ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" IAŞI
Tomul LII, s.I, Matematică, 2006, f.1

QUASI CONFORMALLY FLAT CONTACT METRIC MANIFOLDS

BY

A. BHATTACHARYYA

Abstract. In this paper some types of quasi conformally flat contact metric manifolds have been studied.

Mathematics Subject Classification 2000: 53C15.
Key words: Quasi conformally flat contact metric manifolds, contact strongly pseudo convex integrable CR manifolds.

1. Introduction. In [2] Blair and Koufogiorgos obtained that a conformally flat contact metric manifold on which Q commutes with the fundamental collineation φ is of constant curvature 1. In [6] Sharma improved this result. In a recent paper Ghosh, Koufogiorgos and Sharma simplified the result obtained in [6]. In this paper we show that, if M is a contact strongly pseudo-convex integrable CR manifold of dimension $n > 3$ which is quasi conformally flat, then it is of constant curvature 1.

It is also shown that if M is a quasi conformally flat contact metric manifold such that ξ is an eigenvector of the Ricci operator at each point and its sectional curvature K has the property that $K(\xi, X) + K(\xi, \varphi X)$ is a function independent of the choice of X, then it has constat curvature 1.

2. Preliminaries. $A(2n+1)$ dimensional differentiable manifold M is said to be a contact metric manifold if there exists a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$, everywhere on M, a unique vector field ξ, a Riemannian metric g on M and a $(1,1)$ tensor field φ such that

$$\eta(\xi) = 1 \text{ and } d\eta(\xi, X) = 0, \tag{2.1}$$

$$d\eta(X,Y) = g(X,\varphi Y), \;\; \eta(X) = g(X,\xi), \;\; \varphi^2 X = -X + \eta(X)\xi. \tag{2.2}$$

Consequently we have,

$$\varphi\xi = 0, \;\; \eta\circ\varphi = 0, \;\; g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.3}$$

If J denote the restriction of φ to the contact subbundle D, then $J^2 = -1$ and $G(X,Y) = -d\eta(X,JY)$, $\forall X,Y \in D$, define an almost Hermitian structure on D. Then (M,η,J) is a strongly pseudo convex CR manifold [7]. If the complex distribution $\{X - iJX : X \in D\}$ is integrable then (M,η,J) is called a contact strongly pseudo-convex integrable CR manifold.

The integrability condition is given by (see [7]),

$$(\nabla_X\varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \tag{2.4}$$

where ∇ is the Riemannian connection of g and $h = \frac{1}{2}L_\xi\varphi$ (L denotes the Lie differentiation).

In a contact metric manifold we have [1], [4]

$$\nabla_X\xi = -\varphi X - \varphi hX \tag{2.5}$$

$$\nabla_\xi\varphi = 0 \tag{2.6}$$

$$\ell - \varphi\ell\varphi = -2(h^2 + \varphi^2) \tag{2.7}$$

$$Tr\ell = g(Q\xi,\xi) = 2n - Trh^2 \tag{2.8}$$

$$\nabla_\xi h = \varphi(I - h^2 - \ell) \tag{2.9}$$

where ℓ is a (1,1) tensor field defined as

$$\ell = R(\bullet\,,\xi)\xi$$

in [4], R is the curvature tensor field of ∇ and Q is the (1,1) tensor field defined by formula (2.14).

We know that h and ℓ are both self adjoint and satisfy

$$h\xi = \ell\xi = 0 \tag{2.10}$$

$$Trh = Trh\varphi = 0 \tag{2.11}$$

$$h\varphi = -\varphi h. \tag{2.12}$$

A Riemannian manifold M of dimension m is quasi conformally flat iff, $\tilde{C}$ defined by

$$\begin{aligned} &\tilde{C}(X,Y)Z = aR(X,Y)Z \\ &+b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &-\frac{r}{m}\left(\frac{a}{m-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{2.13}$$

where

$$g(QX,Y) = S(X,Y) \tag{2.14}$$

S is the Ricci-tensor field, and a,b are non-zero constants, vanishes for $m > 3$ [3].

