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TRANSIENT AND STEADY–STATE SOLUTIONS FOR POROUS THERMOELASTIC PLATES

BY

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Abstract. This paper is concerned with the dynamic theory for bending of thermoelastic plates of Mindlin–type made from a material with voids. We establish a representation of Galerkin type for the solution of the field equations. Then, we study the time–harmonic oscillations of porous thermoelastic plates. Finally, the propagation of flexural waves in an infinite plate is discussed.

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1. Introduction. Porous materials play an important role in several branches of science and technology, such as petroleum industry, civil engineering and biomechanics. An approach for the study of the mechanical behaviour of porous media is made by the theory of elastic materials with voids established by NUNZIATO and COWIN (1979) and COWIN and NUNZIATO (1983). This theory is based on the assumption that the bulk mass density ρ of an elastic material with voids is given by the product $\rho = \nu\gamma$, where ν is the volume fraction field ($0 < \nu \leq 1$) and γ is the mass density of the matrix material. Thus, the volume fraction field ν represents an additional degree of kinematical freedom, which describes the porosity of the body. We mention that the Nunziato–Cowin theory for elastic materials with voids can also be regarded as a special case of the theory of materials with microstructure (see e.g., CAPRIZ and PODIO–GUIDUGLI, 1981; CAPRIZ, 1989) or as a particular case of the theory of microstretch elastic solids (see ERINGEN, 1991 and 1999). The linear theory of thermoelastic materials with voids was developed by IEŞAN (1986).

In the last fifty years, many articles and books have investigated the theory of plates which accounts for transverse shear deformation in the flexural motion (see e.g., MINDLIN, 1951; ERINGEN, 1967; LAGNESE and LIONS, 1989; CIARLET, 1997). The time–harmonic oscillations in elastic and thermoelastic plates of Mindlin–type were considered by SCHIAVONE and CONSTANDA (1993) and SCHIAVONE and TAIT (1995). Using the Nunziato–Cowin theory, BÎRSAN (2003) has presented a bending theory for porous thermoelastic plates.

In the present paper, we study the dynamic equations of bending as established by BÎRSAN (2003). First, we prove a representation of Galerkin type for the (transient) solution of these equations. Then, we consider the steady time–harmonic oscillations of porous thermoelastic plates and establish some representation formulae for the amplitude of the oscillations in question. Finally, the propagation of flexural waves in an infinite plate is studied. We show that these waves are dispersed and attenuated, and we discuss the influence of the porosity and thermal fields on the waves propagation.

2. Equations of bending. In this section we present a review of the basic equations for the bending of porous thermoelastic plates and we establish a representation of Galerkin type for the solution.

In what follows we refer to the work of BÎRSAN (2003), where the bending equations for isotropic and homogeneous porous thermoelastic plates with transverse shear deformation have been deduced in detail.

Let us consider a body made from a linearly thermoelastic material with voids which occupies a region B of the three–dimensional Euclidean space, in its reference configuration. We assume that B is the interior of a right cylinder of length h_0 with open cross–section Σ. The length h_0 is assumed to be small enough such that the body occupying B represent a thin plate of constant thickness h_0. The plate is referred to a system of rectangular Cartesian axes $Ox_1x_2x_3$, which is chosen such that the plane x_1Ox_2 is the middle plane.

For the linearly thermoelastic material with voids considered, we denote by $\boldsymbol{u}$ the displacement vector field, φ designates the change in volume fraction field (see COWIN and NUNZIATO, 1983; IEŞAN, 1986) and θ represents the temperature field measured from the constant absolute temperature T_0 of the reference state. Throughout this paper, the Latin subscripts are understood to range over the values $\{1, 2, 3\}$, while Greek subscripts are confined to the range $\{1,2\}$.

Let $\mathcal{T}$ be a given interval of time. In the bending theory of Mindlin–type plates we assume that (see MINDLIN, 1951; ERINGEN, 1967; CIARLET, 1997)

$$
\begin{aligned}
u_\alpha &= x_3 v_\alpha (x_1,x_2,t), \quad u_3 = w(x_1,x_2,t), \\
\varphi &= x_3 \psi (x_1,x_2,t), \quad \theta = x_3 T (x_1,x_2,t),
\end{aligned} \tag{2.1}
$$

for any $(x_1,x_2) \in \Sigma$, $x_3 \in (-h_0/2, h_0/2)$ and $t \in \mathcal{T}$.

The linear strain measures $\varepsilon_{\alpha\beta}$ and γ_α are given by

$$
\varepsilon_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}), \quad \gamma_\alpha = v_\alpha + w,_\alpha , \tag{2.2}
$$

where subscripts preceded by a comma stand for partial differentiation with respect to the corresponding coordinate.

The constitutive equations for isotropic and homogeneous porous thermoelastic plates are (see BÎRSAN, 2003)

$$
\begin{aligned}
M_{\alpha\nu} &= I\left[\lambda \varepsilon_{\rho\rho}\delta_{\alpha\nu} + 2\mu\varepsilon_{\alpha\nu} + b\psi\delta_{\alpha\nu} - \beta T\delta_{\alpha\nu}\right], \quad N_\alpha = \mu\gamma_\alpha, \\
H_\nu &= \alpha I \psi,_\nu , \quad G = -I\left(b\varepsilon_{\rho\rho} + \xi\psi - mT\right), \\
\sigma &= I\left(\beta\varepsilon_{\rho\rho} + m\psi + aT\right), \quad Q_\alpha = kIT,_\alpha , \quad \Gamma = \alpha\psi, \quad R = kT,
\end{aligned} \tag{2.3}
$$

where $\lambda, \mu, b, \beta, \alpha, \xi, m, a$ and k are the constant constitutive coefficients of the thermoelastic material with voids, $\delta_{\alpha\beta}$ is the Kronecker symbol and $I = h_0^3/12$.

