ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII “AL.I. CUZA” IAŞI
Tomul LII, s.I, Matematică, 2006, f.1

A NOTE ON PRIMITIVE Γ-RING

BY

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Abstract. In this paper we have introduced the notion of associated module of a ΓS-module. We have studied some properties of associated module of a ΓS-module and with the help of this notion we have studied primitive Γ-rings.

Mathematics Subject Classification 2000: 16D30, 16P99.

Key words: Γ-ring, left operator ring of a Γ-ring, ΓS-module, associated module of a ΓS-module, irreducible, faithful ΓS-module, primitive Γ-ring.

1. Introduction. The notion of primitive Γ-ring was introduced by Luh [6]. He defined primitive Γ-ring as follows: A Γ-ring S is right primitive if its right operator ring is right primitive and $S\Gamma x = 0$ implies that $x = 0$. If S contains left unity and right unity then according to J.Luh a Γ-ring S is right (resp.left) primitive if and only if its right (resp. left) operator ring is right (resp. left) primitive. Later Ravisankar and Shukla [7] defined right primitive Γ-ring as a Γ-ring which admits a faithful irreducible right $S\Gamma$-module. He showed that a Γ-ring S is right primitive if and only if it is right primitive in the sense of Luh. Dually we can get that a Γ-ring S is left primitive if and only if its left operator ring L is left primitive; but the problem of establishment of the relation between a right primitive Γ-ring S and the right primitive left operator ring L of S was till open. In this paper we have proved that a Γ-ring S with left and right unities is right primitive if and only its left operator ring L is right primitive. Thus the relations among the right primitive Γ-ring S, right primitive left operator ring L and right primitive right operator ring R have been completely determined. In the first section of this paper we have introduced the notion of associated module of a ΓS-module and studied it. We have obtained that the lattice of all ΓS-submodules of a ΓS-module M and the lattice of all submodules of its associated L-module $M^{\#}$ are isomorphic. In the next section using the notion of associated module of a ΓS-module M we have proved the following result: A Γ-ring S with right and left unities is right primitive if and only if L is right primitive if and only if R is right primitive.

2. Some basic definitions and examples

Definition 2.1 [1]. *Let S and Γ be two additive abelian groups . S is called a Γ-ring if there exists a mapping $f : S \times \Gamma \times S \longrightarrow S, f(a, \alpha, b)$ is denoted by $a\alpha b$, $a, b \in S$, $\alpha \in \Gamma$, satisfying the following conditions for all $a, b, c \in S$ and for all $\alpha, \beta, \gamma \in \Gamma$; $(a + b)\alpha c = a\alpha c + b\alpha c, a(\alpha + \beta)b = a\alpha b + a\beta b, a\alpha(b + c) = a\alpha b + a\alpha c$, and $a\alpha(b\beta c) = (a\alpha b)\beta c$.*

Definition 2.2 [2]. *A subset A of a Γ-ring S is called a left (resp. right) ideal of S if A is an additive subgroup of S and $s\alpha a \in A$ (resp. $a\alpha s \in A$) for all $s \in S, \alpha \in \Gamma, a \in A$. If A is a left and a right ideal of S, then A is called a two sided ideal of S or simply an ideal of S.*

Definition 2.3 [8]. *Let S be a Γ ring and F be the free abelian group generated by $\Gamma \times S$. Then $A = \{\sum_i n_i(\gamma_i, x_i) \in F : a \in S \Rightarrow \sum_i n_i a\gamma_i x_i = 0\}$ is a subgroup of F. Let $R = F/A$ be the factor group of F by A. Let us denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. It can be verified that $[\alpha, x] + [\beta, x] = [\alpha + \beta, x]$ and $[\alpha, x] + [\alpha, y] = [\alpha, x + y]$, for all $\alpha, \beta \in \Gamma$ and $x, y \in S$. We define a multiplication in R by $\sum_i[\alpha_i, x_i] \sum_j[\beta_j, y_j] = \sum_{i,j}[\alpha_i, x_j\beta_j y_j]$. Then R forms a ring. This ring R is called the right operator ring of the Γ ring S. Similarly we can construct left operator ring L of S. For subsets $N \subseteq S$, $\phi \subseteq \Gamma$, we denote by $[\phi, N]$ the set of all finite sums $\sum_i[\gamma_i, x_i]$ in R where $\gamma_i \in \phi$ and $x_i \in N$ and we denote by $[(\Phi, N)]$ the set of all elements $[\phi, x]$ in R, where $\phi \in \Phi$ and $x \in N$. Thus in particular, $R = [\Gamma, S]$ and $L = [S, \Gamma]$. If there exists an element $\sum_i[\delta_i, e_i] \in R$ such that $\sum_i x\delta_i e_i = x$ for every element x of S then it is called the right unity of S. It can be verified that $\sum_i[\delta_i, e_i]$ is the unity of R. Similarly we can define the left unity $\sum_j[f_j, \gamma_j]$ which is the unity of the left operator ring L.*

