

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Abstract. We show that all the important connections involved in the study of foliations on semi-Riemannian (Riemannian) manifolds come from the Schouten-Van Kampen and Vrănceanu connections. In particular, it is presented a characterization of totally geodesic foliations with bundle-like metric by means of Vrănceanu connection. Also, the Schur Theorem is extended to foliated manifolds of constant transversal Vrănceanu curvature.

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Key words: Schouten-Van Kampen and Vrănceanu connections, foliated semi-Riemannian manifolds, transversal Vrănceanu curvature.

1. Introduction. The Schouten-Van Kampen connection and the Vrănceanu connections have been introduced in the third decade of last century for a study of non-holonomic manifolds (cf. [8], [12]). Here we present the basic properties of these connections in the context of foliations on semi-Riemannian manifolds.

In the first section we develop a theory of adapted tensor fields on a foliated manifold. Then in the second section we obtain the local coefficients of the Levi-Civita connection with respect to an adapted frame field on a foliated semi-Riemannian manifold. The Schouten-Van Kampen and Vrănceanu connections induced by the Levi-Civita connection are studied in Section 3. Here we introduce both the structural and transversal covariant derivatives with respect to these connections and obtain new characterizations of some classes of foliations. In particular, we show that the Schouten-Van Kampen connection coincides with the Vrănceanu connection if and only if the manifold is locally a semi-Riemannian product. In the last section we define the Vrănceanu sectional curvature of the transversal distribution and extend the Schur Theorem to foliated semi-Riemannian manifolds. Finally, we show the existence of foliated Riemannian manifolds of positive constant transversal Vrănceanu curvature and present an open problem.

1. Adapted tensor fields on a foliated manifold. Let M be an $(n+p)$-dimensional manifold endowed with an n-foliation $\mathcal{F}$. Denote by $\mathcal{D}$ the tangent distribution to $\mathcal{F}$ and suppose that there exists a complementary distribution $\overline{\mathcal{D}}$ to $\mathcal{D}$ in the tangent bundle TM of M, that is, we have

$$TN = \mathcal{D} \oplus \overline{\mathcal{D}}. \tag{1.1}$$

We call $\mathcal{D}$ and $\overline{\mathcal{D}}$ the **structural distribution** and the **transversal distribution** on the foliated manifold $(M, \mathcal{F})$. The paracompactness of M guarantees the existence of $\overline{\mathcal{D}}$. Actually, starting with the next section where a semi-Riemannian metric is considered on M, a canonical $\overline{\mathcal{D}}$ is defined, and thus the study depends on both the foliation and the metric.

Throughout the paper all manifolds are paracompact and all mappings are smooth (differentiable of class C^∞). We denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $F(M)$-module of smooth vector fields on M. Also, we use the Einstein convention, that is, repeated indices with one upper index and one lower index denotes summation over their ranges. If not stated otherwise, we shall use the following ranges for indices: $i, j, k, ... \in \{1, ..., n\}$, $\alpha, \beta, \gamma, ... \in \{n+1, ..., n+p\}$, $a, b, c, ... \in \{1, ..., n+p\}$.

The purpose of this section is the develop a tensor calculus adapted to the decomposition (1.1). To this end, we first construct an adapted frame field on $(M, \mathcal{F})$ as follows. Let $\{(\mathcal{U}, \varphi) : (x^i, x^\alpha)\}$ be a foliated local chart on $(M, \mathcal{F})$, that is, (x^i) are the leaf coordinates. If $\{(\widetilde{\mathcal{U}}, \widetilde{\varphi}), \widetilde{x}^i, \widetilde{x}^\alpha\}$ is another foliated chart with $\mathcal{U} \cap \widetilde{\mathcal{U}} \neq \emptyset$, then we have

$$\widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^{n+p}), \ \ \widetilde{x}^\alpha = \widetilde{x}^\alpha(x^{n+1}, ..., x^{n+p}). \tag{1.2}$$

As $\mathcal{D}$ is integrable, it is locally spanned by the natural frame field $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\}$. Also, a non-holonomic frame field

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - A^i_\alpha \frac{\partial}{\partial x^i}, \ \ \alpha \in \{n+1, ..., n+p\}, \tag{1.3}$$

can be considered for the transversal distribution $\overline{\mathcal{D}}$, where A^i_α are local functions satisfying

$$A^i_\alpha \frac{\partial \widetilde{x}^j}{\partial x^i} = \widetilde{A}^j_\beta \frac{\partial \widetilde{x}^\beta}{\partial x^\alpha} + \frac{\partial \widetilde{x}^j}{\partial x^\alpha}\,, \tag{1.4}$$

with respect to (1.2). Moreover, we have

$$\text{(a)}\ \frac{\partial}{\partial x^i} = \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\partial}{\partial \widetilde{x}^j} \quad \text{and} \quad \text{(b)}\ \frac{\delta}{\delta x^\alpha} = \frac{\partial \widetilde{x}^\beta}{\partial x^\alpha}\frac{\delta}{\delta \widetilde{x}^\beta}\,, \tag{1.5}$$

with respect to (1.2). We call $\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\}$ the **adapted frame field** on $(M, \mathcal{F})$. Such frame fields have been used by Reinhart [6] and Vaisman [10] in their works on foliations.

Next we consider the dual vector bundles $\mathcal{D}^*$ and $\overline{\mathcal{D}}^*$ to $\mathcal{D}$ and $\overline{\mathcal{D}}$ respectively. Then an **adapted tensor field** of type $(q, s; r, t)$ on the foliated manifold $(M, \mathcal{F})$ (cf. Bejancu-Farran [1]) is an $F(M) - (q + s + r + t)$-multilinear mapping

$$T : \Gamma(\mathcal{D}^*)^q \times \Gamma(\mathcal{D})^s \times \Gamma(\overline{\mathcal{D}}^*)^r \times \Gamma(\overline{\mathcal{D}})^t \longrightarrow F(M).$$

To define the local components of T we consider the **dual adapted frame field** $\{\delta x^i, dx^\alpha\}$ on $(M, \mathcal{F})$, where we set

$$\delta x^i = dx^i + A^i_\alpha dx^\alpha.$$

Now we can state the following.

Theorem 1.1. *There exists an adapted tensor field T of type $(q, s; r, t)$ on $(M, \mathcal{F})$ if and only if on the domain of each foliated chart there exist $n^{q+s}p^{r+t}$ smooth functions $T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t}$ satisfying*

$$\begin{aligned} &T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t}\frac{\partial \widetilde{x}^{k_1}}{\partial x^{i_1}}\cdots\frac{\partial \widetilde{x}^{k_q}}{\partial x^{i_q}}\frac{\partial \widetilde{x}^{\gamma_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial \widetilde{x}^{\gamma_r}}{\partial x^{\alpha_r}} \\ &= \widetilde{T}^{k_1\dots k_q\gamma_1\dots\gamma_r}_{h_1\dots h_s\varepsilon_1\dots\varepsilon_t}\frac{\partial \widetilde{x}^{h_1}}{\partial x^{j_1}}\cdots\frac{\partial \widetilde{x}^{h_s}}{\partial x^{j_s}}\frac{\partial \widetilde{x}^{\varepsilon_1}}{\partial x^{\beta_1}}\cdots\frac{\partial \widetilde{x}^{\varepsilon_t}}{\partial x^{\beta_t}}\,, \end{aligned} \tag{1.6}$$

with respect to (1.2).

Moreover, the local components of T with respect to the above adapted frame fields are given by

(1.7)
$$T^{i_1 \ldots i_q \alpha_1 \ldots \alpha_r}_{j_1 \ldots j_s \beta_1 \ldots \beta_t} = T\left(\delta x^{i_1}, ..., \delta x^{i_q}, \frac{\partial}{\partial x^{j_1}}, \cdots, \frac{\partial}{\partial x^{j_s}}, dx^{\alpha_1}, ..., dx^{\alpha_r}, \frac{\delta}{\delta x^{\beta_1}}, \cdots, \frac{\delta}{\delta x^{\beta_t}}\right).$$

In particular, sections of $\mathcal{D}$ and $\overline{\mathcal{D}}$ are adapted tensor fields of types $(1,0;0,0)$ and $(0,0;1,0)$ respectively. Similarly, sections of $\mathcal{D}^*$ and $\overline{\mathcal{D}}^*$ are adapted tensor fields of types $(0,1;0,0)$ and $(0,0;0,1)$ respectively. Most of the geometric objects from the next sections are adapted tensor fields on $(M, \mathcal{F})$.