The equations of motion for the bending of porous thermoelastic plates are

$$
\begin{aligned}
M_{\beta\alpha,\beta} - h_0 N_\alpha + f_\alpha &= \rho_0 I \ddot{v}_\alpha , \\
N_{\alpha,\alpha} + f &= \rho_0 \ddot{w}, \\
H_{\alpha,\alpha} + G - h_0\Gamma + l &= \rho_0 \varkappa I \ddot{\psi},
\end{aligned} \tag{2.4}
$$

and the energy equation is

$$
Q_{\alpha,\alpha} - h_0 R + S = T_0 \dot{\sigma}, \tag{2.5}
$$

where a superposed dot denotes differentiation with respect to the time, ρ_0 is the reference mass density and the coefficient $\varkappa$ is the equilibrated inertia. We assume that ρ_0 and $\varkappa$ are strictly positive. In (2.4), the fields $f_\alpha\,, f$ characterize the effects of the external body forces and of the stresses acting on the surfaces $x_3 = \pm h_0/2$, while l represents the effects of the extrinsic equilibrated body forces and of the equilibrated stresses acting on $x_3 = \pm h_0/2$. The field S which appear in (2.5) characterizes the bending effects of the external heat supply and of the heat flux on the surfaces $x_3 = \pm h_0/2$ (see Bîrsan, 2003). The functions $f_\alpha\,, f, l$ and S are prescribed fields.

If we substitute the geometrical relations (2.2) and the constitutive equations (2.3) into the balance equations (2.4) and (2.5), then we obtain the equations of bending expressed in terms of the functions $v_\alpha\,, w, \psi$ and T. These equations are

(2.6)

$$\begin{gathered} I\left[\mu\varDelta v_\alpha + (\lambda+\mu)\, v_{\nu,\nu\alpha} + b\psi_{,\alpha} - \beta T_{,\alpha}\right] - \mu h_0\left(v_\alpha + w_{,\alpha}\right) + f_\alpha = \rho_0 I \ddot{v}_\alpha\,, \\ \mu\varDelta w + \mu v_{\alpha,\alpha} + f = \rho_0 \ddot{w}\,, \\ I\left(\alpha\varDelta\psi - b v_{\nu,\nu} + mT\right) - \left(\xi I + \alpha h_0\right)\psi + l = \rho_0\varkappa I\ddot{\psi}\,, \\ I\left(k\varDelta T - \beta T_0 \dot{v}_{\alpha,\alpha} - mT_0\dot{\psi}\right) - kh_0 T + S = aIT_0\dot{T}\,, \end{gathered}$$

where $\varDelta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplace operator. To the field equations presented above we must adjoin boundary conditions and initial conditions.

In what follows, we establish a representation of Galerkin type for the solution of the bending equations (2.6) for porous thermoelastic plates.

Assume that the constitutive coefficients satisfy the inequalities

$$\lambda+\mu>0, \quad \mu>0, \quad \alpha>0, \quad k>0$$

and let $c_1\,, c_2\,, c_3$ be defined by

$$c_1 = \sqrt{\frac{\lambda+2\mu}{\rho_0}}, \quad c_2 = \sqrt{\frac{\mu}{\rho_0}}, \quad c_3 = \sqrt{\frac{\alpha}{\rho_0\varkappa}}\,. \tag{2.7}$$

Here c_1 and c_2 are the velocities of dilatational and rotational waves, respectively, in an infinite elastic medium (see e.g., Gurtin, 1972), while c_3 denotes the wave velocity of a wave carrying a change in volume fraction field (see Puri and Cowin, 1985). For the sake of simplicity, we adopt the notations

$$\chi = \sqrt{\frac{h_0}{I}}\,, \quad \zeta = \xi + \alpha\,\frac{h_0}{I}\,. \tag{2.8}$$

Let $\Box_n$ and Λ_r $(n = 1,2,3,4,\ r = 1,2,3)$ be the differential operators defined by

(2.9)

$$\Box_1 = (\lambda + 2\mu)\left(\Delta - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right) - \mu\chi^2, \quad \Box_2 = \mu\left(\Delta - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right),$$

$$\Box_3 = \alpha\left(\Delta - \frac{1}{c_3^2}\frac{\partial^2}{\partial t^2}\right) - \zeta, \quad \Box_4 = \beta\,\Box_3 + bm,$$

$$\Lambda_1 = \frac{k}{T_0}\left(\Delta - \chi^2\right) - a\,\frac{\partial}{\partial t}, \quad \Lambda_2 = b\,\Lambda_1 - m\beta\,\frac{\partial}{\partial t}, \quad \Lambda_3 = \Box_3\Lambda_1 + m^2\frac{\partial}{\partial t}.$$

We denote by $\Box$ the operator

$$\Box = \Box_2\left(b\Delta\,\Lambda_2 + \Box_1\Lambda_3 - \beta\Delta\,\Box_4\,\frac{\partial}{\partial t}\right) + \mu^2\chi^2\Delta\,\Lambda_3\,. \tag{2.10}$$

Using the associated matrices method presented by Moisil (1952) we are led to the following Galerkin representation for the solution of the bending equations.

