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QUASI-LARGE SUBGROUPS IN MODULAR AND SEMISIMPLE GROUP RINGS

BY

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Abstract. In the present paper we find in an explicit form quasi-large subgroups of the normed unit groups in commutative group rings.

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1. Introduction. Suppose G is an abelian multiplicative group with p-component of torsion G_p, R is a commutative ring with identity of prime characteristic p and K is a field of the first kind with respect to p of characteristic different from p.

As usual, throughout this article, RG and KG denote group rings over R and K, respectively, with normalized Sylow p-components $S(RG)$ and $S(KG)$. For a subgroup H of G, the symbol $I(RG;H)$ will always designate the relative augmentation ideal of RG with respect to H. For simplicity of the exposition, we put $S(RG;H) = 1 + I_p(RG;H)$ where the subscript p describes the nil-radical of $I(RG;H)$, that is the nil-ideal of $I(RG;H)$ consisting of all nilpotent elements.

All other unexplained notions and notation are the same or identical as in the cited below, in the references, bibliography.

The following is essential for our further investigation.

Definition. ([1]) *The subgroup H of G is said to be quasi-large if $G = HB$ for every basic subgroup B of G.*

Thus, each quasi-large and fully invariant (= completely characteristic in other terms) subgroup is large (cf. [14]). Notice that the large subgroups of G have satisfactory description of their structure (e.g. [14]).

If H is quasi-large in G and $H \leq \Gamma \leq G$, then Γ is also quasi-large in G (see cf. [1], p. 652, property (3)), so the quasi-large subgroups can be infinitely many; they are not isomorphic as some plain examples show.

Following the comprehensive characterization of quasi-large subgroups, given by Benabdallah-Wilson in ([1], p. 651, Theorem 2.1), we conclude that the following necessary and sufficient condition is realized.

Criterion (Benabdallah-Wilson, 1978). *For an arbitrary subgroup $H \leq G$, H is quasi-large in G if and only if there exists a monotone increasing sequence $n_1, n_2, \dots$ in $\mathbb{N}$ such that $G^{p^{n_i}}[p] \subset H^{p^i}$ for all $i \in \mathbb{N}$ or, equivalently, there exists a monotone increasing sequence $m_1, m_2, \dots$ in $\mathbb{N}$ such that $G^{p^{m_i}}[p^i] \subset H$ for all $i \in \mathbb{N}$.*

We shall use in the sequel this criterion applied to commutative modular group rings.

In ([14], p. 22, Exercise 12 due to R. S. Pierce) is stated that if G is an abelian p-group and there is a basic subgroup $B \neq G$ (thus G is unbounded) such that $G = HB$ for some fully invariant subgroup H, then H is large in G in the sense of [14]. Adapting this Pierce's assertion, if we restrict in the above Definition the condition "for every basic subgroup" to "for some basic subgroup", we call such a subgroup H *almost quasi-large* in G. It is apparent that each quasi-large subgroup is almost quasi-large, while the reverse is, probably, demonstrably untrue.

In [5], we have explored when certain subgroups of $S(RG)$ are either fully invariant or large. In [4], [6], [8] and [10, 11, 12], we have classified the basic and p-basic subgroups in a modular and a semisimple aspect as well as the p-basic subgroups of "p-torsion"groups defined as in [2]. Besides, in [4] and [7,9], we have established in an explicit mode some high subgroups in commutative modular group rings.

The main goal motivated this research study is to determine the kind of some major subgroups of $S(RG)$ and $S(KG)$, that subgroups are quasi-large or almost quasi-large. Since the form of quasi-large subgroups in the semisimple situation is so difficult to be completely described, although it can be obtained by virtue of the techniques in [13] and [8], we shall bound our attention to subgroups of $S(KG)$ of a special sort. In order to do the presented, we utilize our own results documented in [4], [8] and [11]. And so, we come to the main results distributed for convenience into two sections.

2. Main results

I. Modular case. We now proceed by proving the following central attainment.

Theorem 1. *Suppose G is an abelian group with a quasi-large subgroup H of G_p. Then*

(1) $S(RG;H)$ *is quasi-large in* $S(RG)$, *provided* R *is without nilpotents.*

Proof. Referring to the necessary and sufficient condition of Benabdallah-Wilson, listed above, for an arbitrary $i \in \mathbb{N}$ we infer that $S^{p^{m_i}}(RG)[p^i] = S(R^{p^{m_i}}G^{p^{m_i}})[p^i] = S(R^{p^{m_i}}G^{p^{m_i}}; G^{p^{m_i}}[p^i]) \subset S(RG;H)$ where the second equality holds owing to [11], and where $m_1, m_2, \ldots$ is an increasing monotone sequence of positive integers. This completes the proof. □

After this, we treat the another question for almost quasi-large subgroups. First, we need a key technicality.