Definition 2.4 [2]. *For $P \subseteq R$, we define $P^* = \{a \in S : [\Gamma, a] \subseteq P\}$*

and for any $Q \subseteq S$, $Q^{*/} = \{\sum_i [\alpha_i, x_i] \in R : S(\sum_i [\alpha_i, x_i]) \subseteq Q\}$, *where* $S(\sum_i [\alpha_i, x_i]) = \{\sum_i s\alpha_i x_i : \text{for all } s \in S\}$. *Similarly for* $N \subseteq L$ *and* $T \subseteq S$ *we can define* N^+ *and* $T^{+/}$ *respectively.*

Remark 2.5 [2]. If P is a right (resp. left) ideal of R then P^* is a right (resp. left) ideal of S. If Q is a right (resp. left) ideal of S then $Q^{*/}$ is a right (resp. left) ideal of R. In particular, $\{0_R\}^* = \{0_S\}$ and $\{0_S\}^{*/} = \{0_R\}$.

Remark 2.6 [2]. If 0_S is the zero element of S, then $[0_S, \alpha]$ and $[\alpha, 0_S]$ are the zeros in L and R respectively for any $\alpha \in \Gamma$.
Henceforth S will stand for a Γ-ring with left and right unities, unless otherwise stated.

3. Properties of ΓS modules via its associated modules

Definition 3.1 [7]. *Let* S *be a* Γ*-ring. An additive abelian group* M *is called a right* ΓS *module (or just a* ΓS *module), if there exists a mapping* $\Phi : M \times \Gamma \times S \to M$ *described by* $(m, \alpha, x) \mapsto m\alpha x$ *such that* 1) $(m+n)\alpha x = m\alpha x + n\alpha x$, 2) $m\alpha(x+y) = m\alpha x + m\alpha y$, 3) $m\beta(x\alpha y) = (m\beta x)\alpha y$, 4) $0_M \alpha x = 0_M = m\alpha 0_S$, *for all* $x, y \in S$, $\alpha, \beta \in \Gamma$ *and* $m, n \in M$ (0_M *and* 0_S *are the zeros of the groups* M *and* S *respectively).*

A ΓS-module M is called unitary if S has a right unity $\sum_{j=1}^{m} [\gamma_j, f_j] \in R$ such that $\sum_{j=1}^{m} a\gamma_j f_j = a$, for all $a \in M$. Throughout this paper M stands for a unitary ΓS-module.

Definition 3.2. *Let* M *be a unitary right* ΓS*-module and* E *be the free additive commutative group generated by* $M \times \Gamma$. *Then* $B = \{\sum_i n_i(m_i, \alpha_i) \in E : a \in S \Rightarrow \sum_i n_i m_i \alpha_i a = 0_M\}$, *is a subgroup of* E. *Let* E/B *be the factor group of* E *by* B. *Let us denote the coset* $\sum_{i=1}^{n} (x_i, \gamma_i) + B$ *by* $\sum_{i=1}^{n} < x_i, \gamma_i >$. *Let* N *and* Δ *be non-empty subsets of* M *and* Γ *respectively, then we denote by* $< N, \Delta >$ *the set* $\{\sum_{i=1}^{n} < n_i, \alpha_i >$: n *is a positive integer,* $n_i \in N_i, \alpha_i \in \Delta$ *for* $i = 1, 2, ..., n\}$. *For any* $m \in M$ *and*