Theorem 1. *Let*

(2.11)

$$v_\gamma = \Box V_\gamma - \frac{\partial}{\partial x_\gamma}\Bigg\{\left[\Box_2\left(b\Lambda_2 + (\lambda+\mu)\,\Lambda_3 - \beta\Box_4\frac{\partial}{\partial t}\right) + \mu^2\chi^2\Lambda_3\right]\frac{\partial V_\beta}{\partial x_\beta}$$

$$-\mu\chi^2\Lambda_3 W + \Box_2\left(\Lambda_2\,\Psi - \Box_4\,\Theta\right)\Bigg\},$$

$$w = -\mu\left(\Box_2 - \mu\chi^2\right)\Lambda_3\frac{\partial V_\gamma}{\partial x_\gamma} + \left(b\Delta\Lambda_2 + \Box_1\Lambda_3 - \beta\Delta\,\Box_4\frac{\partial}{\partial t}\right)W +$$

$$+\mu\Delta\left(\Lambda_2\,\Psi - \Box_4\,\Theta\right),$$

$$\psi = \Lambda_2\left[\Box_2\left(\Box_2 - \mu\chi^2\right)\frac{\partial V_\gamma}{\partial x_\gamma} + \mu\chi^2\Delta W\right]$$

$$+\left[\left(\Box_1\Box_2 + \mu^2\chi^2\Delta\right)\Lambda_1 - \beta^2\Delta\,\Box_2\frac{\partial}{\partial t}\right]\Psi - \left[\left(m\Box_1 - b\beta\Delta\right)\Box_2 + m\mu^2\chi^2\Delta\right]\Theta,$$

$$T = \Box_4\frac{\partial}{\partial t}\left[\Box_2\left(\Box_2 - \mu\chi^2\right)\frac{\partial V_\gamma}{\partial x_\gamma} + \mu\chi^2\Delta W\right]$$

$$+\left[\left(m\Box_1 - b\beta\Delta\right)\Box_2 + m\mu^2\chi^2\Delta\right]\frac{\partial}{\partial t}\,\Psi + \left[\left(\Box_1\Box_2 + \mu^2\chi^2\Delta\right)\Box_3 + b^2\Delta\,\Box_2\right]\Theta,$$

where V_α are functions of class C^{10} on $\Sigma \times \mathcal{T}$ and W, Ψ, Θ are functions of class C^8 on $\Sigma \times \mathcal{T}$ which satisfy the equations

(2.12)

$$\left(\Box_2 - \mu\chi^2\right) \Box V_\alpha = -\frac{1}{I} f_\alpha\,, \quad \Box W = -f, \quad \Box \Psi = -\frac{1}{I}\, l, \quad \Box \Theta = -\frac{1}{IT_0}\, S\,.$$

Then, (v_1, v_2, w, ψ, T) represents a solution of the bending equations (2.6).

Proof. Taking into account the relations (2.9)–(2.12) we obtain

$$\begin{aligned}
&\mu\Delta w + \mu v_{\gamma,\gamma} - \rho_0 \ddot{w} = \Box_2 w + \mu v_{\gamma,\gamma} \\
&\quad -\mu \left[(\lambda + \mu)\,\Delta - \Box_1 + \left(\Box_2 - \mu\chi^2\right)\right] \Box_2 \Lambda_3 \frac{\partial V_\gamma}{\partial x_\gamma} \\
&\quad + \left[\Box_2 \left(b\Delta\Lambda_2 + \Box_1\Lambda_3 - \beta\Delta\Box_4 \frac{\partial}{\partial t}\right) + \mu^2\chi^2\Delta\Lambda_3\right] W = \Box W = -f
\end{aligned}$$

and so the equation $(2.6)_2$ is satisfied.

Using (2.11) and the notations (2.9), (2.10), we derive that

$$\begin{aligned}
&\alpha\Delta\psi - b v_{\gamma,\gamma} + mT - \zeta\psi - \rho_0 \varkappa \ddot{\psi} = \Box_3 \psi - b v_{\gamma,\gamma} + mT \\
&= \Bigl[b\,(\lambda + \mu)\,\Delta\Lambda_3 - b\Box_1\Lambda_3 + \left(\Box_2 - \mu\chi^2\right)\Box_3\Lambda_2 \\
&+ m\left(\Box_2 - \mu\chi^2\right)\Box_4 \frac{\partial}{\partial t}\Bigr] \Box_2 \frac{\partial V_\gamma}{\partial x_\gamma} + \mu\chi^2 \left(\Box_3\Lambda_2 - b\Lambda_3 + m\Box_4 \frac{\partial}{\partial t}\right) W \\
&+ \left[\Box_2 \left(b\Delta\Lambda_2 + \Box_1\Lambda_3 - \beta\Delta\Box_4 \frac{\partial}{\partial t}\right) + \mu^2\chi^2\Delta\Lambda_3\right] \Psi \\
&- \left[(m\Box_1 - b\beta\Delta)\,\Box_3 + b\Delta\Box_4 - m\left(b^2\Delta + \Box_1\Box_3\right)\right] \Box_2 \Theta = \Box \Psi = -\frac{1}{I}\, l
\end{aligned}$$

and hence, the equation $(2.6)_3$ is verified.

On the other hand, from (2.9)–(2.12) it follows that

$$\begin{aligned}
&\frac{k}{T_0}\Delta T - \beta\dot{v}_{\gamma,\gamma} - m\dot{\psi} - \frac{k}{T_0}\chi^2 T - a\dot{T} = \Lambda_1 T - \beta\dot{v}_{\gamma,\gamma} - m\dot{\psi} \\
&\quad = \beta\left[(\lambda + \mu)\,\Delta - \Box_1 + \left(\Box_2 - \mu\chi^2\right)\right] \Box_2 \Lambda_3 \frac{\partial^2 V_\gamma}{\partial t\,\partial x_\gamma} +
\end{aligned}$$

$$+\mu\chi^2\left(\Box_4\Lambda_1-\beta\Lambda_3-m\Lambda_2\right)\Delta\frac{\partial}{\partial t}W+\beta\left(-b\Lambda_1+\Lambda_2+m\beta\frac{\partial}{\partial t}\right)\Delta\Box_2\frac{\partial}{\partial t}\Psi$$

$$+\left[\Box_2\left(b\Delta\Lambda_2+\Box_1\Lambda_3-\beta\Delta\Box_4\frac{\partial}{\partial t}\right)+\mu^2\chi^2\Delta\Lambda_3\right]\Theta=\Box\Theta=-\frac{1}{IT_0}\,S.$$

Consequently, the functions v_γ, w, ψ and T verify equation $(2.6)_4$.