Lemma 1. *Let G be an abelian p-group so that $G = AC$ for some subgroups A and C, and let R be a commutative unitary ring of prime characteristic p. Then $S(RG) = S(RA)S(RG;C)$.*

Proof. Evidently, the left hand-side contains the right one. To obtain the converse implication, given $x \in S(RG)$. Hence $x = f_1a_1c_1 + \ldots + f_ta_tc_t$ where $f_1, \ldots, f_t \in R$ with $f_1 + \ldots + f_t = 1$; $a_1, \ldots, a_t \in A$; $c_1, \ldots, c_t \in C$. Thus, we discover, $x = 1+f_1a_1c_1+\ldots+f_ta_tc_t-f_1-\ldots-f_t = 1+f_1(a_1c_1-1)+\ldots+f_t(a_tc_t-1) = 1+f_1((c_1-1)a_1+(a_1-1))+\ldots+f_t((c_t-1)a_t+(a_t-1)) = 1 + f_1(a_1-1) + \ldots + f_t(a_t-1) + f_1a_1(c_1-1) + \ldots + f_ta_t(c_t-1)$. Because it is obvious that $1 + f_1(a_1-1) + \ldots + f_t(a_t-1) \in 1 + I_p(RA;A) = S(RA)$ and $f_1a_1(c_1-1) + \ldots + f_ta_t(c_t-1) \in I(RG;C)$, we finally derive that $x \in S(RA) + I(RG;C) = S(RA)(1 + I(RG;C)) = S(RA)S(RG;C)$, as required. This finishes the proof of the relationship. □

Along similar lines, we have the following.

Theorem 2. *Suppose G is an abelian p-group with an almost quasi-large subgroup H. Then*

(2) $S(RH)$ *is an almost quasi-large subgroup of* $S(RG)$.

Proof. Employing the ratio from the lemma, we detect that $S(RG) = S(RH).S(RG; B)$ provided $G = HB$ for some basic subgroup B of G. Besides, according to [11], we deduce $S(RG; B) \subseteq B_{S(RG)}$, a basic subgroup of $S(RG)$. Henceforth $S(RG) = S(RH).B_{S(RG)}$ and, in virtue of the definition, the proof is completed. □

II. Semisimple case. Foremost, we show the decomposition structure of a large subgroup in $S(KG)$.

Proposition 1. *If L is a large subgroup of $S(KG)$, then*

$$L = [dS(KG)] \times [L/dS(KG)], \tag{3}$$

where $dS(KG)$ is the maximal divisible subgroup of $S(KG)$, and $L/dS(KG)$ is a large subgroup of a separable p-group.

Proof. Applying subsequently [14] together with [13], we have $L \supseteq S^1(KG) = dS(KG)$. Therefore, it follows directly from [14] that, $L = S^1(KG) \times [L/S^1(KG)]$. But it is a routine exercise to check that $L/S^1(KG)$ is a large subgroup of $S(KG)/S^1(KG)$ which, complying with [13], is isomorphic to the separable group $S(K(G/G^1))$. This ends the proof. □

Remark. In view of the last proposition and foregoing comments, we have described the second complementary factor in appropriate terms of the whole group $S(KG)$, whence so does L.

It is long-known that any element of $S(KG)$ can uniquely be written as the sum $x_1e_1 + \dots x_te_t$ where $x_1, \dots, x_t \in KG$ and $e_1, \dots, e_t$ are minimal ortogonal idempotents in KG (that are $e_i^2 = e_i$ for $1 \leq i \leq t$ with $e_1 + \dots + e_t = 1$ and $e_ie_j = 0$ for $1 \leq i \neq j \leq t$). So, we may apply the same reasoning as that used at the end of the proof of the dependence from Theorem 1 to conclude a claim analogous to the modular one. Reciprocally, repeating the same procedure as in Theorem 2, we can yield an analogous claim even for the semisimple direction. Nevertheless, as we have already mentioned above, both the statement and the description in an explicit kind of such parallel claims is of some technical difficulty.

We terminate the paper with

Concluding discussion. As we have already seen, the theorems was proved by dropping off the explicit forms of basic subgroups of $S(RG)$ or $S(KG)$.

In closing, we note that the definition stated by Benabdallah-Wilson in [1] for quasi-large subgroups may be successfully expanded to the so-called λ-*quasi-large subgroups,* where λ is an arbitrary ordinal number. To this alternative aim, it suffices to replace the word "basic subgroup" via "λ-basic subgroup". Thus, consuming the results obtained by us in [3], we can extract in the modular way certain extensions to the preceding affirmations. However, there is no a real advantage in this and it is a technical difference only.

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