3. Quasi conformally flat contact strongly pseudo convex integrable CR-manifold. Let M be a $(2n+1)$ dimensional contact pseudo convex integrable CR-manifold, which is quasi conformally flat. Therefore $\tilde{C} = 0$ i.e.

(3.1)

$$\begin{aligned} R(X,Y)Z &= -\frac{b}{a}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &+\frac{r}{2n+1}(\frac{a}{2n} + 2b)[g(Y,Z)X - g(X,Z)Y]. \end{aligned}$$

Putting $X = \xi$, $Y = e_i$ and $Z = \varphi e_i$, where e_i define an orthonormal local basis of the Lie algebra of vector fields on M and using (2.2) and (2.3) we obtain,

(3.2)

$$R(\xi, e_i)\varphi e_i = -\frac{b}{a}[(S(e_i,\varphi e_i)\xi + g(e_i,\varphi e_i)Q\xi] + \frac{r}{2n+1}(\frac{a}{2n} + 2b)g(e_i,\varphi e_i).$$

Again in contact strongly pseudo convex integrable CR-manifold, the following relation holds [5],

(3.3)

$$
\begin{aligned}
&R(X,Y)\varphi Z-\varphi R(X,Y)Z=\{g(\varphi R(X,Y)\xi,Z)+\eta(X)g(\varphi Y+\varphi hY,Z)\\
&-\eta(Y)g(\varphi X+\varphi hX,Z)\}\xi-g(Y+hY,Z)(\varphi X+\varphi hX)\\
&+g(X+hX,Z)(\varphi Y+\varphi hY)+g(\varphi X+\varphi hX,Z)(Y+hY)\\
&-g(\varphi Y+\varphi hY,Z)(X+hX)-\eta(Z)\{\varphi R(X,Y)\xi\\
&+\eta(X)(\varphi Y+\varphi hY)-\eta(Y)(\varphi X+\varphi hX)\}.
\end{aligned}
$$

Substituting ξ for X and using (2.1), (2.2) and (2.3) we have,

$$R(\xi,Y)\varphi Z-\varphi R(\xi,Y)Z=\eta(Z)\varphi\ell Y-g(\varphi\ell Y,Z)\xi, \tag{3.4}$$

where ℓ is defined earlier. Putting $Y=Z=e_i$ and using $Tr\,\varphi\ell$=0 we get after summing over $i=1,2,\ldots,2n+1$,

$$\sum_{i=1}^{2n+1}R(\xi,e_i)\varphi e_i=\varphi Q\xi. \tag{3.5}$$

Again summing over $i=1,2,\ldots,2n+1$, to (3.2) we get

$$\sum_{i=1}^{2n+1}R(\xi,e_i)\varphi e_i=-\frac{b}{a}(\varphi Q\xi). \tag{3.6}$$

From (3.5) and (3.6) we have,

$$(a+b)\varphi Q\xi=0 \tag{3.7}$$

i.e. $\varphi Q\xi=0$ for $(a+b)\neq 0$.

Thus we can state,

Theorem 3.1. *If M is a contact strongly pseudo-convex integrable CR-manifold of dim>3 which is quasi conformally flat, then it is of constant curvature 1, provided $a+b\neq$0, where a and b are constants defined as in definition of quasi conformal curvature tensor.*

Next we consider that M be a contact metric manifold such that ξ is an eigenvector of the Ricci operator at each point and $K(\xi,X)+K(\xi,\varphi X)$ is a function independent of the choice of X, where K is the sectional curvature of M.