In an analogous manner, we can show that the equation $(2.6)_1$ is also satisfied. This completes the proof. □

3. Time–harmonic oscillations. Oscillation problems for classical elastic and thermoelastic plates of Mindlin–type were considered in Schiavone and Constanda (1993) and Schiavone and Tait (1995). In this section, we study the steady time–harmonic oscillations of porous thermoelastic plates and establish some representations of the solution for this case. Let us assume that $f_\alpha\,, f\,, l\,, S$ are separable with respect to position and time and that they are periodic in time, i.e.

(3.1)

$$f_\alpha=\operatorname{Re}\left[f_\alpha^0\left(x_1,x_2\right)\exp\left(-i\omega t\right)\right],\quad f=\operatorname{Re}\left[f^0\left(x_1,x_2\right)\exp\left(-i\omega t\right)\right],$$
$$l=\operatorname{Re}\left[l^0\left(x_1,x_2\right)\exp\left(-i\omega t\right)\right],\quad S=\operatorname{Re}\left[S^0\left(x_1,x_2\right)\exp\left(-i\omega t\right)\right],$$

where f_α^0, f^0, l^0 and S^0 are complex–valued prescribed functions and $i=\sqrt{-1}$.

We study the solutions of the bending equations (2.6) having the form

$$\begin{aligned}v_\alpha\left(x_1,x_2,t\right)&=\operatorname{Re}\left[v_\alpha^*\left(x_1,x_2\right)\exp\left(-i\omega t\right)\right],\\ w\left(x_1,x_2,t\right)&=\operatorname{Re}\left[w^*\left(x_1,x_2\right)\exp\left(-i\omega t\right)\right],\\ \psi\left(x_1,x_2,t\right)&=\operatorname{Re}\left[\psi^*\left(x_1,x_2\right)\exp\left(-i\omega t\right)\right],\\ T\left(x_1,x_2,t\right)&=\operatorname{Re}\left[T^*\left(x_1,x_2\right)\exp\left(-i\omega t\right)\right],\end{aligned}\tag{3.2}$$

where v_α^*, w^*, ψ^* and T^* are unknown complex–valued functions. The constant $\omega\in\mathbb{R}$ is the frequency of the oscillation.

In what follows, we employ the notations

$$\tau^2=\frac{\omega^2}{c_2^2}-\chi^2,\quad \zeta_0=\alpha\,\frac{\omega^2}{c_3^2}-\zeta,\quad k_0=\frac{k}{i\omega T_0}\,.\tag{3.3}$$

In view of (3.2) and (3.3), from the system of equations (2.6) we obtain

$$\mu\Delta v_\alpha^*+\left(\lambda+\mu\right)v_{\beta,\beta\alpha}^*+\mu\tau^2v_\alpha^*-\mu\chi^2w_{;\alpha}^*+b\psi_{;\alpha}^*-\beta T_{;\alpha}^*+\frac{1}{I}f_\alpha^0=0,$$

$$
\begin{aligned}
&\mu\varDelta w^* + \rho_0\omega^2 w^* + \mu v^*_{\gamma,\gamma} + f^0 = 0,\\
&\alpha\varDelta\psi^* + \zeta_0\psi^* - b v^*_{\gamma,\gamma} + mT^* + \frac{1}{I}\, l^0 = 0,\\
&k_0\varDelta T^* + \left(a - k_0\chi^2\right) T^* + \beta v^*_{\gamma,\gamma} + m\psi^* + \frac{1}{i\omega T_0 I}\, S^0 = 0.
\end{aligned}
\tag{3.4}
$$

We introduce the differential operators D_n and $\varOmega_r$ $(n = 1,2,3,4,\ r = 1,2,3)$ defined by

(3.5)

$$
\begin{aligned}
&D_1 = (\lambda + 2\mu)\,\varDelta + \mu\tau^2, \quad D_2 = \mu\varDelta + \rho_0\omega^2,\\
&D_3 = \alpha\varDelta + \zeta_0, \quad D_4 = \beta D_3 + bm,\\
&\varOmega_1 = i\omega\left(k_0\varDelta + a - k_0\chi^2\right), \quad \varOmega_2 = b\varOmega_1 + i\omega m\beta, \quad \varOmega_3 = D_3\varOmega_1 - i\omega m^2,
\end{aligned}
$$

and we denote by D the operator

$$
D = D_2\left(b\varDelta\varOmega_2 + D_1\varOmega_3 + i\omega\beta\varDelta D_4\right) + \mu^2\chi^2\varDelta\varOmega_3\,. \tag{3.6}
$$

As a consequence of Theorem 1, we obtain the following Galerkin representation for the solution of the system (3.4).