Lemma 1.1. *Let $(M, \mathcal{F})$ be a foliated manifold and $\left\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right\}$ be an adapted frame field. Then we have*

$$\text{(1.8)} \qquad \text{(a)} \left[\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right] = I_\alpha{}^i{}_\beta \frac{\partial}{\partial x^i}, \quad \text{(b)} \left[\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^i}\right] = A_i{}^j{}_\alpha \frac{\partial}{\partial x^j},$$

where we put

$$\text{(1.9)} \qquad \text{(a)}\ I_\alpha{}^i{}_\beta = \frac{\delta A^i_\alpha}{\delta x^\beta} - \frac{\delta A^i_\beta}{\delta x^\alpha}, \quad \text{(b)}\ A_i{}^j{}_\alpha = \frac{\partial A^j_\alpha}{\partial x^i}.$$

Proof. By direct calculations using properties of the Lie bracket and (1.3). □

Now, by using (1.5) in (1.8a) we deduce that

$$I_\alpha{}^i{}_\beta \frac{\partial \widetilde{x}^j}{\partial x^i} = \widetilde{I}_\gamma{}^j{}_\mu \frac{\partial \widetilde{x}^\gamma}{\partial x^\alpha} \frac{\partial \widetilde{x}^\mu}{\partial x^\beta}.$$

Comparing with (1.6) we deduce that $\{I_\alpha{}^i{}_\beta\}$ are the local components of an adapted tensor field I of type $(1,0;0,2)$. Moreover, I is skew-symmetric with respect to $\alpha\beta$ and as a consequence of (1.8a) we obtain the following.

Proposition 1.1. *The transversal distribution $\overline{\mathcal{D}}$ on $(M, \mathcal{F})$ is integrable if and only if I vanishes identically on M.*

For this reason the adapted tensor field $I = \{I_\alpha{}^i{}_\beta\}$ was called the **integrability tensor field** of $\overline{\mathcal{D}}$.

On the other hand, by using (1.8b) and (1.5) we infer that

$$A_i{}^j{}_\alpha \frac{\partial \widetilde{x}^k}{\partial x^j} = \widetilde{A}_h{}^k{}_\beta \frac{\partial \widetilde{x}^h}{\partial x^i} \frac{\partial \widetilde{x}^\beta}{\partial x^\alpha} + \frac{\delta}{\delta x^\alpha}\left(\frac{\partial \widetilde{x}^k}{\partial x^i}\right). \tag{1.10}$$

Hence, in general, $\{A_i{}^j{}_\alpha\}$ do not define an adapted tensor field. However, as we shall see in the next sections, these functions play an important role as local coefficients of the Vrănceanu connection (see (3.4b)).

2. The Levi-Civita connection on a foliated semi-Riemannian manifold. Let $(M, g, \mathcal{F})$ be a foliated semi-Riemannian manifold, where M is an $(n+p)$-dimensional manifold, g is a semi-Riemannian metric and $\mathcal{F}$ is a non-degenerate n-foliation on M. Thus the leaves of $\mathcal{F}$ are non-degenerate submanifolds of M with respect to g. Then the complementary orthogonal distribution $\overline{\mathcal{D}}$ to $\mathcal{D}$ in TM is non-degenerate too. From now on we consider $\overline{\mathcal{D}}$ as the transversal distribution to $\mathcal{F}$.

Denote by $\widetilde{\nabla}$ the Levi-Civita connection defined by g on M. It is the purpose of this section to obtain all the local coefficients of $\widetilde{\nabla}$ with respect to an adapted frame field $\left\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right\}$. First, we put

$$\text{(a) } g_{\alpha\beta} = g\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right), \quad \text{(b) } g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad \text{(c) } g_{\alpha i} = g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^i}\right). \tag{2.1}$$

It is easy to see that $g_{\alpha\beta}$ and g_{ij} define adapted tensor fields of type $(0,0;0,2)$ and $(0,2;0,0)$ respectively. Moreover, $g_{\alpha\beta}$ and g_{ij} are the local components of the semi-Riemannian metrics induced by g on $\overline{\mathcal{D}}$ and $\mathcal{D}$ respectively. Now, we replace $\frac{\partial}{\partial x^\alpha}$ from (2.1c) by $\frac{\delta}{\delta x^\alpha} + A_\alpha^j \frac{\partial}{\partial x^j}$ (see (1.3)) and by using (2.1b) we obtain

$$g_{\alpha i} = A_\alpha^j g_{ji}. \tag{2.2}$$

Denote by g^{ij} the entries of the inverse matrix of $[g_{ij}]$, and from (2.2) we deduce that the transversal distribution $\overline{\mathcal{D}}$ is locally spanned by $\left\{\frac{\delta}{\delta x^\alpha}\right\}$ from (1.3), where A_α^i are given by

$$A_\alpha^i = g_{\alpha j} g^{ji}. \tag{2.3}$$

Next, we express $\widetilde{\nabla}$ locally as follows:

$$
(2.4)\qquad
\begin{aligned}
&\text{(a)} && \widetilde{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = F_\alpha{}^\gamma{}_\beta \frac{\delta}{\delta x^\gamma} + G_\alpha{}^i{}_\beta \frac{\partial}{\partial x^i}\,,\\
&\text{(b)} && \widetilde{\nabla}_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial x^i} = H_i{}^\gamma{}_\alpha \frac{\delta}{\delta x^\gamma} + K_i{}^j{}_\alpha \frac{\partial}{\partial x^j}\,,\\
&\text{(c)} && \widetilde{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\delta}{\delta x^\alpha} = L_\alpha{}^\gamma{}_i \frac{\delta}{\delta x^\gamma} + M_\alpha{}^j{}_i \frac{\partial}{\partial x^j}\,,\\
&\text{(d)} && \widetilde{\nabla}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = N_i{}^\gamma{}_j \frac{\delta}{\delta x^\gamma} + P_i{}^k{}_j \frac{\partial}{\partial x^k}\,,
\end{aligned}
$$

where $(F_\alpha{}^\gamma{}_\beta, G_\alpha{}^i{}_\beta, H_i{}^\gamma{}_\alpha, K_i{}^j{}_\alpha, L_\alpha{}^\gamma{}_i, M_\alpha{}^j{}_i, N_i{}^\gamma{}_j, P_i{}^k{}_j)$ are its local coefficients with respect to the adapted frame field $\left\{\frac{\partial}{\partial x^i}\,, \frac{\delta}{\delta x^\alpha}\right\}$. As it is well-known, $\widetilde{\nabla}$ is a torsion-free and metric connection, that is, we have

$$
(2.5)\qquad \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X = [X, Y], \quad \forall X, Y \in \Gamma(TM),
$$

and

$$
(2.6)\qquad X(g(Y, Z)) = g(\widetilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}_X Z), \quad \forall X, Y, Z \in \Gamma(TM).
$$

Moreover, $\widetilde{\nabla}$ is determined by g as follows (cf. O'Neill [5], p.61)

$$
(2.7)\qquad
\begin{aligned}
2g(\widetilde{\nabla}_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y))\\
&\quad + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y),
\end{aligned}
$$

for any $X, Y, Z \in \Gamma(TM)$.

Lemma 2.1. *Let $(M, g, \mathcal{F})$ be a foliated semi-Riemannian manifold. Then the local coefficients of the Levi-Civita connection with respect to the adapted frame field $\left\{\frac{\partial}{\partial x^i}\,, \frac{\delta}{\delta x^\alpha}\right\}$ satisfy the identities:*

$$
(2.8)\qquad
\begin{aligned}
&\text{(a) } F_\alpha{}^\gamma{}_\beta = F_\beta{}^\gamma{}_\alpha, \quad \text{(b) } G_\beta{}^i{}_\alpha - G_\alpha{}^i{}_\beta = I_\alpha{}^i{}_\beta, \quad \text{(c) } H_i{}^\gamma{}_\alpha = L_\alpha{}^\gamma{}_i,\\
&\text{(d) } K_i{}^j{}_\alpha - M_\alpha{}^j{}_i = A_i{}^j{}_\alpha, \quad \text{(e) } H_i{}^\gamma{}_\alpha = -g^{\gamma\beta} G_\beta{}^j{}_\alpha g_{ji},\\
&\text{(f) } N_i{}^\gamma{}_j = N_j{}^\gamma{}_i = -g^{\gamma\alpha} M_\alpha{}^h{}_i g_{hj}, \quad \text{(g) } P_i{}^k{}_j = P_j{}^k{}_i,
\end{aligned}
$$

where $g^{\gamma\alpha}$ are the entries of the inverse matrix of $[g_{\alpha\beta}]$.