$\alpha \in \Gamma$ we write $< m, \alpha >$ instead of $< \{m\}, \{\alpha\} >$. This is to observe that $< 0_M, \alpha > = < 0_M, \beta >$ for any $\alpha, \beta \in \Gamma$. A multiplication from right of the elements of E/B by the elements of left operator ring L of S is defined as follows: for any $\sum_{i=1}^{m} < m_i, \alpha_i > \in E/B$ and $\sum_{j=1}^{n}[x_j, \beta_j] \in L$, $\sum_{i=1}^{m} < m_i, \alpha_i > \sum_{j=1}^{n}[x_j, \beta_j] = \sum_{i,j} < m_i\alpha_i x_j, \beta_j >$. This multiplication is well defined. In fact if $\sum_{i=1}^{m} < m_i, \alpha_i > = \sum_{j=1}^{n} < n_j, \beta_j >$ in E/B and $\sum_{k=1}^{p}[x_k, \gamma_k] = \sum_{l=1}^{q}[y_i, \delta_l]$ in L, then $\sum_{i=1}^{m} m_i\alpha_i x = \sum_{j=1}^{n} n_j\beta_j x$ for all $x \in S$ and $\sum_{k=1}^{p} x_k\gamma_k x = \sum_{l=1}^{q} y_l\delta_l x$ for all $x \in S$. Now $\sum_{i,k}(m_i\alpha_i x_k)\gamma_k x = \sum_{i,k} m_i\alpha_i(x_k\gamma_k x) = \sum_{i=1}^{m} m_i\alpha_i(\sum_{k=1}^{p} x_k\gamma_k x) = \sum_{i=1}^{m} m_i\alpha_i(\sum_{l=1}^{q} y_l\delta_l x) = \sum_{j=1}^{n} n_j\beta_j(\sum_{l=1}^{q} y_l\delta_l x) = \sum_{j,l}(n_j\beta_j y_l)\delta_l x$ for all $x \in S$. So $\sum_{i=1}^{m} < m_i, \alpha_i > \sum_{k=1}^{p}[x_k, \gamma_k] = \sum_{j=1}^{n} < n_j, \beta_j > \sum_{l=1}^{q}[y_l, \delta_l]$. With respect to this multiplication, the additive commutative group E/B is a unitary right L-module. We call it the associated L-module of the ΓS-module M and denote it by $M^{\#}$. This is to be noted that $< 0_M, \gamma > = 0_{M^{\#}}$ for all $\gamma \in \Gamma$ and $< M, \Gamma > = M^{\#}$.

Definition 3.3. *For a submodule P of the associated module $M^{\#}$ of the ΓS-module M, we define $P^{+} = \{m \in M :< m, \Gamma > \subseteq P\}$ where $< m, \Gamma > = \{\sum_{i=1}^{n} < m, \alpha_i >$: n is a positive integer, $\alpha_i \in \Gamma$, for $i = 1, 2, ..., n\}$. For a submodule N of a ΓS-module M, $N^{+/} = \{\sum_{i=1}^{m} < m_i, \alpha_i > \in M^{\#} : (\sum_{i=1}^{m} < m_i, \alpha_i >)S \subseteq N\}$ where $(\sum_{i=1}^{m} < m_i, \alpha_i >)S = \{\sum_{i,j} m_i\alpha_i x_j : x_j \in S, j = 1, 2, ..., n$ for some positive integer $m, n\}$.*

Lemma 3.4. i) *$< m, \Gamma > \subseteq P$ if and only if $< m, \alpha > \in P$, for all $\alpha \in \Gamma$, where $P \subseteq M^{\#}$ and $m \in M$.*

ii) *$(\sum_{i=1}^{m} < m_i, \alpha_i >)S \subseteq N$ if and only if $\sum_{i=1}^{m} m_i\alpha_i x \in N$, for all $x \in S$ where m is a positive integer and $N \subseteq M$.*

Proof. i) Let $< m,\Gamma >\subseteq P$ that is $\{\sum\limits_{i=1}^{n} < m,\alpha_i >$: n is a positive integer, $\alpha_i \in \Gamma$, for $i = 1, 2, ..., n\} \subseteq P$. Then $< m,\alpha >\in P$ for all $\alpha \in \Gamma$. Conversely, let $< m,\alpha >\in P$ for all $\alpha \in \Gamma$. Then $\sum\limits_{i=1}^{n} < m,\alpha_i >= < m, \sum\limits_{i=1}^{n} \alpha_i >\in P$. Hence $< m,\Gamma >\subseteq P$.