Theorem 2. *Let*

(3.7)

$$
\begin{aligned}
v^*_\gamma = {}& D\, V^0_\gamma - \frac{\partial}{\partial x_\gamma}\Big\{ \left[D_2\left(b\varOmega_2 + (\lambda+\mu)\,\varOmega_3 + i\omega\beta D_4\right) + \mu^2\chi^2\varOmega_3\right]\frac{\partial V^0_\beta}{\partial x_\beta}\\
& -\mu\chi^2\varOmega_3 W^0 + D_2\left(\varOmega_2\varPsi^0 - D_4\varTheta^0\right)\Big\},\\
w^* = {}& -\mu^2\left(\varDelta + \tau^2\right)\varOmega_3\frac{\partial V^0_\gamma}{\partial x_\gamma} + \left(b\varDelta\varOmega_2 + D_1\varOmega_3 + i\omega\beta\varDelta D_4\right) W^0\\
& +\mu\varDelta\left(\varOmega_2\varPsi^0 - D_4\varTheta^0\right),\\
\psi^* = {}& \mu\varOmega_2\left[\left(\varDelta + \tau^2\right) D_2\frac{\partial V^0_\gamma}{\partial x_\gamma} + \chi^2\varDelta W^0\right] + \left[\left(D_1 D_2 + \mu^2\chi^2\varDelta\right)\varOmega_1 + \right.\\
& \left. + i\omega\beta^2\varDelta D_2\right]\varPsi^0 - \left[(mD_1 - b\beta\varDelta)\, D_2 + m\mu^2\chi^2\varDelta\right]\varTheta^0,\\
T^* = {}& -i\omega\mu D_4\left[\left(\varDelta + \tau^2\right) D_2\frac{\partial V^0_\gamma}{\partial x_\gamma} + \chi^2\varDelta W^0\right] - i\omega[(mD_1 - b\beta\varDelta)\, D_2\\
& + m\mu^2\chi^2\varDelta]\,\varPsi^0 + \left[\left(D_1 D_2 + \mu^2\chi^2\varDelta\right) D_3 + b^2\varDelta D_2\right]\varTheta^0,
\end{aligned}
$$

where $V_\gamma^0 \in C^{10}(\Sigma)$ *and* $W^0, \Psi^0, \Theta^0 \in C^8(\Sigma)$ *are complex–valued functions which satisfy the equations*

(3.8)

$$\left(\Delta+\tau^2\right) D\,V_\gamma^0=-\frac{1}{\mu I}f_\gamma^0,\ D\,W^0=-f^0,\ D\,\Psi^0=-\frac{1}{I}\,l^0,\ D\Theta^0=-\frac{1}{IT_0}\,S^0.$$

Then, $(v_1^*, v_2^*, w^*, \psi^*, T^*)$ *represents a solution of the system of equations* (3.4) *.*

For the remainder of this paper, we consider the case of null external body loads and heat supply. Thus, we assume that

$$f_\alpha^0=f^0=l^0=S^0=0. \tag{3.9}$$

In what follows, we establish a decomposition which is valid for any solution of the equations (3.4) in the absence of external body loads and heat supply.

Theorem 3. *Let* $U=(v_1^*, v_2^*, w^*, \psi^*, T^*)$ *be a solution of the homogeneous system of equations* (3.4)*. Then*

(i) *there exist the functions* $v^{(1)}$ *and* $v^{(2)}$ *such that*

$$v_\alpha^*=v^{(1)},_\alpha+\epsilon_{\alpha\beta}\,v^{(2)},_\beta\,,\quad D\,v^{(1)}=0,\quad \left(\Delta+\tau^2\right)v^{(2)}=0, \tag{3.10}$$

where $\epsilon_{\alpha\beta}$ *is the two–dimensional alternator;*

(ii) *the solution* U *has the following decomposition*

$$U=U^{(1)}+U^{(2)},\quad D\,U^{(1)}=\hat{0},\quad \left(\Delta+\tau^2\right)U^{(2)}=\hat{0}, \tag{3.11}$$

where $U^{(1)}=\left(v^{(1)},_1\,,v^{(1)},_2\,,w^*,\psi^*,T^*\right),U^{(2)}=\left(v^{(2)},_2\,,-v^{(2)},_1\,,0,0,0\right)$ *and* $\hat{0}=(0,0,0,0,0)$.

Proof. If we substitute the formula

$$\Delta v_\alpha^*=v_\beta^*,_{\beta\alpha}-\epsilon_{\alpha\beta}\,\epsilon_{\gamma\delta}\,v_\delta^*,_{\gamma\beta}$$

into $(3.4)_1$, then we obtain

$$v_\alpha^*=\frac{1}{\mu\tau^2}\left[-(\lambda+2\mu)\,v_\beta^*,_\beta+\mu\chi^2w^*-b\psi^*+\beta T^*\right],_\alpha+\frac{1}{\tau^2}\,\epsilon_{\alpha\beta}\,\epsilon_{\gamma\delta}\,v_\delta^*,_{\gamma\beta}\ . \tag{3.12}$$

We define the functions $v^{(1)}$ and $v^{(2)}$ by

$$
\begin{aligned}
v^{(1)} &= \frac{1}{\mu\tau^2}\big[-(\lambda+2\mu)\, v^*_{\beta},_{\beta} + \mu\chi^2 w^* - b\psi^* + \beta T^*\big], \\
v^{(2)} &= \frac{1}{\tau^2}\, \epsilon_{\gamma\delta}\, v^*_{\delta},_{\gamma}\ .
\end{aligned} \tag{3.13}
$$

In view of (3.12) and (3.13), we see that $(3.10)_1$ holds.

Let us differentiate the equation $(3.4)_1$ with respect to x_γ and multiply it by $\epsilon_{\gamma\alpha}$. Hence, we derive that

$$\mu(\varDelta + \tau^2)(\epsilon_{\gamma\alpha}\, v^*_{\alpha},_{\gamma}) = 0$$

and taking into account $(3.13)_2$, we prove the relation $(3.10)_3$.

