

**EXISTENCE RESULTS FOR NONLINEAR FUNCTIONAL  
INTEGRAL EQUATIONS VIA NONLINEAR ALTERNATIVE  
OF LERAY-SCHAUDER TYPE**

BY

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**Abstract.** In this paper an existence theorem for a certain nonlinear functional integral equations of mixed type is proved using the nonlinear alternatives recently proved by DHAGE [2] and DHAGE and REGAN [3]. An existence theorem for the functional integral equation that has been considered in NTOUYAS and TSAMATOS [9] is also obtained under some weaker conditions than that given in [9]. Finally applications are given to some nonlinear functional initial and boundary value problems of ordinary differential equations for proving the existence results. Our results of the present paper complement the work in DHAGE and REGAN [3].

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**Key words:** Fixed point theorem, functional integral equation.

**1. Introduction.** Nonlinear integral equations have been a topic of great interest among the mathematicians working in the field of nonlinear analysis since long time. See KRASNOSELSKII [7] and the references given therein. Nonlinear functional integral equations have also been discussed in the literature, see for example, SUBRAMANYAM and SUNDERSANAM [9], NTOUYAS and TSAMATOS [9] and DHAGE and REGAN [3] etc. In the present paper, we study a nonlinear functional interval equation of mixed type for the existence result. In particular, for a given closed and bounded interval  $J = [0, 1]$  in  $\mathbb{R}$ , the set of all real numbers, we discuss the following nonlinear functional integral equation (in short FIE)

$$(1.1) \quad x(t) = q(t) + \int_0^{\mu(t)} k(t, s) f(s, x(\theta(s))) ds$$

$$+ \int_0^{\sigma(t)} v(t, s)g(s, x(\eta(s))) ds, \quad t \in J$$

for  $t \in J$ , where  $q : J \rightarrow \mathbb{R}$ ,  $k, v : J \times J \rightarrow \mathbb{R}$ ,  $f, g : J \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\mu, \theta, \sigma, \eta : J \times J \rightarrow J$ .

The FIE (1.1) is general in the sense that it includes the well-known Volterra and Hammerstein integral equations as special cases which have been extensively studied in the literature for various aspects of the solution. The existence for the FIE (1.1) is generally proved by using a fixed point theorem of KRASNOSELSKII [7], but here in the present paper we obtain the existence result via the nonlinear alternatives recently developed in DHAGE [2] and DHAGE and REGAN [3].

Before stating the nonlinear alternatives we give some preliminary definitions needed in the sequel.

Let  $X$  denote a Banach space and let  $T : X \rightarrow X$  be a mapping. Then  $T$  is called a *nonlinear contraction* on  $X$  if there is a continuous function  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that

$$(1.2) \quad \|Tx - Ty\| \leq \psi(\|x - y\|)$$

for all  $x, y \in X$ , where  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a continuous function satisfying  $\psi(r) < r$  for  $r > 0$ . In particular if  $\psi(r) = \alpha r$ ,  $0 \leq \alpha < 1$ , then  $T$  is called a contraction on  $X$ . Again  $T$  is called **compact** if  $T(X)$  is a compact subset of  $X$ . Similarly  $T : X \rightarrow X$  is called **totally bounded** if  $T$  maps a bounded subset of  $X$  into the relatively compact subset of  $X$ . Finally  $T : X \rightarrow X$  is called **completely continuous** operator if it is continuous and totally bounded operator on  $X$ . It is clear that every compact operator is totally bounded, but the converse may not be true. However these two notions are equivalent on a bounded subset of  $X$ .

**Theorem 1.1.** (Dhage [2]) *Let  $U$  and  $\bar{U}$  denote respectively the open and closed subsets in a Banach space  $X$  such that  $0 \in U$  and let  $A, B : X \rightarrow X$  be two operators satisfying*

- (a)  *$A$  is nonlinear contraction, and*
- (b)  *$B$  is completely continuous.*

*Then either*

- (i) *the operator equation  $Ax + Bx = x$  has a solution in  $\bar{U}$ , or*
- (ii) *there exists an element  $u \in \partial U$  such that  $\lambda A\left(\frac{u}{\lambda}\right) + \lambda Bu = u$  for some  $\lambda \in (0, 1)$ , where  $\partial U$  is the boundary of  $U$ .*