ii) Let $(\sum\limits_{i=1}^{m} < m_i,\alpha_i >)S \subseteq N$ that is $\{\sum\limits_{i,j} m_i\alpha_i x_j : x_j \in S, j = 1, 2, ..., n$ for some positive integer $m, n\} \subseteq N$. Then $\sum\limits_{i} m_i\alpha_i x \in N$ for all $x \in S$. Conversely, let $\sum\limits_{i=1}^{m} m_i\alpha_i x \in N$, for all $x \in S$ where m is a positive integer and $N \subseteq M$. Then $\sum\limits_{i,j} m_i\alpha_i x_j = \sum\limits_{i} m_i\alpha_i(\sum\limits_{j} x_j) \in N$ for $i = 1, 2, .., m,\ j = 1, 2, ...n$, and $x_j \in S$. Thus $(\sum\limits_{i=1}^{m} < m_i,\alpha_i >)S \subseteq N$.

Proposition 3.5. *Let $M^{\#}$ be the associated L-module of a ΓS-module M. Then the following statements hold:*

i) *If P be a submodule of $M^{\#}$, then P^+ is a ΓS-submodule of M.*

ii) *If N is a ΓS-submodule of M, then $N^{+/}$ is a submodule of $M^{\#}$.*

Proof. i) Since P is a submodule of $M^{\#}$, $0_{M^{\#}} =< 0_M,\alpha >\in P$, for all $\alpha \in \Gamma$. $0_M \in P^+$; so P^+ is non-empty. Let $m_1, m_2 \in P^+$ then $< m_1,\alpha >, < m_2,\alpha >\in P$ for all $\alpha \in \Gamma$. So $< m_1 - m_2,\alpha > = < m_1,\alpha > - < m_2,\alpha >\in P$ as P is a submodule of $M^{\#}$. Hence $m_1 - m_2 \in P^+$. Suppose $m \in P^+$, $\alpha \in \Gamma$ and $x \in S$. Then $< m,\alpha >\in P$, so $< m\alpha x,\beta > = < m,\alpha > [x,\beta] \in P$ as P is a submodule of $M^{\#}$ for all $\beta \in \Gamma$. Hence $m\alpha x \in P^+$. So P^+ is a ΓS-submodule of M.

ii) Since N is a ΓS-submodule of M, $0_M\alpha x = 0_M \in N$; where $\alpha \in \Gamma$ and $x \in S$; hence $< 0_M,\alpha >\in N^{+/}$. Thus $N^{+/}$ is non-empty.
Let $\sum\limits_{i=1}^{m} < m_i,\alpha_i >, \sum\limits_{j=1}^{n} < n_j,\beta_j >\in N^{+/}$. Then $\sum\limits_{i} m_i\alpha_i x, \sum\limits_{j} n_j\beta_j x \in N$ for all $x \in S$. Whence $\sum\limits_{i} m_i\alpha_i x - \sum\limits_{j} n_j\beta_j x \in N$. Thus $\sum\limits_{i=1}^{m} < m_i,\alpha_i > - \sum\limits_{j=1}^{n} < n_j,\beta_j >\in N^{+/}$. Let $\sum\limits_{i=1}^{m} < m_i,\alpha_i >\in N^{+/}$, $\sum\limits_{k=1}^{n} [y_k,\beta_k] \in L$. Now

$\sum_i m_i\alpha_i x \in N$ and $\sum_k y_k\beta_k x \in S$ for all $x \in S$. Thus $\sum_{i,k} m_i\alpha_i(y_k\beta_k x) \in N$ for all $x \in S$. Hence $\sum_i < m_i, \alpha_i > \sum_k [y_k, \beta_k] \in N^{+/}$. This proves $N^{+/}$ is a submodule of $M^{\#}$.