On the other hand, if we differentiate $(3.4)_1$ with respect to x_α and use the notations $u^* = v^*_{\beta},_{\beta}$ and (3.5), then the system of equations (3.4) becomes

$$
\begin{aligned}
& D_1 u^* - \mu\chi^2 \varDelta w^* + b\varDelta\psi^* - \beta\varDelta T^* = 0\,, \\
& \mu\, u^* + D_2 w^* = 0\,, \\
& -b\, u^* + D_3\psi^* + m\, T^* = 0\,, \\
& i\omega\beta\, u^* + i\omega m\, \psi^* + \varOmega_1 T^* = 0\,.
\end{aligned} \tag{3.14}
$$

We apply the associated matrices method for the system of partial differential equations (3.14) and we deduce that the functions u^* , w^* , ψ^* and T^* satisfy the relations

$$Du^* = 0, \quad Dw^* = 0, \quad D\psi^* = 0, \quad D\,T^* = 0, \tag{3.15}$$

where D is the differential operator defined in (3.6).

From equations (3.13) and (3.15) it follows that $Dv^{(1)} = 0$, i.e. $(3.10)_2$ holds.

By virtue of $(3.10)_{2,3}$, we have

$$D(v^{(1)},_{\alpha}) = 0\,, \quad (\varDelta + \tau^2)(\epsilon_{\alpha\beta}\, v^{(2)},_{\beta}) = 0\,,$$

and, consequently, the assertion (ii) of the theorem is also true. The proof is complete. □

For later reference, let us denote by $(-\lambda_k^2)$, $k = 1,2,3,4$, the roots of the polynomial P defined by the determinant

(3.16)

$$P(Z) = \begin{vmatrix} (\lambda+2\mu)\,Z + \mu\tau^2 & -\mu\chi^2 Z & bZ & -\beta Z \\ \mu & \mu Z + \rho_0\omega^2 & 0 & 0 \\ -b & 0 & \alpha Z + \zeta_0 & m \\ \beta & 0 & m & k_0 Z + a - k_0\chi^2 \end{vmatrix} .$$

The sign of the complex numbers λ_k is chosen such that $\operatorname{Im}\lambda_k \geq 0$, $k = 1,2,3,4$.

In view of the definition (3.6), we observe that the differential operator D can be written in the form

$$D = C_0(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)(\Delta + \lambda_4^2),$$

where C_0 is the constant $C_0 = i\omega\alpha k_0\mu(\lambda+2\mu)$.

From Theorem 3, we obtain the following result.

Corollary 1. *Assume that the numbers* λ_k^2 $(k = 1,2,3,4)$ *are distinct. Then, any solution* $U = (v_1^*, v_2^*, w^*, \psi^*, T^*)$ *of the system of equations* (3.4), *in the case of null external body loads and heat supply, can be written in the form*

$$U = U^{(2)} + \sum_{j=1}^{4} W^{(j)}, \tag{3.17}$$

where $U^{(2)}$ *and* $W^{(j)}$ $(j = 1,2,3,4)$ *are vector–valued functions of dimension five which satisfy the equations*

$$(\Delta + \tau^2)\,U^{(2)} = \hat{0}, \qquad (\Delta + \lambda_j^2)\,W^{(j)} = \hat{0} \quad (j \text{ not summed}). \tag{3.18}$$

Proof. Let us introduce the differential operators Q_j, $j = 1,2,3,4$, defined by

$$Q_j = \prod_{k\neq j,\ k=1}^{4} \frac{\Delta + \lambda_k^2}{\lambda_k^2 - \lambda_j^2} . \tag{3.19}$$

It follows that

$$\sum_{j=1}^{4} Q_j = \mathbf{1}, \tag{3.20}$$

where **1** represents the identical operator.

In view of $(3.11)_2$ and (3.19), we deduce that

$$(\varDelta + \lambda_j^2)\, Q_j\, U^{(1)} = \hat{0} \quad (j \text{ not summed}). \tag{3.21}$$

If we denote by $W^{(j)}$ the functions

$$W^{(j)} = Q_j\, U^{(1)}, \quad j = 1, 2, 3, 4,$$

then from (3.20) and (3.21) we obtain that

$$U^{(1)} = \sum_{j=1}^{4} W^{(j)}, \qquad (\varDelta + \lambda_j^2)\, W^{(j)} = \hat{0} \quad (j \text{ not summed}). \tag{3.22}$$

By virtue of Theorem 3 and of relations (3.22), we conclude that the equalities (3.18) and (3.19) hold. This completes the proof. □

Remark. The representation established in Theorem 3 is analogous with the decomposition of the solution given by Mindlin (1951) for the bending of elastic plates with transverse shear deformation.

Using the above results, one can follow the method described in Schiavone and Constanda (1993) and Schiavone and Tait (1995) to prove a uniqueness theorem for exterior boundary value problems associated with the steady time–harmonic oscillations of porous thermoelastic plates.

4. Plane flexural waves in an infinite plate. We consider flexural motions of an infinite porous thermoelastic plate in the absence of external body loads and heat supply.

Thus, in what follows we determine the solutions of the bending equations (2.6) of the form

$$\begin{aligned} v_\alpha &= \mathrm{Re}\left[v_\alpha^*\left(x_1\right)\exp\left(-i\omega t\right)\right], \quad w = \mathrm{Re}\left[w^*\left(x_1\right)\exp\left(-i\omega t\right)\right], \\ \psi &= \mathrm{Re}\left[\psi^*\left(x_1\right)\exp\left(-i\omega t\right)\right], \quad T = \mathrm{Re}\left[T^*\left(x_1\right)\exp\left(-i\omega t\right)\right]. \end{aligned} \tag{4.1}$$