We also consider M is quasi conformally flat. Putting $Y = Z = \xi$ in (3.1) and using (2.9) and (3.8) we have,

$$(\varphi(\nabla_\xi h) - h^2 - \varphi^2)X = \frac{r}{2n+1}(\frac{a}{2n} + 2b)(X - \eta(X)\xi) - \frac{b}{a}\{QX + (Tr\ell)X - 2(Tr\ell)\eta(X)\xi\}. \tag{3.9}$$

Action of φ on both sides yields

$$(-\nabla_\xi h - \varphi h^2 + \varphi)X = \frac{r}{2n+1}(\frac{a}{2n} + 2b)\varphi X - \frac{b}{a}(\varphi QX + (Tr\ell)\varphi X). \tag{3.10}$$

Substituting hX for X gives an equation

$$(-\nabla_\xi h - \varphi h^2 + \varphi)hX = \frac{r}{2n+1}(\frac{a}{2n} + 2b)\varphi hX - \frac{b}{a}[\varphi QhX + (Tr\ell)\varphi hX] \tag{3.11}$$

Action of h on (3.10) gives,

$$h(-\nabla_\xi h - \varphi h^2 + \varphi)X = \frac{r}{2n+1}(\frac{a}{2n} + 2b)h\varphi X - \frac{b}{a}[h\varphi QX + (Tr\ell)h\varphi X]. \tag{3.12}$$

Adding (3.11) and (3.12) and using (2.12) we get,

$$(\nabla_\xi h^2)X = \frac{b}{a}(h\varphi QX + \varphi QhX]. \tag{3.13}$$

Replacing X by φX in (3.9) and using (2.2), (2.3) we have,

$$\nabla_\xi hX - \varphi h^2 X + \varphi X = \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\varphi X - \frac{b}{a}[Q\varphi X + (Tr\ell)\varphi X]. \tag{3.14}$$

Adding (3.10) and (3.14) we get

$$-2(\varphi h^2 - \varphi)X = \frac{2r}{2n+1}\left(\frac{a}{2n} + 2b\right)\varphi X - \frac{b}{a}[Q\varphi X + \varphi QX) - \frac{2b}{a}(Tr\ell)\varphi X. \tag{3.15}$$

Subtracting (3.14) from (3.10) we obtain,

$$2(\nabla_\xi h)X = \frac{b}{a}[Q\varphi X - \varphi QX]. \tag{3.16}$$

Differentiating $Q\xi = (Tr\ell)\xi$ along an arbitrary vector field X and using (2.5) we get,

$$(\nabla_X Q)\xi = Q(\varphi X + \varphi hX) + (XTr\ell)\xi - (Tr\ell)(\varphi X + \varphi hX). \tag{3.17}$$

As $\tilde{C} = 0, div\tilde{C} = 0$ and hence

$$\begin{aligned} &g((\nabla_X Q)Y, Z) - g((\nabla_Y Q)X, Z) \\ &= \frac{1}{2b(2n+1)}\left(\frac{a}{2n} + 2b\right)\{(Xr)g(Y,Z) - (Yr)g(X,Z)\}. \end{aligned} \tag{3.18}$$

Putting $Y = Z = \xi$ and using (3.17) we find,

$$\begin{aligned} XTr\ell \; &- \frac{1}{2b(2n+1)}\left(\frac{a}{2n} + 2b\right)Xr \\ &= \left\{\xi Tr\ell - \frac{1}{2b(2n+1)}\left(\frac{a}{2n} + 2b\right)\xi r\right\}\eta(X). \end{aligned} \tag{3.19}$$

Applying exterior derivation on (3.19) and using Poincare lemma: $d^2 = 0$ and then replacing X, Y respectively by φX,φY we get

$$\xi Tr\ell = \frac{1}{2b(2n+1)}\left(\frac{a}{2n} + 2b\right)\xi r. \tag{3.20}$$

Hence from (3.19) we obtain,

$$XTr\ell = \frac{1}{2b(2n+1)}\left(\frac{a}{2n} + 2b\right)Xr. \tag{3.21}$$

Further putting $Y = \xi$ in (3.18) and using (3.17) and (3.21) we get

$$\begin{aligned} (\nabla_\xi Q)X \; &= Q\varphi(X + hX) - (Tr\ell)(\varphi(X + hX)) \\ &+ \frac{1}{2b(2n+1)}\left(\frac{a}{2n} + 2b\right)(\xi r)X. \end{aligned} \tag{3.22}$$