Proof. By using (1.8a) and (2.4a) in (2.5) we obtain (2.8a) and (2.8b). In a similar way, we use (2.4b), (2.4c) and (1.8b) in (2.5) and derive (2.8c) and (2.8d). Now, taking into account that $\mathcal{D}$ and $\overline{\mathcal{D}}$ are orthogonal distributions, from (2.6) we infer that

$$g\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^\alpha}}\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\beta}\right) + g\left(\frac{\partial}{\partial x^i}, \widetilde{\nabla}_{\frac{\delta}{\delta x^\alpha}}\frac{\delta}{\delta x^\beta}\right) = 0.$$

Then by using (2.4b), (2.4a), (2.1a) and (2.1b) we obtain (2.8e). The second equality in (2.8f) is deduced in a similar way. Finally, (2.8g) and the first equality in (2.8f) are obtained from (2.5) via (2.4d). □

Now, by direct calculations in (2.4) using (1.5) we deduce that $\{G_\alpha{}^i{}_\beta\}$, $\{H_i{}^\gamma{}_\alpha\}$, $\{L_\alpha{}^\gamma{}_i\}$, $\{M_\alpha{}^j{}_i\}$ and $\{N_i{}^\gamma{}_j\}$ define adapted tensor fields on M. Moreover, from Lemma 2.1 we see that the adapted tensor fields $G = (G_\alpha{}^i{}_\beta)$ and $N = (N_i{}^\gamma{}_j)$ determine all the other adapted tensor fields from (2.4). We call G (resp. N) the **$\overline{\mathcal{D}}$-second fundamental form** (resp. **$\mathcal{D}$-second fundamental form**) on $(M, g, \mathcal{F})$.

Theorem 2.1. *Let $(M, g, \mathcal{F})$ be a foliated semi-Riemannian manifold. Then the Levi-Civita connection $\widetilde{\nabla}$ is completely determined by the following local coefficients:*

$$(2.9)\quad \begin{aligned}
&\text{(a)}\quad F_\alpha{}^\gamma{}_\beta = \frac{1}{2}\, g^{\gamma\mu}\left(\frac{\delta g_{\mu\alpha}}{\delta x^\beta} + \frac{\delta g_{\mu\beta}}{\delta x^\alpha} - \frac{\delta g_{\alpha\beta}}{\delta x^\mu}\right),\\
&\text{(b)}\quad G_\alpha{}^i{}_\beta = -\frac{1}{2}\left(g^{ij}\,\frac{\partial g_{\alpha\beta}}{\partial x^j} + I_\alpha{}^i{}_\beta\right),\\
&\text{(c)}\quad K_i{}^j{}_\alpha = \frac{1}{2}\, g^{jk}\left(\frac{\delta g_{ik}}{\delta x^\alpha} + A_i{}^h{}_\alpha g_{hk} - A_k{}^h{}_\alpha g_{hi}\right),\\
&\text{(d)}\quad N_i{}^\gamma{}_j = -\frac{1}{2}\, g^{\gamma\alpha}\left(\frac{\delta g_{ij}}{\delta x^\alpha} - A_i{}^k{}_\alpha g_{kj} - A_j{}^k{}_\alpha g_{ki}\right),\\
&\text{(e)}\quad P_i{}^k{}_j = \frac{1}{2}\, g^{kh}\left(\frac{\partial g_{hi}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^h}\right).
\end{aligned}$$

Proof. First, we take $X = \dfrac{\delta}{\delta x^\beta}$, $Y = \dfrac{\delta}{\delta x^\alpha}$, $Z = \dfrac{\delta}{\delta x^\mu}$ in (2.7) and by using (2.4a), (1.8a) and (2.1a) we obtain (2.9a). If we take the same X and Y but $Z = \frac{\partial}{\partial x^j}$, then (2.7) yields (2.9b) via (2.1a), (2.1b), (2.4a) and (1.8). The other formulas are obtained in a similar way. □

3. Schouten-Van Kampen and Vrănceanu connections. Let $(M, g, \mathcal{F})$ be an $(n+p)$-dimensional foliated semi-Riemannian manifold, where M, g and $\mathcal{F}$ are considered as in Section 2. Denote by $\widetilde{\nabla}$ the Levi-Civita connection on (M, g) and by P and $\overline{P}$ the projection morphisms of $\Gamma(TM)$ on $\Gamma(\mathcal{D})$ and $\Gamma(\overline{\mathcal{D}})$ respectively. Then, two remarkable linear connections are defined on M as follows:

$$\nabla_X Y = P\widetilde{\nabla}_X PY + \overline{P}\widetilde{\nabla}_X \overline{P}Y, \tag{3.1}$$

and

$$\nabla^*_X Y = P\widetilde{\nabla}_{PX} PY + \overline{P}\widetilde{\nabla}_{\overline{P}X} \overline{P}Y + P[\overline{P}X, PY] + \overline{P}[PX, \overline{P}Y], \tag{3.2}$$

for any $X, Y \in \Gamma(TM)$. It is easy to see that the distributions $\mathcal{D}$ and $\overline{\mathcal{D}}$ are parallel with respect to both linear connections ∇ and ∇^*. For historical reasons, which we shall explain later on in this section, we call ∇ and ∇^* the **Schouten-Van Kampen connection** and the **Vrănceanu connection** respectively. The coordinate-free formulas (3.1) and (3.2) for ∇ and ∇^* have been first given by Ianuş [3].

Our approach of non-degenerate foliations on semi-Riemannian manifolds by using the connections $\widetilde{\nabla}$, ∇ and ∇^* is a continuation of the study initiated by Vrănceanu [11] for non-holonomic spaces. In the modern terminology we can say that a non-holonomic space is a manifold endowed with a non-integrable distribution. At that time (the third decade of the last century), the study of non-holonomic spaces was entirely performed by using local coordinates. Here we initiate such a study for the geometry of both the foliation and the ambient manifold, via Schouten-Van Kampen and Vrănceanu connections.

Lemma 3.1. *The Schouten-Van Kampen and Vrănceanu connections are locally given by the following formulas:*

$$\begin{aligned}
&\text{(a)}\ \nabla_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = F_\alpha{}^\gamma{}_\beta \frac{\delta}{\delta x^\gamma}, \quad &&\text{(b)}\ \nabla_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial x^i} = K_i{}^j{}_\alpha \frac{\partial}{\partial x^j},\\
&\text{(c)}\ \nabla_{\frac{\partial}{\partial x^i}} \frac{\delta}{\delta x^\alpha} = L_\alpha{}^\gamma{}_i \frac{\delta}{\delta x^\gamma}, \quad &&\text{(d)}\ \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = P_i{}^k{}_j \frac{\partial}{\partial x^k},
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\text{(a)}\ \nabla^*_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = F_\alpha{}^\gamma{}_\beta \frac{\delta}{\delta x^\gamma}, \quad &&\text{(b)}\ \nabla^*_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial x^i} = A_i{}^j{}_\alpha \frac{\partial}{\partial x^j},\\
&\text{(c)}\ \nabla^*_{\frac{\partial}{\partial x^i}} \frac{\delta}{\delta x^\alpha} = 0, \quad &&\text{(d)}\ \nabla^*_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = P_i{}^k{}_j \frac{\partial}{\partial x^k},
\end{aligned} \tag{3.4}$$

respectively, where $(F_\alpha{}^\gamma{}_\beta, K_i{}^j{}_\alpha, L_\alpha{}^\gamma{}_i, P_i{}^k{}_j)$ *are the local coefficients from* (2.4) *and* $A_i{}^j{}_\alpha$ *are given by* (1.9b).

Proof. By direct calculations in (3.1) and (3.2) using (2.4) and (1.8b).□

Taking into account (2.9), (2.8c) and (2.8e), from Lemma 3.1 we deduce the following.