From Section 3, we see that the unknown functions v_α^*, w^*, ψ^* and T^* must satisfy the homogeneous system of equations (3.4). Since these functions depend only on x_1 , in our case the system (3.4) uncouples. Indeed, we obtain that v_2^* verifies the equation

$$\left(\frac{d^2}{dx_1^2} + \tau^2\right) v_2^* = 0, \tag{4.2}$$

whereas the functions v_1^* , w^* , ψ^* and T^* satisfy the system of equations

$$
\begin{aligned}
&\left[(\lambda+2\mu)\,\frac{d^2}{dx_1^2}+\mu\tau^2\right]v_1^*-\mu\chi^2\frac{dw^*}{dx_1}+b\,\frac{d\psi^*}{dx_1}-\beta\,\frac{dT^*}{dx_1}=0,\\
&\mu\,\frac{dv_1^*}{dx_1}+\left(\mu\frac{d^2}{dx_1^2}+\rho_0\omega^2\right)w^*=0,\\
&-b\,\frac{dv_1^*}{dx_1}+\left(\alpha\,\frac{d^2}{dx_1^2}+\zeta_0\right)\psi^*+mT^*=0,\\
&\beta\,\frac{dv_1^*}{dx_1}+m\psi^*+\left(k_0\,\frac{d^2}{dx_1^2}+a-k_0\chi^2\right)T^*=0.
\end{aligned}
\tag{4.3}
$$

Let us consider first the equation (4.2) and determine the function v_2 .

If the frequency ω is such that $\omega^2\geq\dfrac{\mu h_0}{\rho_0 I}$, then

$$
\tau=\left(\frac{\omega^2}{c_2^2}-\chi^2\right)^{1/2}\in\mathbb{R}.
$$

Hence, from $(4.1)_1$ and (4.2) we get

$$
v_2=\mathrm{Re}\left\{K_1\exp\left[-i\omega\left(t-\frac{\tau}{\omega}x_1\right)\right]+K_2\exp\left[-i\omega\left(t+\frac{\tau}{\omega}x_1\right)\right]\right\},
\tag{4.4}
$$

where $K_1,K_2\in\mathbb{C}$ are arbitrary constants. The two terms in relation (4.4) represent (straight–crested) harmonic flexural waves travelling along the Ox_1 axis in the positive and negative directions (see Chadwick, 1960) with the phase velocity

$$
\frac{\omega}{\tau}=\omega\left(\frac{\omega^2}{c_2^2}-\chi^2\right)^{-1/2}.
$$

We remark that these waves are subject to dispersion, in the sense that the phase velocity is a function of the frequency ω.

If ω satisfies $\omega^2<\dfrac{\mu h_0}{\rho_0 I}$, then (4.2) yields

(4.5)

$$
v_2=\mathrm{Re}\left\{\left[K_3\exp\left(x_1\sqrt{\chi^2-\frac{\omega^2}{c_2^2}}\right)+K_4\exp\left(-x_1\sqrt{\chi^2-\frac{\omega^2}{c_2^2}}\right)\right]\exp\left(-i\omega t\right)\right\},
$$

where K_3 and K_4 are arbitrary constants. In this case, the waves (4.5) are attenuated and the attenuation coefficient is equal to $\left(\chi^2 - \dfrac{\omega^2}{c_2^2}\right)^{1/2}$.

We notice that the waves described in (4.4), (4.5) are not influenced by the temperature and porosity fields, but depend only on the elastic properties of the material.

Let us turn our attention to the determination of the functions v_1^*, w^*, ψ^* and T^* which satisfy the system of equations (4.3).

We deduce that the functions $\dfrac{dv_1^*}{dx_1}$, w^*, ψ^* and T^* verify the equation

$$\left(\frac{d^2}{dx_1^2} + \lambda_1^2\right)\left(\frac{d^2}{dx_1^2} + \lambda_2^2\right)\left(\frac{d^2}{dx_1^2} + \lambda_3^2\right)\left(\frac{d^2}{dx_1^2} + \lambda_4^2\right) y = 0. \tag{4.6}$$

In view of $(4.3)_1$, we have

$$v_1^* = -\frac{1}{\mu\tau^2}\,\frac{d}{dx_1}\left[(\lambda + 2\mu)\,\frac{dv_1^*}{dx_1} - \mu\chi^2 w^* + b\psi^* - \beta T^*\right]$$

and, hence, the function v_1^* also satisfies the equation (4.6).

Assume that λ_k^2 $(k = 1, 2, 3, 4)$ are distinct numbers. Then, the equation (4.6) admits the following fundamental system of solutions

$$\{\exp(\lambda_k x_1), \exp(-\lambda_k x_1) \mid k = 1, 2, 3, 4\}.$$

Since v_1^*, w^*, ψ^* and T^* are solutions of the equations (4.6), they can be written in the form

$$\begin{aligned}
v_1^* &= \sum_{k=1}^{4}\left[A_k^{(1)} \exp(i\lambda_k x_1) + A_k^{(2)} \exp(-i\lambda_k x_1)\right],\\
w^* &= \sum_{k=1}^{4}\left[B_k^{(1)} \exp(i\lambda_k x_1) + B_k^{(2)} \exp(-i\lambda_k x_1)\right],\\
\psi^* &= \sum_{k=1}^{4}\left[C_k^{(1)} \exp(i\lambda_k x_1) + C_k^{(2)} \exp(-i\lambda_k x_1)\right],\\
T^* &= \sum_{k=1}^{4}\left[D_k^{(1)} \exp(i\lambda_k x_1) + D_k^{(2)} \exp(-i\lambda_k x_1)\right],
\end{aligned} \tag{4.7}$$

where $A_k^{(\alpha)}$, $B_k^{(\alpha)}$, $C_k^{(\alpha)}$ and $D_k^{(\alpha)}$ are some arbitrary constants ($k = 1, 2, 3, 4$, $\alpha = 1, 2$). If we substitute the expressions (4.7) into (4.3), we obtain that the quadruples $(X_1, X_2, X_3, X_4) = \left(i\lambda_k A_k^{(\alpha)}, B_k^{(\alpha)}, C_k^{(\alpha)}, D_k^{(\alpha)}\right)$ are solutions of the following system of algebraic equations, for every $k = 1, 2, 3, 4$ (not summed) and $\alpha = 1, 2$