Since $g((\nabla_\xi Q)X, Y)$ is symmetric in X, Y, (3.22) becomes

$$(Q\varphi + \varphi Q)X + Q\varphi hX + h\varphi QX = 2(Tr\ell)\varphi X. \tag{3.23}$$

Adding (3.22) and (3.23) yields,

$$\begin{aligned} (\nabla_\xi Q)X \; &= -\varphi(Q - hQ)X + (Tr\ell)\varphi(X - hX) \\ &+ \frac{1}{2b(2n+1)}(\frac{a}{2n} + 2b)(\xi r)X. \end{aligned} \tag{3.24}$$

Differentiating (3.15) along ξ and using (2.6) and (3.21) we get,

$$\begin{aligned} -2\varphi(\nabla_\xi h^2)X \; &= -\frac{b}{a}[(\nabla_\xi Q)\varphi X + \varphi(\nabla_\xi Q)X] \\ &+ 2\left\{\frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right) - \frac{b}{a}\right\}(\xi r)\varphi X. \end{aligned} \tag{3.25}$$

Now the action of φ in (3.22) gives,

$$\begin{aligned}\varphi(\nabla_\xi Q)X &= \varphi Q\varphi(X+hX)-(Tr\ell)\varphi^2(X+hX)\\ &+\frac{1}{2b(2n+1)}\left(\frac{a}{2n}+2b\right)(\xi r)\varphi X.\end{aligned} \tag{3.26}$$

Again putting φX in place of X in (3.24) and using (2.2), we have,

$$\begin{aligned}(\nabla_\xi Q)\varphi X &= -\varphi(Q-hQ)\varphi X+(Tr\ell)\varphi(\varphi X-h\varphi X)\\ &+\frac{1}{2b(2n+1)}\left(\frac{a}{2n}+2b\right)(\xi r)\varphi X\end{aligned} \tag{3.27}$$

putting (3.26) and (3.27) in (3.25) we get,

(3.28)

$$\begin{aligned}-2\varphi(\nabla_\xi h^2)X &= -\frac{b}{a}(\varphi Q\varphi hX+\varphi hQ\varphi X)\\ &+\left[\left\{\frac{2r}{2n+1}-\frac{1}{a(2n+1)}\right\}\left(\frac{a}{2n}+2b\right)-\frac{2b}{a}\right](\xi r)\varphi X.\end{aligned}$$

Applying φ again to (3.28) and using (2.2), (2.3) we obtain,

$$2(\nabla_\xi h^2)X=\frac{b}{a}(Q\varphi h+hQ\varphi)X. \tag{3.29}$$

Subtracting (3.13) from (3.29) we get

$$\nabla_\xi h^2=0. \tag{3.30}$$

Hence replacing X by φX and using (3.8). We get from (3.29),

$$Qh=hQ. \tag{3.31}$$

Using (3.31) in (3.23) and then subtracting (3.15) from the resulting equation we get,

$$\begin{aligned}&2\varphi(h^2X-X)=(Q\varphi-\varphi Q)hX+\left(1+\frac{b}{a}\right)(Q\varphi X+\varphi QX)\\ &+\left\{2Tr\ell\left(1+\frac{b}{a}\right)+\frac{2r}{2n+1}\left(\frac{a}{2n}+2b\right)\right\}\varphi X.\end{aligned} \tag{3.32}$$

By the hypothesis $K(\xi,X)+K(\xi,\varphi X)=2k$ and using (2.7), (2.8) we find,

$$h^2=(k-1)\varphi^2 \ \text{ where } \ k=\frac{Tr\ell}{2n}. \tag{3.33}$$

Replacing X by hX in (3.32) and applying (2.2), (2.12), (3.31) we get after a brief calculation,

$$Q\varphi - \varphi Q = \frac{2\{k + Tr\ell(1 + \frac{b}{a}) + \frac{r}{2n+1}(\frac{a}{2n} + 2b)\}}{k-1}\varphi h, \tag{3.34}$$

where $k = \frac{Tr\ell}{2n}$. Now if $Tr\ell - 2n = 0$ on M then by virtue of (2.8) $h = 0$ and hence M is of constant curvature 1. So let $Tr\ell - 2n \neq 0$ in some open neighbourhood N. Using (3.33) in (3.15) and after simplification we get,

$$Q\varphi + \varphi Q = \frac{a}{b}\left\{\frac{2r}{2n+1}\left(\frac{a}{2n} + 2b\right) - \frac{2b}{a}(Tr\ell) - \frac{Tr\ell}{n}\right\}\varphi. \tag{3.35}$$