**Corollary 1.1.** (Dhage and Regan [3]) *Let  $B(0, r)$  and  $B[0, r]$  denote respectively the open and closed balls in a Banach space  $X$  and let  $A, B : X \rightarrow X$  be two operators satisfying*

- (a)  *$A$  is contraction, and*
- (b)  *$B$  is completely continuous.*

*Then either*

- (i) *the operator equation  $Ax + Bx = x$  has a solution in  $B[0, r]$ , or*
- (ii) *there exists an element  $u \in X$  with  $\|u\| = r$  such that  $\lambda A(\frac{u}{\lambda}) + \lambda Bu = u$  for some  $\lambda \in (0, 1)$ .*

In the following section we prove some existence theorems for the FIE (1.1) under certain growth conditions on the functions involved in it in the form of functional inequalities. These sufficient conditions are motivated by the priori bounds on the solutions of the FIE (1.1) which ultimately yields that every solution of FIE (1.1) if exists is bounded on  $J$ . Such sufficient conditions on the growth of the functions are not new and have been used in the literature since long time. See DHAGE and REGAN [3], GRANAS et. al. [5] and the references therein.

**2. Main results.** Let  $M(J, \mathbb{R})$  and  $B(J, \mathbb{R})$  respectively denote the spaces of measurable and bounded real-valued functions on  $J$ . We shall seek the solution of the FIE(1.1) in the space  $BM(J, \mathbb{R})$  of all bounded and measurable real-valued functions on  $J$ . Define a norm  $\|\cdot\|$  in  $BM(J, \mathbb{R})$  by

$$\|x\| = \max_{t \in J} |x(t)|.$$

Clearly  $BM(J, \mathbb{R})$  becomes a Banach space with this norm. We need the following definition in the sequel.

**Definition 2.1.** *A mapping  $\beta : J \times \mathbb{R} \rightarrow \mathbb{R}$  is said to satisfy the Carathéodory condition or simply is called a Carathéodory if*

- (i)  *$t \rightarrow \beta(t, x)$  is measurable for each  $x \in \mathbb{R}$ ,*
- (ii)  *$x \rightarrow \beta(t, x)$  is continuous almost everywhere for  $t \in J$  and*
- (iii) *for each real number  $r > 0$ , there exists a function  $h_r \in L^1(J, \mathbb{R})$  such that*

$$|\beta(t, x)| \leq h_r(t), \quad \text{a.e. } t \in J$$

*for all  $x \in \mathbb{R}$  with  $|x| \leq r$ .*

We consider the following hypotheses:

- ( $H_0$ ) The functions  $\mu, \theta, \sigma, \eta : J \rightarrow J$  are continuous.

(H<sub>1</sub>) The function  $q : J \rightarrow \mathbb{R}$  is bounded and measurable.

(H<sub>2</sub>) The functions  $k, v : J \times J \rightarrow J$  are continuous.

(H<sub>3</sub>) The function  $f(t, x)$  is continuous and there exists a function  $\alpha \in L^1(J, \mathbb{R})$  such that  $\alpha(t) > 0$ , a.e.  $t \in J$  and

$$|f(t, x) - f(t, y)| \leq \alpha(t)|x - y|, \quad \text{a.e. } t \in J$$

for all  $x, y \in \mathbb{R}$ .

(H<sub>4</sub>) The function  $f(t, x)$  is continuous and satisfies

$$|f(t, x) - f(t, y)| \leq \frac{|x - y|}{K + |x - y|}, \quad \text{a.e. } t \in J$$

for all  $x, y \in \mathbb{R}$ , where  $K = \sup_{t, s \in J} |k(t, s)|$ .

(H<sub>5</sub>) The function  $g(t, x)$  is  $L^1$ -Carathéodory.

(H<sub>6</sub>) There exists a continuous and nondecreasing function  $\Omega : [0, \infty) \rightarrow [0, \infty)$  and a function  $\phi \in L^1(J, \mathbb{R})$  such that  $\phi(t) > 0$ , a.e.  $t \in J$  and

$$|g(t, x)| \leq \phi(t)\Omega(|x|), \quad \text{a.e. } t \in J$$

for all  $x \in \mathbb{R}$ .