Corollary 3.1. (i) *The Schouten-Van Kampen connections is locally given by* $(F_\alpha{}^\gamma{}_\beta, K_i{}^j{}_\alpha, L_\alpha{}^\gamma{}_i, P_i{}^k{}_j)$*, where*

$$L_\alpha{}^\gamma{}_i = \frac{1}{2} g^{\gamma\beta}\left(\frac{\partial g_{\alpha\beta}}{\partial x^i} + g_{ij} I_\beta{}^j{}_\alpha\right), \tag{3.5}$$

and $F_\alpha{}^\gamma{}_\beta$, $K_i{}^j{}_\alpha$ *and* $P_i{}^k{}_j$ *are given by* (2.9a), (2.9c) *and* (2.9e) *respectively.*

(ii) *The Vrănceanu connection is locally given by* $(F_\alpha{}^\gamma{}_\beta, A_i{}^j{}_\alpha, P_i{}^k{}_j)$*, where* $F_\alpha{}^\gamma{}_\beta, A_i{}^j{}_\alpha$ *and* $P_i{}^k{}_j$ *are given by* (2.9a), (1.9b) *and* (2.9e) *respectively.*

Before we continue our study on foliations by means of connections $\widetilde{\nabla}$, ∇ and ∇^* let us make the following comment. In the book of Vrănceanu [13] at page 235 we find the formulas (60) and (61) which represent exactly the above two connections ∇ and ∇^* given by (3.3) and (3.4) respectively. The only difference is that the Vrănceanu's presentation of these connections is done in the most general case. That is to say that the ambient manifold M is endowed with a linear connection and no one of the complementary distributions is supposed to be integrable. In the same reference Vrănceanu quoted Schouten-Van Kampen [8] and Vrănceanu [12] for the connections given by (60) and (61) respectively. As the above connections ∇ and ∇^* given locally by (3.3) and (3.4) are deduced from formulas (60) and (61) in Vrănceanu's book [13] when $\mathcal{D}$ is integrable, we are entitled to call them the Schouten-Van Kampen and Vrănceanu connections respectively.

Certainly, the above connections have been used before in studying foliations (especially when M is a Riemannian manifold). We quote here some references where these connections appeared under different names. First, in the book of Reinhart [7] at page 147, the **almost product connection** ∇^G and the **adapted connection** $\nabla^{\mathcal{F}}$ are just the Schouten-Van Kampen and Vrănceanu connections respectively. In the book of Tondeur [9] at page 21, the connection given by (3.3) coincides, in our notations, with

$\nabla^*_X \overline{P}Y$, which is the restriction of Vrănceanu connection to the transversal distribution. Also, Vaisman [10] constructed on a non-holonomic Riemannian manifold a linear connection named the **second connection** (the first connection being the Levi-Civita connection). Comparing the formulas from (1.31) on page 51 in Vaisman [10] with formulas from (61) in Vrănceanu [13], p. 235 we see that they represent the same connection. In particular, when M is a foliated Riemannian manifold, the above local coefficients of Vrănceanu connection have been first obtained by Vaisman [10]. Finally, we note that the **Bott connection** (see Tondeur [9], p.19) is expressed, in our notations, by $\nabla^*_{PX}\overline{P}Y$ and therefore represents the restriction of Vrănceanu connection to $\Gamma(\mathcal{D}) \times \Gamma(\overline{\mathcal{D}})$.

Next, by using the Schouten-Van Kampen and Vrănceanu connections we define two types of covariant derivatives for adapted tensor fields on $(M, g, \mathcal{F})$. Let $T = \left(T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t}\right)$ be an adapted tensor field of type $(q, s; r, t)$. Then the **Schouten-Van Kampen transversal covariant derivative** and the **Vrănceanu transversal covariant derivative** of T are given by

$$
\begin{aligned}
T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t|\gamma} &= \frac{\delta T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t}}{\delta x^\gamma} \\
&+ \sum_{x=1}^{q} T^{i_1\dots h\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t} K_h{}^{i_x}{}_\gamma + \sum_{y=1}^{r} T^{i_1\dots i_q\alpha_1\dots\varepsilon\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t} F_\varepsilon{}^{\alpha_y}{}_\gamma \\
&- \sum_{z=1}^{s} T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots h\dots j_s\beta_1\dots\beta_t} K_{j_z}{}^h{}_\gamma - \sum_{u=1}^{t} T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\varepsilon\dots\beta_t} F_{\beta_u}{}^\varepsilon{}_\gamma,
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t|^*\gamma} &= \frac{\delta T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t}}{\delta x^\gamma} \\
&+ \sum_{x=1}^{q} T^{i_1\dots h\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t} A_h{}^{i_x}{}_\gamma + \sum_{y=1}^{r} T^{i_1\dots i_q\alpha_1\dots\varepsilon\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t} F_\varepsilon{}^{\alpha_y}{}_\gamma \\
&- \sum_{z=1}^{s} T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots h\dots j_s\beta_1\dots\beta_t} A_{j_z}{}^h{}_\gamma - \sum_{u=1}^{t} T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\varepsilon\dots\beta_t} F_{\beta_u}{}^\varepsilon{}_\gamma,
\end{aligned} \tag{3.7}
$$

respectively. Similarly, the **Schouten-Van Kampen structural covariant derivative** and the **Vrănceanu structural covariant derivative** of

T are given by

$$
\begin{aligned}
T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t\|k} &= \frac{\partial T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t}}{\partial x^k} \\
&+ \sum_{x=1}^{q} T^{i_1\ldots h\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t} P_h{}^{i_x}{}_k + \sum_{y=1}^{r} T^{i_1\ldots i_q\alpha_1\ldots\varepsilon\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t} L_\varepsilon{}^{\alpha_y}{}_k \\
&- \sum_{z=1}^{s} T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots h\ldots j_s\beta_1\ldots\beta_t} P_{j_z}{}^h{}_k - \sum_{u=1}^{t} T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\varepsilon\ldots\beta_t} L_{\beta_u}{}^\varepsilon{}_k,
\end{aligned} \tag{3.8}
$$

and

$$
\begin{aligned}
T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t\|^*k} &= \frac{\partial T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t}}{\partial x^k} \\
&+ \sum_{x=1}^{q} T^{i_1\ldots h\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t} P_h{}^{\alpha_x}{}_k - \sum_{z=1}^{s} T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots h\ldots j_s\beta_1\ldots\beta_t} P_{j_z}{}^h{}_k,
\end{aligned} \tag{3.9}
$$

respectively.

Remark 3.1. It is noteworthy that the transversal (resp. structural) covariant derivative of an adapted tensor field of type $(q, s; r, t)$ is an adapted tensor field of type $(q, s; r, t + 1)$ (resp. $(q, s + 1; r, t)$.

From (3.8) and (3.9) we obtain

$$
\begin{aligned}
&T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t\|k} - T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t\|^*k} \\
&= \sum_{y=1}^{r} T^{i_1\ldots i_q\alpha_1\ldots\varepsilon\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t} L_\varepsilon{}^{\alpha_y}{}_k - \sum_{u=1}^{t} T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\varepsilon\ldots\beta_t} L_{\beta_u}{}^\varepsilon{}_k.
\end{aligned} \tag{3.10}
$$

Also, by using (2.8d), from (3.6) and (3.7) we deduce that

$$
\begin{aligned}
&T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t|\gamma} - T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t|^*\gamma} \\
&= \sum_{x=1}^{q} T^{i_1\ldots h\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t} M_\gamma{}^{i_x}{}_h - \sum_{z=1}^{s} T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots h\ldots j_s\beta_1\ldots\beta_t} M_\gamma{}^h{}_{j_z}.
\end{aligned} \tag{3.11}
$$

In particular, from (3.9) it follows that the Vrănceanu structural covariant derivative of a **transversal tensor field** $T = \left(T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_t}\right)$ coincides with the partial derivative, that is, we have

$$T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_t\|^*k} = \frac{\partial T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_t}}{\partial x^k}\,. \tag{3.12}$$

Lemma 3.2. (i) *The Schouten-Van Kampen and Vrănceanu structural covariant derivatives of* $g_{ij}, g^{ij}, g_{\alpha\beta}$ *and* $g^{\alpha\beta}$ *are given by*

$$\text{(a) } g_{ij\|k} = g^{ij}{}_{\|k} = 0, \quad \text{(b) } g_{\alpha\beta\|k} = g^{\alpha\beta}{}_{\|k} = 0, \tag{3.13}$$

and

$$\begin{gathered} \text{(a) } g_{ij\|^*k} = g^{ij}{}_{\|^*k} = 0, \ \text{(b) } g_{\alpha\beta\|^*k} \frac{\partial g_{\alpha\beta}}{\partial x^k} = -g_{kj}(G_\alpha{}^j{}_\beta + G_\beta{}^j{}_\alpha), \\ \text{(c) } g^{\alpha\beta}{}_{\|^*k} = \frac{\partial g^{\alpha\beta}}{\partial x^k} = g^{\alpha\mu} g^{\beta\varepsilon} g_{kj}(G_\mu{}^j{}_\varepsilon + G_\varepsilon{}^j{}_\mu), \end{gathered} \tag{3.14}$$

respectively.