Theorem 3.6. *The lattices of all ΓS-submodules of a ΓS-module M and its associated L-module $M^{\#}$ are isomorphic via the mapping $N \to N^{+/}$ for a submodule N of M.*

Proof. Let N be a ΓS-submodule of a ΓS-module M and $n \in (N^{+/})^+$. Then $< n, \alpha > \in N^{+/}$, for all $\alpha \in \Gamma$, whence $n\alpha x \in N$, for all $x \in S$ and for all $\alpha \in \Gamma$. Thus $n = \sum_{j=1}^{m} n\gamma_j f_j \in N$ where $\sum_{j=1}^{m} [\gamma_j, f_j]$ is the right unity of S. So $(N^{+/})^+ \subseteq N$.

Conversely, let $n \in N$; then $n\alpha x \in N$ for all $x \in S$, which implies that $< n, \alpha > \in N^{+/}$ for all $\alpha \in \Gamma$. So $n \in (N^{+/})^+$. Thus $N \subseteq (N^{+/})^+$. Hence $(N^{+/})^+ = N$. Now suppose that P is a submodule of $M^{\#}$. Let $\sum_{j=1}^{m} < m_j, \alpha_j > \in (P^+)^{+/}$. Then $\sum_{j=1}^{m} m_j\alpha_j x \in P^+$, for all $x \in S$, i.e $< \sum_{j=1}^{m} m_j\alpha_j x, \alpha > = \sum_{j=1}^{m} < m_j, \alpha_j > [x, \alpha] \in P$ for all $x \in S$ for all $\alpha \in \Gamma$. Since P is a submodule of $M^{\#}$, we have $\sum_{j=1}^{m} < m_j, \alpha_j > = \sum_{j=1}^{m} < m_j, \alpha_j > \sum_{i=1}^{n} [e_i, \delta_i] \in P$, where $\sum_{i=1}^{n} [e_i, \delta_i]$ is the left unity of S. Thus $(P^+)^{+/} \subseteq P$. Conversely, let $\sum_{i=1}^{m} < m_i, \alpha_i > \in P$. Since P is a submodule of $M^{\#}$, $< \sum_{i=1}^{m} m_i\alpha_i x, \alpha > = \sum_{i=1}^{m} < m_i\alpha_i x, \alpha > = \sum_{i=1}^{m} < m_i, \alpha_i > [x, \alpha] \in P$ for all $x \in S$ and for all $\alpha \in \Gamma$. This implies that $\sum_{i=1}^{m} m_i\alpha_i x \in P^+$ for all $x \in S$. Hence $\sum_{i=1}^{m} < m_i, \alpha_i > \in (P^+)^{+/}$. Therefore $P \subseteq (P^+)^{+/}$. Hence $(P^+)^{+/} = P$. This shows that the mapping $N \to N^{+/}$ is a bijective mapping. Let N and K be two ΓS-submodules of M such that $N \subseteq K$. Then $\sum_{i=1}^{m} < m_i, \alpha_i > \in N^{+/}$ implies $\sum_{i=1}^{m} m_i\alpha_i x \in N \subseteq K$, for all $x \in S$. Thus

$\sum_{i=1}^{m} m_i, \alpha_i >\in K^{+/}$. So $N^{+/} \subseteq K^{+/}$. Let P, Q be two sub-modules of $M^{\#}$ such that $P \subseteq Q$. Then for any $m \in P^{+}, < m, \alpha >\in P \subseteq Q$, for all $\alpha \in \Gamma$ which implies $m \in Q^{+}$. Hence $P^{+} \subseteq Q^{+}$. So the mapping $N \to N^{+/}$ where N is a ΓS-submodule of M is a lattice isomorphism. □

Corollary 3.7. *The lattice of all ΓS-submodules of a ΓS-module M is complete.*

Proof. This follows from theorem 3.6 and from the fact that the lattice of all submodules of a module over a ring is complete. □

Definition 3.8. *The annihilator of a subset N of a ΓS-module M is the set $A(N) = \{x \in S | \ a\alpha x = 0_M$ for all $\alpha \in \Gamma$ and for all $a \in N\}$.*

Lemma 3.9. *If $M^{\#}$ is the associated L-module of a ΓS-module M then $(A(M^{\#}))^{+} = A(M)$.*