(4.8)

$$\begin{aligned}
&\left[(\lambda{+}2\mu)\left(-\lambda_k^2\right) + \mu\tau^2\right] X_1 - \mu\chi^2\left(-\lambda_k^2\right) X_2 + b\left(-\lambda_k^2\right) X_3 - \beta\left(-\lambda_k^2\right) X_4{=}0,\\
&\mu X_1 + \left[\mu\left(-\lambda_k^2\right) + \rho_0\omega^2\right] X_2 = 0,\\
&-bX_1 + \left[\alpha\left(-\lambda_k^2\right) + \zeta_0\right] X_3 + mX_4 = 0,\\
&\beta X_1 + mX_3 + \left[k_0\left(-\lambda_k^2\right) + a - k_0\chi^2\right] X_4 = 0.
\end{aligned}$$

Since $\left(-\lambda_k^2\right)$ are the roots of the polynomial P defined in (3.16), it follows that the determinant of the system (4.8) is zero, for every $k = 1, 2, 3, 4$. Then, from $(4.8)_{2,3,4}$ we derive

$$\begin{aligned}
&B_k^{(\alpha)} = \frac{i\lambda_k}{\lambda_k^2 - \left(\omega^2/c_2^2\right)}\, A_k^{(\alpha)}, \qquad C_k^{(\alpha)} = \frac{a_k}{d_k}\, A_k^{(\alpha)},\\
&D_k^{(\alpha)} = \frac{b_k}{d_k}\, A_k^{(\alpha)}, \quad k = 1, 2, 3, 4 \text{ (not summed)},\ \alpha = 1, 2\,,
\end{aligned} \tag{4.9}$$

where a_k, b_k and d_k denote the constant expressions

(4.10)

$$\begin{aligned}
&a_k = i\lambda_k\left[ab + m\beta - bk_0\left(\lambda_k^2 + \chi^2\right)\right], \quad b_k = -i\lambda_k\left[bm + \beta\left(\zeta_0 - \alpha\lambda_k^2\right)\right],\\
&d_k = \left(\zeta_0 - \alpha\lambda_k^2\right)\left[a - k_0\left(\lambda_k^2 + \chi^2\right)\right] - m^2, \quad k = 1, 2, 3, 4 \text{ (not summed)}.
\end{aligned}$$

Let s_k and r_k ($k = 1, 2, 3, 4$) be the real numbers defined by the relations

$$\lambda_k = \frac{\omega}{s_k} + i\, r_k\,. \tag{4.11}$$

By virtue of (4.7), (4.9) and (4.11), we find the following solution

$$\begin{aligned}
v_1 = \sum_{k=1}^{4} \operatorname{Re}\Bigg\{ & A_k^{(1)} \exp\left(-r_k x_1\right) \exp\left[-i\omega\left(t - \frac{x_1}{s_k}\right)\right]\\
& + A_k^{(2)} \exp\left(r_k x_1\right) \exp\left[-i\omega\left(t + \frac{x_1}{s_k}\right)\right]\Bigg\},
\end{aligned}$$

(4.12)

$$w = \sum_{k=1}^{4} \mathrm{Re}\left\{ \frac{i\lambda_k}{\lambda_k^2 - (\omega^2/c_2^2)} \left[A_k^{(1)} \exp(-r_k x_1) \exp\left[-i\omega\left(t - \frac{x_1}{s_k}\right)\right] \right.\right.$$
$$\left.\left. + A_k^{(2)} \exp(r_k x_1) \exp\left[-i\omega\left(t + \frac{x_1}{s_k}\right)\right]\right]\right\},$$
$$\psi = \sum_{k=1}^{4} \mathrm{Re}\left\{ \frac{a_k}{d_k} \left[A_k^{(1)} \exp(-r_k x_1) \exp\left[-i\omega\left(t - \frac{x_1}{s_k}\right)\right] \right.\right.$$
$$\left.\left. + A_k^{(2)} \exp(r_k x_1) \exp\left[-i\omega\left(t + \frac{x_1}{s_k}\right)\right]\right]\right\},$$
$$T = \sum_{k=1}^{4} \mathrm{Re}\left\{ \frac{b_k}{d_k} \left[A_k^{(1)} \exp(-r_k x_1) \exp\left[-i\omega\left(t - \frac{x_1}{s_k}\right)\right] \right.\right.$$
$$\left.\left. + A_k^{(2)} \exp(r_k x_1) \exp\left[-i\omega\left(t + \frac{x_1}{s_k}\right)\right]\right]\right\},$$

where $A_k^{(\alpha)}$ $(k = 1, 2, 3, 4\ ,\ \alpha = 1, 2)$ are arbitrary constants.

The relations (4.12) describe four types of flexural waves which propagate in the positive and negative directions of the Ox_1 axis with the phase velocities

$$s_k = \frac{\omega}{\mathrm{Re}\,\lambda_k}, \quad k = 1, 2, 3, 4.$$

These waves are subject to dispersion, since the phase velocities depend on the frequency ω. We also notice that the waves are attenuated and the attenuation coefficients are given by

$$r_k = \mathrm{Im}\,\lambda_k, \quad k = 1, 2, 3, 4.$$

From relations (4.12) we observe that the propagation of these waves depends on the elastic and thermal properties of the material, as well as on the porosity of the plate.

In conclusion, the solutions of the bending equations of the form (4.1) are given by (4.4), (4.5), (4.12) and represent the harmonic flexural waves travelling in direction of the Ox_1 axis.

Remark. The propagation of flexural waves in classical elastic Mindlin–type plates is discussed in CONSTANDA and SCHIAVONE (1994), where a comparison with results of the classical plate theory is also presented.

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