(ii) *The Schouten-Van Kampen and Vrănceanu transversal covariant derivatives of* $g_{ij}, g^{ij}, g_{\alpha\beta}$ *and* $g^{\alpha\beta}$ *are given by*

$$\text{(a) } g_{ij|\gamma} = g^{ij}{}_{|\gamma} = 0, \quad \text{(b) } g_{\alpha\beta|\gamma} = g^{\alpha\beta}{}_{|\gamma} = 0, \tag{3.15}$$

and

$$\begin{gathered} \text{(a) } g_{ij|^*\gamma} = -2 g_{\gamma\alpha} N_i{}^\alpha{}_j, \quad \text{(b) } g^{ij}{}_{|^*\gamma} = 2 g_{\gamma\alpha} N_h{}^\alpha{}_k g^{hi} g^{kj}, \\ \text{(c) } g_{\alpha\beta|^*\gamma} = g^{\alpha\beta}{}_{|^*\gamma} = 0, \end{gathered} \tag{3.16}$$

respectively.

Proof. First (3.13) and (3.15) follow from (2.6) by using (2.4) and (3.6)-(3.9). Then by using (3.13a) and (3.10) we obtain (3.14a). Similarly, (3.16c) is obtained from (3.11) by using (3.13b). The first equalities in (3.14b) and (3.14c) are deduced from (3.12). By using (3.10) and (3.13b) we obtain

$$g_{\alpha\beta\|^*k} = g_{\varepsilon\beta} L_\alpha{}^\varepsilon{}_k + g_{\alpha\varepsilon} L_\beta{}^\varepsilon{}_k.$$

Then we use (2.8c) and (2.8e) and obtain the second equality in (3.14b), which implies the second equality in (3.14c). In a similar way, by using (3.11) and (2.8f) we deduce (3.16a) and (3.16b). □

From (3.13) and (3.15) we see that the Schouten-Van Kampen is a metric connection. On the contrary, (3.14) and (3.16) show us that the Vrănceanu connection, in general, is not a metric connection. To state necessary and sufficient conditions for ∇^* to be metric we recall some concepts from theory of foliations.

First, if each leaf of $\mathcal{F}$ is totally geodesic immersed in (M, g), then $\mathcal{F}$ is called **totally geodesic**. Then from (2.4d) and (2.8f) we see that $\mathcal{F}$ *is totally geodesic if and only if the* $\mathcal{D}$*-second fundamental form on* $(M, g, \mathcal{F})$ *vanishes identically on* M*, that is, one of the following conditions is satisfied*

$$(3.17) \qquad \text{(a) } N_i{}^{\gamma}{}_j = 0, \text{ or (b) } M_\gamma{}^i{}_j = 0, \quad \begin{array}{l} \forall i, j \in \{1, ..., n\} \\ \forall \gamma \in \{n+1, ..., n+p\}. \end{array}$$

Also, we say that $\mathcal{F}$ is a foliation with **bundle-like metric** g if each geodesic in (M, g) which is tangent to the transversal distribution $\overline{\mathcal{D}}$ at one point remains tangent for its entire length (cf. Reinhart [6]). Then *a necessary and sufficient condition for* g *to be bundle-like for* $\mathcal{F}$ *is that* (see Reinhart [6], p. 122)

$$(3.18) \qquad \frac{\partial g_{\alpha\beta}}{\partial x^i} = 0, \quad \forall \alpha, \beta \in \{n+1, ..., n+p\}, \; i \in \{1, ..., n\}.$$

Then from (3.14b) it follows that g *is bundle-like for* $\mathcal{F}$ *if and only if the* $\overline{\mathcal{D}}$*-second fundamental form on* $(M, g, \mathcal{F})$ *is skew-symmetric*, that is, we have

$$(3.19) \qquad G_\alpha{}^i{}_\beta + G_\beta{}^i{}_\alpha = 0, \; \forall \alpha, \beta \in \{n+1, ..., n+p\}, \; i \in \{1, ..., n\}.$$

Finally, we say that the Vrănceanu connection ∇^* is a **structural** (resp. **transversal**) **metric connection** if we have

$$(3.20) \qquad g_{\alpha\beta\|^* k} = 0 \;\; \text{(resp. } g_{ij\|^*\gamma} = 0\text{)}.$$

Now we can state the following.

Theorem 3.1. *Let* $(M, g, \mathcal{F})$ *be a foliated semi-Riemannian manifold. Then we have the following assertions:*

(i) g *is bundle-like for* $\mathcal{F}$ *if and only if the Vrănceanu connection is a structural metric connection.*

(ii) *$\mathcal{F}$ is totally geodesic if and only if the Vrănceanu connection is a transversal metric connection.*

Proof. The assertion (i) follows by using (3.18), (3.20) and the first equality in (3.14b). The assertion (ii) is a consequence of (3.17), (3.20) and (3.16a). □

Thus we obtain an interesting characterization of a worth class of foliations by means of Vrănceanu connection.

Corollary 3.2. *The Vrănceanu connection ∇^* on a foliated semi-Riemannian manifold $(M, g, \mathcal{F})$ is a metric connection if and only if $\mathcal{F}$ is totally geodesic with bundle-like metric g.*

Next, we denote by T the torsion tensor field of ∇, that is, we have

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],\ \forall X, Y \in \Gamma(TM). \tag{3.21}$$

Then with respect to an adapted frame field $\left\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right\}$ we put:

$$\begin{aligned}
&\text{(a)}\quad T\left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right) = T_\alpha{}^\gamma{}_\beta \frac{\delta}{\delta x^\gamma} + T_\alpha{}^i{}_\beta \frac{\partial}{\partial x^i},\\
&\text{(b)}\quad T\left(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^i}\right) = -T\left(\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right) = T_i{}^\gamma{}_\alpha \frac{\delta}{\delta x^\gamma} + T_i{}^j{}_\alpha \frac{\partial}{\partial x^j},\\
&\text{(c)}\quad T\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) = T_i{}^\gamma{}_j \frac{\delta}{\delta x^\gamma} + T_i{}^k{}_j \frac{\partial}{\partial x^k}.
\end{aligned} \tag{3.22}$$

Lemma 3.3. *The local components of the torsion tensor field of Schouten-Van Kampen connection are given by*

$$\begin{aligned}
&\text{(a) } T_\alpha{}^\gamma{}_\beta = 0, &&\text{(b) } T_\alpha{}^i{}_\beta = I_\alpha{}^i{}_\beta, &&\text{(c) } T_i{}^\gamma{}_\alpha = -L_\alpha{}^\gamma{}_i,\\
&\text{(d) } T_i{}^k{}_\alpha = M_\alpha{}^k{}_i, &&\text{(e) } T_i{}^\gamma{}_j = 0, &&\text{(f) } T_i{}^k{}_j = 0.
\end{aligned} \tag{3.23}$$

Proof. First, by using (3.21), (3.3a), (1.8a) and (2.8a) we obtain

$$T\left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right) = I_\alpha{}^i{}_\beta \frac{\partial}{\partial x^i},$$

which implies (3.23a) and (3.23b) via (3.22a). Similarly, we use (3.21), (3.3b), (3.3c), (1.8b) and (1.8d), and deduce that

$$T\left(\frac{\delta}{\delta x^{\alpha}}, \frac{\partial}{\partial x^{i}}\right) = M_{\alpha}{}^{j}{}_{i}\,\frac{\partial}{\partial x^{j}} - L_{\alpha}{}^{\gamma}{}_{i}\,\frac{\delta}{\delta x^{\gamma}},$$

which together with (3.22b) proves (3.23c) and (3.23d). Finally, (3.23c) and (3.23f) follow from (3.21) by using (3.3d), (2.8g) and (3.23c). □

Now, if T^* denotes the torsion tensor field of ∇^* then its local components are defined by similar equations as in (3.22). Moreover, the following lemma has a similar proof as of Lemma 3.3.

Lemma 3.4. *The local components of the torsion tensor field of the Vrănceanu connection are given by*

(3.24)
$$\text{(a) } T^*{}_{\alpha}{}^{\gamma}{}_{\beta} = 0, \quad \text{(b) } T^*{}_{\alpha}{}^{i}{}_{\beta} = I_{\alpha}{}^{i}{}_{\beta}, \quad \text{(c) } T^*{}_{i}{}^{\gamma}{}_{\alpha} = 0,$$
$$\text{(d) } T^*{}_{i}{}^{k}{}_{\alpha} = 0, \quad \text{(e) } T^*{}_{i}{}^{\gamma}{}_{j} = 0, \quad \text{(f) } T^*{}_{i}{}^{k}{}_{j} = 0.$$

Next, we state the following.