Proof. Let $x \in (A(M^{\#}))^{+}$. Then $[x, \alpha] \in A(M^{\#})$ for all $\alpha \in \Gamma$; hence $< a, \beta > [x, \alpha] = 0_{M^{\#}}$ implies $< a\beta x, \alpha >= 0_{M^{\#}}$ for all $\alpha, \beta \in \Gamma, x \in S$ and $a \in M$. Thus $a\beta x\alpha y = 0_M$ for all $\alpha \in \Gamma, y \in S$ which implies $x\alpha y \in A(M)$. Hence $x \sum_i [\delta_i, e_i] = x \in A(M)$. Thus $(A(M^{\#}))^{+} \subseteq A(M)$. Conversely, let $x \in A(M)$. Then $a\beta x = 0_M$ for all $\beta \in \Gamma$ and for all $a \in M$. Now $< a, \beta > [x, \alpha] =< a\beta x, \alpha >= < 0_M, \alpha >= 0_{M^{\#}}$ for all $a \in M$, for all $\alpha \in \Gamma$. Thus $\sum_{i=1}^{n} < a_i, \beta_i > [x, \alpha] = 0_{M^{\#}}$ for all $\sum_{i=1}^{n} < a_i, \beta_i >\in M^{\#}$ and for all $\alpha \in \Gamma$. Thus $[x, \alpha] \in A(M^{\#})$ i.e $x \in (A(M^{\#}))^{+}$. Hence $A(M) \subseteq (A(M^{\#}))^{+}$. Consequently $A(M) = (A(M^{\#}))^{+}$. □

Corollary 3.10. *For a ΓS-module M, $(A(M))^{+/} = A(M^{\#})$.*

Proof. $A(M)$ and $A(M^{\#})$ are ideals of S and L respectively. Now by the Theorem 1 of [3] and Lemma 3.9 it follows that $A(M)^{+/} = ((A(M^{\#}))^{+})^{+/} = A(M^{\#})$. □

Definition 3.11 [7]. *A ΓS-module M is called faithful if $A(M) = \{0_S\}$.*

Theorem 3.12. *A ΓS-module M is faithful if and only if its associated L-module $M^{\#}$ is faithful.*

Proof. Suppose $M^{\#}$ is a faithful L-module. Then $A(M^{\#}) = \{0_L\}$. This implies that $(A(M^{\#}))^{+} = \{0_L\}^{+} = \{0_S\}$, whence by Lemma 3.9, it follows that $A(M) = \{0_S\}$. So M is faithful. Conversely, suppose M is a faithful ΓS-module. Then $A(M) = \{0_S\}$ i.e $(A(M^{\#}))^{+} = \{0_S\}$. Now $A(M^{\#}) = (A(M^{\#})^{+})^{+\prime} = \{0_S\}^{+\prime} = \{0_L\}$. This proves that $M^{\#}$ is a faithful L-module.

4. Primitive ideals and primitive Γ-ring

Definition 4.1 [5]. *A ΓS-module M is called irreducible if $M\Gamma S \neq 0$ and it has no nonzero proper submodule.*

Proposition 4.2. *A ΓS-module M is irreducible if and only if its associated L-module $M^{\#}$ is irreducible.*

Proof. Let $M^{\#}$ be irreducible. If possible let, N be a submodule of M such that $\{0_M\} \subset N \subset M$. Then $N^{+\prime}$ is a submodule of $M^{\#}$ and $\{0_{M^{\#}}\} = \{0_M\}^{+\prime} \subset N^{+\prime} \subset M^{+\prime} = M^{\#}$ by Theorem 3.6. This contradicts the fact that $M^{\#}$ is irreducible. So M is irreducible. Similarly the converse follows. □

Definition 4.3 [7]. *A Γ-ring S is called right primitive if it admits a faithful irreducible right ΓS- module.*

Theorem 4.4. *A Γ-ring S is right primitive if and only if its left operator ring L is right primitive.*