**Theorem 2.1.** *Assume that the hypotheses (H<sub>0</sub>)-(H<sub>3</sub>) and (H<sub>5</sub>)-(H<sub>6</sub>) hold. Suppose that there exists a real number  $r > 0$  such that*

$$(2.1) \quad r > \frac{\|q\| + KF + V\|\phi\|_{L^1}\Omega(r)}{1 - K\|\alpha\|_{L^1}}, \quad K\|\alpha\|_{L^1} < 1,$$

where  $K = \sup_{t, s \in J} |k(t, s)|$ ,  $V = \sup_{t, s \in J} |v(t, s)|$  and  $F = \int_0^\alpha |f(s, 0)| ds$ . Then the FIE (1.1) has a solution on  $J$ .

**Proof.** Define an open ball  $B(0, r)$  in the Banach space  $X = BM(J, \mathbb{R})$  centered at the origin and of radius  $r > 0$ , where  $r$  satisfies the inequalities in (2.1). Now consider the two mappings  $A$  and  $B$  on  $BM(J, \mathbb{R})$  defined by

$$(2.2) \quad Ax(t) = q(t) + \int_0^{\mu(t)} k(t, s)f(s, x(\theta(s))) ds, \quad t \in J$$

and

$$(2.3) \quad Bx(t) = \int_0^{\sigma(t)} v(t, s)g(s, \eta(s)) ds, \quad t \in J.$$

Since the function  $\mu$  and  $\sigma$  are continuous in view of the hypotheses  $(H_0)$ - $(H_5)$ ,  $A$  and  $B$  define the operators  $A, B : BM(J, \mathbb{R}) \rightarrow BM(J, \mathbb{R})$ . We shall show that the operators  $A$  and  $B$  satisfy all the conditions of Theorem 1.1. First show that  $A$  is a contraction on  $BM(J, \mathbb{R})$ . Let  $x, y \in BM(J, \mathbb{R})$ . Then by  $(H_4)$ ,

$$\begin{aligned} |Ax(t) - Ay(t)| &\leq \int_0^{\mu(t)} |k(t, s)| |f(s, x(\theta(s))) - f(s, y(\theta(s)))| ds \\ &\leq K \int_0^1 \alpha(s) \|x - y\| = K \|\alpha\|_{L^1} \|x - y\|. \end{aligned}$$

Taking the maximum over  $t$ ,

$$\|Ax - Ay\| \leq K \|\alpha\|_{L^1} \|x - y\|$$

for all  $x, y \in BM(J, \mathbb{R})$ , where  $K \|\alpha\|_{L^1} < 1$ . So  $A$  is a contraction on  $BM(J, \mathbb{R})$ . Next we show that  $B$  is completely continuous on  $BM(J, \mathbb{R})$ . Using the standard arguments as in Granas et. al. [5], it is shown that  $B$  is a continuous operator on  $BM(J, \mathbb{R})$ . Let  $S$  be a bounded set in  $BM(J, \mathbb{R})$  and let  $\{x_n\}$  be a sequence in  $S$ . Then there exists a constant  $r > 0$  such that  $\|x_n\| \leq r$  for all  $n \in N$ . Now by  $(H_6)$ ,

$$\begin{aligned} |Bx_n(t)| &\leq \int_0^{\sigma(t)} |v(t, s)| |g(s, x_n(\eta(s)))| ds \\ &\leq \int_0^{\sigma(t)} V \phi(s) \Omega(|x_n(\eta(s))|) ds \leq V \|\phi\|_{L^1} \Omega(r), \end{aligned}$$

i.e.,  $\|Bx_n\| \leq M$  for all  $n \in N$ , where  $M = V \|\phi\|_{L^1} \Omega(r)$ . This shows that  $\{Bx_n\}$  is a uniformly bounded sequence in  $BM(J, \mathbb{R})$ . Now we show  $\{Bx_n\}$  is an equi-continuous set. Let  $t, \tau \in J$ . Then by (2.3),

$$\begin{aligned} &|Bx_n(t) - Bx_n(\tau)| \\ &\leq \left| \int_0^{\sigma(t)} v(t, s) g(s, x(\eta(s))) ds - \int_0^{\sigma(\tau)} v(\tau, s) g(s, x(\eta(s))) ds \right| \\ &\leq \left| \int_0^{\sigma(t)} v(t, s) g(s, x(\eta(s))) ds - \int_0^{\sigma(t)} v(\tau, s) g(s, x(\eta(s))) ds \right| \end{aligned}$$