Theorem 3.2. (i) *The Schouten-Van Kampen connection is torsion-free if and only if $\overline{\mathcal{D}}$ is integrable and M is locally a semi-Riemannian product $N \times \overline{N}$, where N and $\overline{N}$ are leaves of $\mathcal{D}$ and $\overline{\mathcal{D}}$ respectively.*

(ii) *The Vrănceanu connection is torsion-free if and only if $\overline{\mathcal{D}}$ is integrable.*

Proof. Suppose ∇ is torsion-free. Then taking into account that ∇ is also a metric connection (see (3.13) and (3.15)), by uniqueness of Levi-Civita connection we deduce that $\nabla = \widetilde{\nabla}$. Thus both distributions $\mathcal{D}$ and $\overline{\mathcal{D}}$ are parallel with respect to $\widetilde{\nabla}$. Hence $\overline{\mathcal{D}}$ is integrable and M is locally a semi-Riemannian product as in the assertion (i). Conversely, suppose $\overline{\mathcal{D}}$ is integrable and M is a locally semi-Riemannian product $N \times \overline{N}$. Then by Proposition 1.1 we have $I_{\alpha}{}^{i}{}_{\beta} = 0$ which implies $T_{\alpha}{}^{i}{}_{\beta} = 0$ via (3.23b). Taking into account that both N and $\overline{N}$ are totally geodesic immersed in (M, g), from (2.4a) and (2.4d) we infer that $G_{\alpha}{}^{i}{}_{\beta} = 0$ and $N_{i}{}^{\alpha}{}_{j} = 0$. Then by using (2.8e), (2.8c) and (2.8f) we obtain $L_{\alpha}{}^{\gamma}{}_{i} = 0$ and $M_{\alpha}{}^{h}{}_{i} = 0$. Thus by (3.23c) and (3.23d) we deduce that $T_{i}{}^{\gamma}{}_{\alpha} = 0$ and $T_{i}{}^{k}{}_{\alpha} = 0$. Hence by Lemma 3.3 we conclude that ∇ is torsion-free. The assertion (ii) follows by using Lemma 3.4 and Proposition 2.2. □

Theorem 3.3. *Let $(M, g, \mathcal{F})$ be a foliated semi-Riemannian manifold. Then the Schouten-Van Kampen connection coincides with the Vrănceanu connection if and only if $\overline{\mathcal{D}}$ is integrable and M is locally a semi-Riemannian product $N \times \overline{N}$, where N and $\overline{N}$ are leaves of $\mathcal{D}$ and $\overline{\mathcal{D}}$ respectively.*

Proof. We compare (3.3) with (3.4) and deduce that $\nabla = \nabla^*$ if and only if $L_\alpha{}^\gamma{}_i = 0$ and $K_i{}^j{}_k = A_i{}^j{}_\alpha$. By (2.8c), (2.8e), (2.8d) and (2.8f) we see that the above conditions are equivalent to the conditions $G_\alpha{}^i{}_\beta = 0$ and $N_i{}^\alpha{}_j = 0$. Then by using (2.4a), (2.8b) and Proposition 1.1 we deduce that $G_\alpha{}^i{}_\beta = 0$ if and only if $\overline{\mathcal{D}}$ is integrable and its leaves are totally geodesic immersed in (M, g). Finally from (2.4d) we see that $N_i{}^\alpha{}_j = 0$ if and only if $\mathcal{F}$ is totally geodesic. This completes the proof of the theorem. □

4. The transversal Vrănceanu curvature. In the present section we investigate the curvature of the transversal distribution $\overline{\mathcal{D}}$ with respect to the Vrănceanu connection ∇^* on a foliated semi-Riemannian manifold $(M, g, \mathcal{F})$.

First, we consider an adapted frame field $\left\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right\}$ and put:

$$
\text{(4.1)} \quad
\begin{aligned}
&\text{(a)} \quad R^*\left(\frac{\delta}{\delta x^\gamma}, \frac{\delta}{\delta x^\beta}\right)\frac{\delta}{\delta x^\alpha} = R^*{}_\alpha{}^\mu{}_{\beta\gamma}\frac{\delta}{\delta x^\mu},\\
&\text{(b)} \quad R^*\left(\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\beta}\right)\frac{\delta}{\delta x^\alpha} = R^*{}_\alpha{}^\mu{}_{\beta i}\frac{\delta}{\delta x^\mu},\\
&\text{(c)} \quad R^*\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right)\frac{\delta}{\delta x^\alpha} = R^*{}_\alpha{}^\mu{}_{ij}\frac{\delta}{\delta x^\mu},
\end{aligned}
$$

where R^* is the restriction of the curvature tensor field of ∇^* to $\overline{\mathcal{D}}$ given by

$$
\text{(4.2)} \qquad R^*(X,Y)\overline{P}Z = \nabla^*_X\nabla^*_Y\overline{P}Z - \nabla^*_Y\nabla^*_X\overline{P}Z - \nabla^*_{[X,Y]}\overline{P}Z,
$$

for any $X, Y, Z \in \Gamma(TM)$. Then by direct calculations using (4.1), (4.2), (3.4a), (3.4c) and (1.8) we obtain

$$
\text{(4.3)} \quad
\begin{aligned}
&\text{(a)} \quad R^*{}_\alpha{}^\mu{}_{\beta\gamma} = \frac{\delta F_\alpha{}^\mu{}_\beta}{\delta x^\gamma} - \frac{\delta F_\alpha{}^\mu{}_\gamma}{\delta x^\beta} + F_\alpha{}^\varepsilon{}_\beta F_\varepsilon{}^\mu{}_\gamma - F_\alpha{}^\varepsilon{}_\gamma F_\varepsilon{}^\mu{}_\beta,\\
&\text{(b)} \quad R^*{}_\alpha{}^\mu{}_{\beta i} = \frac{\partial F_\alpha{}^\mu{}_\beta}{\partial x^i}, \quad \text{(c)} \quad R^*{}_\alpha{}^\mu{}_{ij} = 0.
\end{aligned}
$$

On the other hand, from Lemma 3.4 we deduce that the torsion tensor field T^* of ∇^* is given by

$$
\begin{aligned}
&\text{(a) } T^*\left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right) = I_\alpha{}^i{}_\beta \frac{\partial}{\partial x^i}, \ \text{(b) } T^*\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) = 0, \\
&\text{(c) } T^*\left(\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right) = -T^*\left(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^i}\right) = 0.
\end{aligned}
\tag{4.4}
$$

Next, we consider the Bianchi identities for ∇^* (cf. Kobayashi-Nomizu [4], p. 135) restricted to $\overline{\mathcal{D}}$, that is, we have

$$
\begin{aligned}
\sum_{(\alpha,\beta,\gamma)} \Bigg\{ &\left(\nabla^*_{\frac{\delta}{\delta x^\alpha}} T^*\right)\left(\frac{\delta}{\partial x^\beta}, \frac{\delta}{\delta x^\gamma}\right) + T^*\left(T^*\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right), \frac{\delta}{\delta x^\gamma}\right) \\
&- R^*\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right)\frac{\delta}{\delta x^\gamma} \Bigg\} = 0,
\end{aligned}
\tag{4.5}
$$

and

$$
\begin{aligned}
\sum_{(\alpha,\beta,\gamma)} \Bigg\{ &\left(\nabla^*_{\frac{\delta}{\delta x^\alpha}} R^*\right)\left(\frac{\delta}{\partial x^\beta}, \frac{\delta}{\delta x^\gamma}\right) \\
&+ R^*\left(T^*\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right), \frac{\delta}{\delta x^\gamma}\right) \Bigg\} \left(\frac{\delta}{\delta x^\nu}\right) = 0,
\end{aligned}
\tag{4.6}
$$

where $\sum_{(\alpha,\beta,\gamma)}$ denotes the cyclic sum with respect to (α, β, γ).

Lemma 4.1. *The local components* $R^*{}_\alpha{}^\mu{}_{\beta\gamma}$ *satisfy the identity*

$$
\sum_{(\alpha,\beta,\gamma)} \{R^*{}_\alpha{}^\mu{}_{\beta\gamma}\} = 0.
\tag{4.7}
$$

Proof. It follows by using (4.1a), (4.4a) and (4.4b) in (4.5) and then taking its $\overline{\mathcal{D}}$-component. □

Next, we put

$$
R^*{}_{\alpha\nu\beta\gamma} = g_{\nu\mu} R^*{}_\alpha{}^\mu{}_{\beta\gamma},
\tag{4.8}
$$

and prove the following.