Proof. Let the left operator ring L be right primitive and M be a faithful and irreducible L-module. Suppose K be the free additive commutative group generated by $M \times S$. Now $A = \{\sum_i n_i(a_i, x_i) \in K \ : \ \alpha \in \Gamma \Rightarrow \sum_i n_i a_i[x_i, \alpha] = 0_M\}$ is a subgroup of K. Let K/A be the factor group of K by A. We denote the coset $\sum_{i=1}^{n}(a_i, x_i) + A$ by $\sum_{i=1}^{n} \ll a_i, x_i \gg$. We define a Γ-scalar multiplication on K/A as follows: $(\sum_{i=1}^{n} \ll a_i, x_i \gg)\alpha x = \sum_{i=1}^{n} \ll a_i[x_i, \alpha], x \gg$ for any $\alpha \in \Gamma, x \in S$. With respect to this multiplication the additive commutative group K/A is a right unitary ΓS-module. We denote it by $M^{\#/}$. We observe that i) $\sum_{i=1}^{n} \ll a_i, x \gg = \ll \sum_{i=1}^{n} a_i, x \gg$, ii) $\sum_{i=1}^{n} \ll a, x_i \gg = \ll a, \sum_{i=1}^{n} x_i \gg$, iii) for any $a \in M,\ x, y \in S,\ \beta \in \Gamma$, $\ll a, x\beta y \gg = \ll a[x,\beta], y > = (\ll a, x \gg)\beta y$. iv) $\ll 0_M, x \gg = \ll a, 0_S \gg = 0_{M^{\#/}}$, for any $x \in S$ and $a \in M$. Now $A(M^{\#/}) = \{x \in S \mid (\sum_{i=1}^{n} \ll a_i, x_i \gg)\alpha x = 0_{M^{\#/}}$, for all $\alpha \in \Gamma$, for all $\sum_{i=1}^{n} \ll a_i, x_i \gg \in M^{\#/}\}$. Let $x \in A(M^{\#/})$. Then $(\sum_{i=1}^{m} \ll a_i, x_i \gg)\alpha x = 0_{M^{\#/}}$ for all $\alpha \in \Gamma$, for all $\sum_{i=1}^{m} \ll a_i, x_i \gg \in M^{\#/}$. Let $\sum_{j=1}^{n} [f_j, \gamma_j]$ be the identity element of L. Hence for $a \in M$, $\ll a, f_j \gg \gamma_j x = 0_{M^{\#/}}$ for all $j = 1, 2, ..., n$ which implies $\ll a, f_j\gamma_j x \gg = 0_{M^{\#/}}$. Thus $a[f_j\gamma_j x, \alpha] = 0_M$ for all $\alpha \in \Gamma$. Hence $a[\sum_{j=1}^{n} f_j\gamma_j x, \alpha] = 0_M$ implies that $[x, \alpha] \in A(M)$ for all $\alpha \in \Gamma$. This proves that $A(M^{\#/}) = \{0_S\}$ as M is a faithful L module. Thus $M^{\#/}$ is a faithful ΓS-module. Now let N be a ΓS-submodule of $M^{\#/}$ such that $N \neq \{0_{M^{\#/}}\}$. Now let $N^{/} = \{m \in M \mid \ll m, x \gg \in N$ for all $x \in S\}$. Clearly, $N^{/}$ is a L-submodule of M. As $N \neq \{0_{M^{\#/}}\}$, $N^{/} \neq \{0_M\}$. Now M is an irreducible L-module of S. Hence $N^{/} = M$. This shows that $\ll m, x \gg \in N$ for all $x \in S$ and all $m \in M$. As N is a submodule of $M^{\#/}$, $N = M^{\#/}$. Hence $M^{\#/}$ is a faithful ΓS-module of S. Thus S has a faithful irreducible ΓS-module $M^{\#/}$. Hence S is right primitive. Conversely suppose that the Γ-ring S is right primitive. Then S admits a faithful irreducible right ΓS-module M. Then by Proposition 3.12 and Proposition 4.2 it follows that L admits a faithful irreducible right module $M^{\#}$. So L is right primitive. □

Theorem 4.5. *Let S be a Γ- ring. Then the following are equivalent:*

i) *S is right primitive;*

ii) *L is right primitive;*

iii) *R is right primitive.*

Proof. i) ⇔ ii) follows from Theorem 4.4 and i) ⇔ iii) follows from

Theorem 1.6[7]. This proves the equivalence. □

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Received: 13.IV.2005
Revised: 22.VIII.2005

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