$$\begin{aligned}
& + \left| \int_0^{\sigma(t)} v(\tau, s)g(s, x(\eta(s))) ds - \int_0^{\sigma(\tau)} v(\tau, s)g(s, x(\eta(s))) ds \right| \\
& \leq \left| \int_0^{\sigma(t)} [v(t, s) - v(\tau, s)]g(s, x(\eta(s))) ds \right| + \left| \int_{\sigma(\tau)}^{\sigma(t)} v(\tau, s)g(s, x(\eta(s))) ds \right| \\
& \leq \int_0^{\sigma(t)} |v(t, s) - v(\tau, s)| |g(s, x(\eta(s)))| ds + \int_{\sigma(\tau)}^{\sigma(t)} |v(\tau, s)| |g(s, x(\eta(s)))| ds \\
& \leq \int_0^1 |v(t, s) - v(\tau, s)| h_r(s) ds + V|p(t) - p(\tau)| \\
& = \|h_r\|_{L^1} \int_0^1 |v(t, s) - v(\tau, s)| ds + V|p(t) - p(\tau)|,
\end{aligned}$$

where  $p(t) = \int_0^{\sigma(t)} h_r(s) ds$ .

From the above inequality it follows that

$$|Bx_n(t) - Bx_n(\tau)| \rightarrow 0 \quad \text{as } t \rightarrow \tau.$$

Hence  $\{Bx_n\}$  is equi-continuous and consequently  $Bx_n$  is compact by Arzela-Ascoli theorem. Thus every sequence  $\{Bx_n\}$  in  $B(S)$  has a convergent subsequence. Therefore  $B(S)$  is compact and  $B$  is completely continuous operator on  $BM(J, \mathbb{R})$ . Thus all the conditions Corollary 1.1 are satisfied and a direct application of it yields that either conclusion (i), or conclusion (ii) of Theorem 1.1 holds. We show that the conclusion (ii) is not possible. Let  $u \in X$  be such that  $\|u\| = r$ . Then we have for any  $\lambda \in (0, 1)$ ,

$$\begin{aligned}
u(t) & = \lambda A\left(\frac{u}{\lambda}\right)(t) + \lambda Bu(t) \\
& = \lambda \left( q(t) + \int_0^{\mu(t)} k(t, s) f\left(s, \frac{u}{\lambda}(\theta(s))\right) ds \right) + \lambda \left( \int_0^{\sigma(t)} v(t, s) g(s, u(\eta(s))) ds \right) \\
& = \lambda q(t) + \lambda \int_0^{\mu(t)} k(t, s) f\left(s, \frac{u}{\lambda}(\theta(s))\right) ds - \lambda \int_0^{\mu(t)} k(t, s) f(s, 0) ds \\
& \quad + \lambda \int_0^{\mu(t)} k(t, s) f(s, 0) ds + \lambda \int_0^{\sigma(t)} v(t, s) g(s, u(\eta(s))) ds \\
& = \lambda q(t) + \lambda \int_0^{\mu(t)} k(t, s) \left[ f\left(s, \frac{u}{\lambda}(\theta(s))\right) - f(s, 0) \right] ds \\
& \quad + \lambda \int_0^{\mu(t)} k(t, s) f(s, 0) ds + \lambda \int_0^{\sigma(t)} v(t, s) g(s, u(\eta(s))) ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|u(t)| &\leq |q(t)| + \lambda \int_0^{\mu(t)} |k(t, s)| \left| f\left(s, \frac{u}{\lambda}(\theta(s))\right) - f(s, 0) \right| ds \\
&\quad + \int_0^{\mu(t)} |k(t, s)| |f(s, 0)| ds + \int_0^{\sigma(t)} |v(t, s)| |g(s, u(\eta(s)))| ds \\
&\leq \|q\| + K \int_0^1 \alpha(s) |u(\theta(s))| ds + KF + V \int_0^1 \phi(s) \Omega(\|u\|) ds \\
&= \|q\| + K \|\alpha\|_{L^1} \|u\| + KF + V \|\phi\|_{L^1} \Omega(\|u\|).
\end{aligned}$$