Lemma 4.2. *The local components* $R^*_{\alpha\nu\beta\gamma}$ *satisfy the identity*

$$R^*_{\alpha\nu\beta\gamma} + R^*_{\nu\alpha\beta\gamma} = I_\gamma{}^i{}_\beta \frac{\partial g_{\alpha\nu}}{\partial x^i} \cdot \tag{4.9}$$

Proof. First, from (3.2) we deduce that

$$\nabla^*_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = \overline{P}\widetilde{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} \cdot \tag{4.10}$$

Then by direct calculations using (1.8a), (3.4c), (4.10) and taking into account that $\widetilde{\nabla}$ is a metric connection we obtain

$$\begin{aligned}
R^*_{\alpha\nu\beta\gamma} &= g\left(R^*\left(\frac{\delta}{\delta x^\gamma}, \frac{\delta}{\delta x^\beta}\right)\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\nu}\right) \\
&= g\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^\gamma}} \overline{P}\widetilde{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\nu}\right) - g\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^\beta}} \overline{P}\widetilde{\nabla}_{\frac{\delta}{\delta x^\gamma}} \frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\nu}\right) \\
&= -g\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha}, \overline{P}\widetilde{\nabla}_{\frac{\delta}{\delta x^\gamma}} \frac{\delta}{\delta x^\nu}\right) + \frac{\delta}{\delta x^\gamma} g\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\nu}\right) \\
&+g\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^\gamma}} \frac{\delta}{\delta x^\alpha}, \overline{P}\widetilde{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\nu}\right) - \frac{\delta}{\delta x^\beta} g\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^\gamma}} \frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\nu}\right) \\
&= -g\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^\gamma}} \overline{P}\widetilde{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\nu} - \widetilde{\nabla}_{\frac{\delta}{\delta x^\beta}} \overline{P}\widetilde{\nabla}_{\frac{\delta}{\delta x^\gamma}} \frac{\delta}{\delta x^\nu}, \frac{\delta}{\delta x^\alpha}\right) \\
&+\frac{\delta}{\delta x^\gamma}\frac{\delta}{\delta x^\beta}(g_{\alpha\nu}) - \frac{\delta}{\delta x^\beta}\frac{\delta}{\delta x^\gamma} g_{\alpha\nu}) = -R^*_{\nu\alpha\beta\gamma} + I_\gamma{}^i{}_\beta \frac{\partial g_{\alpha\nu}}{\partial x^i} \cdot \square
\end{aligned}$$

According to Proposition 1.1 and (3.18) we see that in case $\overline{\mathcal{D}}$ is integrable or g is bundle-like for $\mathcal{F}$, (4.9) becomes an identity that is similar to the one frequently used in the study of semi-Riemannian manifolds of constant curvature (cf. O'Neill [5], p.75).

Theorem 4.1. *Let* $(M, g, \mathcal{F})$ *be a foliated semi-Riemannian manifold such that* g *is a bundle-like metric for* $\mathcal{F}$. *Then we have:*

(i) *The Vrănceanu connection* ∇^* *on* $\overline{\mathcal{D}}$ *is given by*

$$F_\alpha{}^\gamma{}_\beta = \frac{1}{2} g^{\gamma\mu}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right). \tag{4.11}$$

(ii) *The local components of the curvature tensor field of* ∇^* *on* $\overline{\mathcal{D}}$ *are given by*

$$(4.12)\quad \begin{aligned}&(a)\ {R^*}_{\alpha}{}^{\mu}{}_{\beta\gamma} = \frac{\partial F_{\alpha}{}^{\mu}{}_{\beta}}{\partial x^{\gamma}} - \frac{\partial F_{\alpha}{}^{\mu}{}_{\gamma}}{\partial x^{\beta}} + F_{\alpha}{}^{\varepsilon}{}_{\beta} F_{\varepsilon}{}^{\mu}{}_{\gamma} - F_{\alpha}{}^{\varepsilon}{}_{\gamma} F_{\varepsilon}{}^{\mu}{}_{\beta},\\ &(b)\ {R^*}_{\alpha}{}^{\mu}{}_{\beta i} = 0, \quad (c)\ {R^*}_{\alpha}{}^{\mu}{}_{ij} = 0.\end{aligned}$$

Proof. First, by using (3.18) in (2.9a) we obtain (4.11). Then taking into account (3.4c) we obtain (i). Next we note that $F_{\alpha}{}^{\gamma}{}_{\beta}$ from (4.11) are functions of $(x^{n+1}, ..., x^{n+p})$ alone. Thus (4.12) follows from (4.3). □

From (4.12) we see that the curvature tensor field of the restriction of Vrănceanu connection ∇^* to $\overline{\mathcal{D}}$ is given by ${R^*}_{\alpha}{}^{\mu}{}_{\beta\gamma}$ or $R^*_{\alpha\nu\beta\gamma}$. We call $R^*_{\alpha\nu\beta\gamma}$ the *Vrănceanu curvature tensor field* of the transversal distribution $\overline{\mathcal{D}}$.

Theorem 4.2. *Let* $(M, g, \mathcal{F})$ *be a foliated semi-Riemannian manifold such that* g *is a bundle-like metric for* $\mathcal{F}$. *Then the Vrănceanu curvature tensor field of* $\overline{\mathcal{D}}$ *satisfies the identities:*

$$(4.13)\quad \begin{aligned}&(a)\ R^*_{\alpha\nu\beta\gamma} + R^*_{\alpha\nu\gamma\beta} = 0, \qquad (b)\ R^*_{\alpha\nu\beta\gamma} + R^*_{\nu\alpha\beta\gamma} = 0,\\ &(c)\ \sum_{(\alpha,\beta,\gamma)} \{R^*_{\nu\alpha\beta\gamma}\} = 0, \qquad (d)\ R^*_{\alpha\nu\beta\gamma} = R^*_{\beta\gamma\alpha\nu},\\ &(e)\ \sum_{(\alpha,\beta,\gamma)} \{R^*_{\mu\nu\alpha\beta|^*\gamma}\} = 0,\end{aligned}$$

where $|^*$ *represents the Vrănceanu transversal covariant derivative.*

Proof. First, (4.13a) is a well-known property of any curvature tensor of a linear connection. Then (4.13b) follows from (4.9) by using (3.18). Also, by using (4.8) and (4.13b) in (4.7) we obtain (4.13c). Next, (4.13d) follows by a combinatorial exercise using (4.13a), (4.13b) and (4.13c) (see O'Neill [5], p.75). Now, by using (4.4a) and (4.12b) we deduce that

$$R^*\left(T^*\left(\frac{\delta}{\delta x^{\alpha}}, \frac{\delta}{\delta x^{\beta}}\right), \frac{\delta}{\delta x^{\gamma}}\right) = 0.$$

Hence (4.6) becomes

$$\sum_{(\alpha,\beta,\gamma)} \left\{ \left(\nabla^*_{\frac{\delta}{\delta x^{\alpha}}} R^*\right) \left(\frac{\delta}{\delta x^{\beta}}, \frac{\delta}{\delta x^{\gamma}}\right) \left(\frac{\delta}{\delta x^{\nu}}\right) \right\} = 0,$$

which is equivalent to (see (3.7))

$$\sum_{(\alpha,\beta,\gamma)} \left\{ R^{*}{}_{\nu}{}^{\mu}{}_{\beta\gamma|^{*}\alpha} \right\} = 0. \tag{4.14}$$

Thus (4.13e) follows from (4.14) by using (4.8) and taking into account (3.16c). □

Next, we consider a two-dimensional subspace Π_x of $\overline{\mathcal{D}}_x$ and call it a **transversal plane** to the foliation $\mathcal{F}$ at the point $x \in M$. Then Π_x is non-degenerate if and only if

$$\Delta(X,Y) = g(X,X)g(Y,Y) - g(X,Y)^2,$$

is non-zero, where $\{X,Y\}$ is an arbitrary basis for Π_x. If we put

$$X = X^{\alpha}\,\frac{\delta}{\delta x^{\alpha}}\,, \quad Y = Y^{\beta}\,\frac{\delta}{\delta x^{\beta}}\,,$$

then we obtain

$$\Delta(X,Y) = (g_{\alpha\nu}g_{\beta\gamma} - g_{\alpha\beta}g_{\nu\gamma})X^{\alpha}X^{\nu}Y^{\beta}Y^{\gamma}. \tag{4.15}$$

Now, we consider the number

$$K^{*}(X,Y) = \frac{R^{*}_{\alpha\beta\nu\gamma}X^{\alpha}X^{\nu}Y^{\beta}Y^{\gamma}}{\Delta(X,Y)}\,, \tag{4.16}$$

provided Π_x is a non-degenerate transversal plane. By using (4.13a), (4.13b) and (4.13d) we deduce that $K^{*}(X,Y)$ is independent of the basis $\{X,Y\}$ of Π_x. Then we call it the **Vrănceanu sectional curvature** of the transversal distribution at the point x with respect to the transversal plane Π_x. When K^{*} does not depend on the non-degenerate transversal planes in $\overline{\mathcal{D}}$ we say that $\overline{\mathcal{D}}$ is of **scalar Vrănceanu curvature** $K^{*}(x)$.