Adding (3.34) and (3.35) we get,

$$\begin{aligned} 2Q\varphi X &= 2\left\{k + Tr\ell\left(1 + \frac{b}{a}\right) + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right\}\varphi hX \\ &+\frac{a}{b}\left\{\frac{2r}{2n+1}\left(\frac{a}{2n} + 2b\right) - \frac{2b}{a}(Tr\ell) - \frac{Tr\ell}{n}\right\}\varphi X. \end{aligned} \tag{3.36}$$

Replacing X by φX and using (2.2) we have,

$$\begin{aligned} QX &= \left[2Tr\ell - \frac{a}{b}\left\{\frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right) - \frac{Tr\ell}{2n}\right\}\right]\eta(X)\xi \\ &+ \left[\frac{Tr\ell}{2n} + Tr\ell\left(1 + \frac{b}{a}\right) + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right]hX \\ &+\frac{a}{b}\left[\frac{Tr\ell}{2n} + \frac{b}{a}(Tr\ell) - \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right]X. \end{aligned} \tag{3.37}$$

We write coefficient of $\eta(X)\xi$ as α, coefficient of hX as β and coefficient of X as μ. So (3.37) becomes,

$$QX = \alpha\eta(X)\xi + \beta hX + \mu X. \tag{3.38}$$

Finally substituting $Z = \xi$ in (3.1) and using (2.2), (2.14) we get,
(3.39)

$$\begin{aligned} R(X,Y)\xi &= -\frac{b}{a}\{g(QY,\xi)X - g(QX,\xi)Y + \eta(Y)QX - \eta(X)QY\} \\ &-\frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\{\eta(Y)X - \eta(X)Y\} \end{aligned}$$

putting (3.38) in (3.39) we have,

(3.40)

$$R(X,Y)\xi = \left\{-\frac{\alpha b}{a} + 2\mu - \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right\}\{\eta(Y)X - \eta(X)Y\} - \\ -\frac{2b}{a}\beta\{\eta(Y)hX - \eta(X)hY\}.$$

From [4] we know that for any contact metric manifold,

(3.41)

$$2(\nabla_{hX}\varphi)Y = -R(\xi,X)Y - \varphi R(\xi,X)\varphi Y + \varphi R(\xi,\varphi X)Y - R(\xi,\varphi X)\varphi Y \\ +2g(X+hX,Y)\xi - 2\eta(Y)(X+hX).$$

Using (3.40) in (3.41) we get

(3.42)

$$(\nabla_{hX}\varphi)Y = \left\{-\frac{\alpha b}{a} + 2\mu - \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\}\{\eta(Y)X - g(X,Y)\xi\right\} \\ -\eta(Y)(X+hX) + g(X+hX,Y)\xi.$$

Replacing X by hX and using (3.33) we get

$$(\nabla_X\varphi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX).$$

Hence M is a contact strongly pseudo-convex integrable CR-manifold. From Theorem (3.1) it follows that M is of constant curvature 1.

Thus we can state.

Theorem 3.2. *If a contact metric manifold is quasi conformally flat and ξ is an eigenvector of the Ricci operator at each point and $K(\xi,X) + K(\xi,\varphi X)$ is a function independent of choice of X, where K is the sectional curvature of M, then it has constant curvature 1.*

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Received: 23.VIII.2005

Department of Mathematics,
Jadavpur University,
Kolkata-700032
INDIA,
arin1968@indiatimes.com