Taking the maximum over  $t$ ,

$$\|u\| = \frac{\|q\| + KF + V \|\phi\|_{L^1} \Omega(\|u\|)}{1 - K \|\alpha\|_{L^1}}$$

or

$$r \leq \frac{\|q\| + KF + V \|\phi\|_{L^1} \Omega(r)}{1 - K \|\alpha\|_{L^1}},$$

which is a contradiction to the inequality in (2.1). Therefore the conclusion (i) holds and consequently the FIE (1.1) has a solution on  $J$ . This completes the proof.  $\square$

**Theorem 2.2.** *Assume that the hypotheses  $(H_0)$ - $(H_2)$ ,  $(H_4)$  and  $(H_5)$ - $(H_6)$  hold. Suppose that there exists a real number  $r > 0$  such that*

$$(2.4) \quad r > \|q\| + K(1 + F) + V \|\phi\|_{L^1} \Omega(r)$$

where  $K = \sup_{t,s \in J} |k(t, s)|$ ,  $V = \sup_{t,s \in J} |v(t, s)|$  and  $F = \int_0^\alpha |f(s, 0)| ds$ .

Then the FIE (1.1) has a solution on  $J$ .

**Proof.** Define an open ball  $B(0, r)$  in the Banach space  $X = BM(J, \mathbb{R})$  centered at the origin and of radius  $r > 0$ , where  $r$  satisfies the inequalities in (2.4). Now consider the two mappings  $A$  and  $B$  on  $BM(J, \mathbb{R})$  defined by (2.2) and (2.3) respectively. We shall show that  $A$  and  $B$  satisfy all the conditions of Theorem 3.1 on  $\bar{U} = B[0, r]$ .

First we shall show that  $A$  is a nonlinear contraction on  $X$ . Let  $x, y \in X$ . By hypothesis  $(H_4)$ ,

$$\begin{aligned} |Ax(t) - Ay(t)| &\leq \int_0^{\mu(t)} |k(t, s)| |f(s, x(\theta(s))) - f(s, y(\theta(s)))| ds \\ &\leq \int_0^{\mu(t)} |k(t, s)| \left( \frac{|x(\theta(s)) - y(\theta(s))|}{K + |x(\theta(s)) - y(\theta(s))|} \right) ds \\ &\leq K \left( \frac{\|x - y\|}{K + \|x - y\|} \right). \end{aligned}$$

Taking the supremum over  $t$ , we get

$$\|Ax - Bx\| \leq \psi(\|x - y\|),$$

where  $\psi(r) = \frac{Kr}{K+r} < r$ ,  $r > 0$ . This shows that  $A$  is a nonlinear contraction on  $X$ . Again by giving the arguments as in the proof of Theorem 2.1 it is proved that the operator  $B$  is completely continuous on  $B[0, r]$ . Thus all the conditions Theorem 1.1 are satisfied and a direct application of it yields that either conclusion (i), or conclusion (ii) of Theorem 1.1 holds. We show that the conclusion (ii) is not possible. Let  $u \in X$  be such that  $\|u\| = r$ . Then we have for any  $\lambda \in (0, 1)$ ,

$$\begin{aligned} u(t) &= \lambda A\left(\frac{u}{\lambda}\right)(t) + \lambda B u(t) \\ &= \lambda \left( q(t) + \int_0^{\mu(t)} k(t, s) f\left(s, \frac{u}{\lambda}(\theta(s))\right) ds \right) + \lambda \left( \int_0^{\sigma(t)} v(t, s) g(s, u(\eta(s))) ds \right) \\ &= \lambda q(t) + \lambda \int_0^{\mu(t)} k(t, s) f\left(s, \frac{u}{\lambda}(\theta(s))\right) ds - \lambda \int_0^{\mu(t)} k(t, s) f(s, 0) ds \\ &\quad + \lambda \int_0^{\mu(t)} k(t, s) f(s, 0) ds + \lambda \int_0^{\sigma(t)} v(t, s) g(s, u(\eta(s))) ds \\ &= \lambda q(t) + \lambda \int_0^{\mu(t)} k(t, s) \left[ f\left(s, \frac{u}{\lambda}(\theta(s))\right) - f(s, 0) \right] ds \\ &\quad + \lambda \int_0^{\mu(t)} k(t, s) f(s, 0) ds + \lambda \int_0^{\sigma(t)} v(t, s) g(s, u(\eta(s))) ds. \end{aligned}$$