Theorem 4.3. *Let $(M,g,\mathcal{F})$ be a foliated connected $(n+p)$-dimensional semi-Riemannian manifold, where $\mathcal{F}$ is a non-degenerate n-foliation and g is bundle-like for $\mathcal{F}$. Suppose that the transversal p-distribution with $p > 2$ is of scalar Vrănceanu curvature K^{*}. Then K^{*} is a constant.*

Proof. From (4.16) and (4.15) we deduce that the Vrănceanu curvature tensor field is given by

$$R^{*}_{\alpha\beta\nu\gamma} = K^{*}(x)(g_{\alpha\nu}g_{\beta\gamma} - g_{\alpha\beta}g_{\nu\gamma}). \tag{4.17}$$

Moreover, by using (3.18), (4.11), (4.12a) and (4.8) we deduce that K^* is a function of $(x^{n+1}, ..., x^{n+p})$ alone. Then we take the Vrănceanu transversal covariant derivative in (4.17) and by using (3.16c) and (1.3) we deduce that

$$R^*_{\alpha\beta\nu\gamma|^*\mu} = \frac{\partial K^*}{\partial x^\mu}\,(g_{\alpha\nu}g_{\beta\gamma} - g_{\alpha\beta}g_{\nu\gamma}).$$

Finally, following similar calculations as in the proof of Schur Theorem for Riemannian manifold of constant curvature (see EISENHART [2], p.83) we obtain $K^* =$ constant. □

Remark 4.1. If in particular, $\mathcal{F}$ is the trivial foliation by points of M, that is, $\mathcal{D} = \{0\}$, we have $P = 0$, $\overline{P} =$ identity, and from (3.2) we deduce that $\nabla^* = \widetilde{\nabla}$. Thus Theorem 4.3 becomes the well-known Schur Theorem for semi-Riemannian manifolds of constant curvature. □

When K^* is a constant c on M, we say that $(M, g, \mathcal{F})$ is a foliated semi-Riemannian manifold of **constant transversal Vrănceanu curvature**. In this case, from (4.17) we deduce that the Vrănceanu curvature tensor field is given by

$$R^*_{\alpha\beta\nu\gamma} = c(g_{\alpha\nu}g_{\beta\gamma} - g_{\alpha\beta}g_{\nu\gamma}). \tag{4.18}$$

Now we want to relate the curvature of the manifold and the Vrănceanu sectional curvature. First, we put

$$\widetilde{R}_{\alpha\nu\beta\gamma} = \widetilde{g}\left(\widetilde{R}\left(\frac{\delta}{\delta x^\gamma}, \frac{\delta}{\delta x^\beta}\right)\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\nu}\right). \tag{4.19}$$

Then we prove the following.

Lemma 4.3. *Let $(M, g, \mathcal{F})$ be a foliated semi-Riemannian manifold. Then we have*

$$\begin{aligned}\widetilde{R}_{\alpha\nu\beta\gamma} &= R^*_{\alpha\nu\beta\gamma}\\ &+g_{ij}(G_\beta{}^i{}_\gamma G_\nu{}^j{}_\alpha + G_\alpha{}^i{}_\nu G_\nu{}^j{}_\beta - G_\alpha{}^i{}_\beta G_\nu{}^j{}_\gamma - G_\gamma{}^i{}_\beta G_\nu{}^j{}_\alpha).\end{aligned} \tag{4.20}$$

Proof. First, by using (2.4), (1.8) and (4.3a) we obtain

$$\begin{aligned}&\overline{P}\widetilde{R}\left(\frac{\delta}{\delta x^\gamma}, \frac{\delta}{\delta x^\beta}\right)\frac{\delta}{\delta x^\alpha}\\ &= \left(R^*{}_\alpha{}^\mu{}_{\beta\gamma} + G_\alpha{}^i{}_\beta H_i{}^\mu{}_\gamma - G_\alpha{}^i{}_\gamma H_i{}^\mu{}_\beta + I_\beta{}^i{}_\gamma L_\alpha{}^\mu{}_i\right)\frac{\delta}{\delta x^\mu}.\end{aligned} \tag{4.21}$$

Then (4.20) follows from (4.21) by using (4.8), (4.19), (2.8b), (2.8c) and (2.8e). □

Theorem 4.4. *Let M be an open submanifold of the Euclidean space $(\mathbb{R}^{n+p}, g)$ and $(M, g, \mathcal{F})$ be a foliated Riemannian manifold such that g is bundle-like for $\mathcal{F}$. Then we have:*

(i) *At any point of M the Vrănceanu sectional curvature must be non-negative.*

(ii) *If $(M, g, \mathcal{F})$ is of constant transversal Vrănceanu curvature c and the transversal distribution is not integrable, then $c > 0$.*

Proof. Since g is bundle-like for $\mathcal{F}$, from (3.18) and (3.14b) we deduce that ${G_\alpha}^i{}_\beta$ must be skew-symmetric with respect to (α, β). Then taking into account that $\widetilde{R}_{\alpha\nu\beta\gamma} = 0$, from (4.20) we obtain

$$0 = R^*_{\alpha\nu\beta\gamma} X^\alpha X^\beta Y^\nu Y^\gamma - 3g_{ij} {G_\beta}^i{}_\gamma X^\beta X^\gamma {G_\alpha}^j{}_\nu X^\alpha Y^\nu. \tag{4.21}$$

As in the Riemannian case we have $\Delta(X, Y) > 0$, we infer that $K^*(X, Y) \geq 0$, which proves the assertion (i). Now, from the first assertion we deduce that $c \geq 0$ in case $(M, g, \mathcal{F})$ is of constant transversal Vrănceanu curvature c. If $c = 0$, from (4.21) we conclude that ${G_\beta}^i{}_\gamma = 0$ for any $i \in \{1, ..., n\}$ and $\beta, \gamma \in \{n+1, ..., n+p\}$. Then from (2.8b) it follows ${I_\alpha}^i{}_\beta = 0$ and thus (see Proposition 1.1) $\overline{\mathcal{D}}$ is integrable. This contradicts the hypothesis. Thus we must have $c > 0$. □

Then an interesting question arises. Are there examples of foliated Riemannian manifolds of positive constant Vrănceanu curvature. The answer is in the affirmative by the next example.

Consider the Euclidean space $(\mathbb{R}^3, g)$ with Cartesian coordinates (x, y, z). Then define on $M = \{(x, y, z) \in \mathbb{R}^3 : 0 < y + z < \frac{\pi}{2}\}$ the vector fields

$$X = f\frac{\partial}{\partial y} - \frac{\partial}{\partial x} \text{ and } Y = f\frac{\partial}{\partial z} + \frac{\partial}{\partial x},$$

where f is a function on M given by

$$f(x, y, z) = \sqrt{2}\tan(y + z).$$

Consider the distribution $\overline{\mathcal{D}}$ spanned by the linear independent vector fields $\{X, Y\}$. Then it is easy to check that $\overline{\mathcal{D}}$ is not integrable and g is bundle-like for the foliation $\mathcal{F}$ determined by

$$Z = f\,\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - \frac{\partial}{\partial z}\,.$$

Moreover, by direct calculations we obtain

$$g(R^*(X,Y)Y,X)) = \frac{2f^2(t)(f'(t))^2}{2+f^2(t)} \text{ and } \Delta(X,Y) = f^2(t)(2+f^2(t)),$$

where $t = y + z$. Hence $K^* = 1$.

Open problem. *Classify all foliated Riemannian* (*semi-Riemannian*) *manifolds of constant transversal Vrănceanu curvature.*

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