Therefore,

$$\begin{aligned}
|u(t)| &\leq |q(t)| + \lambda \int_0^{\mu(t)} |k(t,s)| \left| f\left(s, \frac{u}{\lambda}(\theta(s))\right) - f(s,0) \right| ds \\
&+ \int_0^{\mu(t)} |k(t,s)| |f(s,0)| ds + \int_0^{\sigma(t)} |v(t,s)| |g(s, u(\eta(s)))| ds \\
&\leq \|q\| + K + KF + V \int_0^1 \phi(s) \Omega(\|u\|) ds \\
&= \|q\| + K + KF + V \|\phi\|_{L^1} \Omega(\|u\|).
\end{aligned}$$

Taking the maximum over  $t$ ,

$$\|u\| \leq \|q\| + K + KF + V \|\phi\|_{L^1} \Omega(\|u\|),$$

or

$$r \leq \|q\| + K + KF + V \|\phi\|_{L^1} \Omega(r),$$

which is a contradiction to the inequality in (2.4). Therefore the conclusion (i) holds and consequently the FIE (1.1) has a solution on  $J$ . This completes the proof.  $\square$

On Taking  $k(t,s) = 1 = v(t,s)$  for all  $t, s \in J$  in the FIE (1.1), we obtain the following existence result for the FIE

$$(2.5) \quad x(t) = q(t) + \int_0^{\mu(t)} f(s, x(\theta(s))) ds + \int_0^{\sigma(t)} g(s, x(\eta(s))) ds.$$

**Corollary 2.1.** *Assume that the hypotheses  $(H_0)$ ,  $(H_2)$ ,  $(H_3)$ - $(H_6)$  hold. Further suppose that there exists a real number  $r > 0$  such that*

$$(2.6) \quad r > \frac{\|q\| + F + \|\phi\|_{L^1} \Omega(r)}{1 - \|\alpha\|_{L^1}},$$

where  $\|\alpha\|_{L^1} < 1$  and  $F = \int_0^1 |f(s,0)| ds$ . Then the FIE (2.7) has a solution on  $J$ .

In [9] NTOUYAS and TSAMATOS considered the following FIE

$$(2.7) \quad x(t) = f(t, x(\theta(t))) + \int_0^{\sigma(t)} g(s, x(\eta(s))) ds, \quad t \in J,$$

where  $f, g : J\mathbb{R} \rightarrow \mathbb{R}$  and  $\mu, \theta, \sigma, \eta : J \times J \rightarrow J$  are continuous. Next we state an existence theorem for the FIE (2.7) under some weaker condition than that given in NTOUYAS and TSAMATOS [9].

**Theorem 2.3.** *Assume that  $(H_5)$ - $(H_6)$  and the hypotheses:  $(H_7)$   $f : J \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and there exists a function  $\alpha \in B(J, \mathbb{R})$  with bound  $\|\alpha\|$  such that for all  $x, y \in \mathbb{R}$ ,*

$$|f(t, x) - f(t, y)| \leq \alpha(t)|x - y|, \quad \text{a.e. } t \in J$$

holds. Further suppose that there exists a real number  $r > 0$  such that

$$(2.8) \quad r > \frac{C + \|\phi\|_{L^1\Omega}(r)}{1 - \|\alpha\|}, \quad \|\alpha\| < 1,$$

where  $C = \sup_{t \in J} |f(t, 0)|$ . Then the FIE (2.7) has a solution on  $J$ .

**Proof.** The proof is obtained by using the arguments similar to Theorem 2.1 with appropriate modifications. We omit the details.  $\square$

**3. Applications.** In this section we shall discuss some applications of the main result of the previous section to the initial and boundary value problem of nonlinear ordinary differential equations.

**3.1. Initial value problems.** Consider the initial value problem of first order ordinary functional differential equations (in short FIVP)

$$(3.1) \quad \begin{cases} x'(t) &= f(t, x(\theta(t))) + g(t, x(\eta(t))), \quad \text{a.e. } t \in J, \\ x(0) &= x_0 \in \mathbb{R}, \end{cases}$$

where  $\theta, \eta : J \rightarrow J$  are continuous and  $f, g : J \times \mathbb{R} \rightarrow \mathbb{R}$ .

By the solution the FIVP (3.1) we mean a function  $x \in AC(J, \mathbb{R})$  that satisfies the equations in (3.1), where  $AC(J, \mathbb{R})$  is the space of all absolutely continuous real-valued functions on  $J$ .

**Theorem 3.1.** *Assume that the hypotheses  $(H_3)$ ,  $(H_5)$ - $(H_6)$  hold. Further if there exists a real number  $r > 0$  such that*

$$(3.2) \quad r > \frac{\|x_0\| + F + \|\phi\|_{L^1\Omega}(r)}{1 - \|\alpha\|_{L^1}}, \quad \|\alpha\|_{L^1} < 1,$$

where  $F = \int_0^1 |f(t, 0)| ds$ . Then the FIVP(3.1) has a solution on  $J$ .

**Proof.** The FIVP (3.1) is equivalent to the FIE

$$(3.3) \quad x'(t) = x_0 + \int_0^t f(s, x(\theta(s))) ds + \int_0^t g(s, x(\eta(s))) ds$$

for  $t \in J$ . Now the desired conclusion follows by an application of Corollary 3.1 with  $q(t) = x_0$ ,  $\mu(t) = t = \sigma$  for all  $t \in J$  since  $AC(J, \mathbb{R}) \subset BM(J, \mathbb{R})$ .  $\square$

Next we consider the functional IVP of neutral type, namely

$$(3.4) \quad \begin{cases} (x(t) - f(t, x(\theta(t))))' = g(t, x(\eta(t))), & \text{a.e. } t \in J, \\ x(0) = x_0 \in \mathbb{R}, \end{cases}$$

where  $\theta, \eta : J \rightarrow J$  are continuous with  $\theta(0) = 0$  and  $f, g : J \times \mathbb{R} \rightarrow \mathbb{R}$ .

By the solution of the IVP (3.4) is a function  $x \in AC(J, \mathbb{R})$  and that satisfies the equations in (3.4).

**Theorem 3.2.** *Assume that the hypotheses  $(H_5)$ - $(H_6)$  and  $(H_7)$  hold. Further suppose that there exists a real number  $r > 0$  such that*

$$(3.5) \quad r > \frac{D + F + \|\phi\|_{L^1\Omega}(r)}{1 - \|\alpha\|}, \quad \|\alpha\| < 1,$$

where  $D = |x_0 - f(0, x_0)| + \sup_{t \in J} |f(s, 0)|$ . Then the functional IVP (3.4) has a solution on  $J$ .

**Proof.** Now the FIVP (3.4) is equivalent to the FIE

$$(3.6) \quad x(t) = x_0 - f(0, x_0) + f(t, x(\theta(t))) + \int_0^t g(s, x(\eta(s))) ds$$

for  $t \in J$ . If we define a function  $k : J \times \mathbb{R} \rightarrow \mathbb{R}$  by  $k(t, x(\theta(t))) = x_0 - f(0, x_0) + f(t, x(\theta(t)))$ , then FIE (3.6) reduces to FIE (2.7) with  $\sigma(t) = t$ ,  $t \in J$ . Now the desired conclusion follows by an application of Theorem 3.2. The proof is complete.  $\square$

**3.2. Boundary value problems.** Consider the functional two point boundary value problems (in short BVPs) of second order differential equations

$$(3.7) \quad \begin{cases} -x''(t) = f(t, x(\theta(t))) + g(t, x(\eta(t))), & \text{a.e. } t \in J, \\ x(0) = 0 = x(1) \end{cases}$$

and

$$(3.8) \quad \begin{cases} -x''(t) = f(t, x(\theta(t))) + g(t, x(\eta(t))), & \text{a.e. } t \in J, \\ x(0) = 0 = x'(1), \end{cases}$$

where  $\theta, \eta : J \rightarrow J$  are continuous and  $f, g : J \times \mathbb{R} \rightarrow \mathbb{R}$ .

By the solution of the BVP (3.7) or (3.8) we mean a function  $x \in AC^1(J, \mathbb{R})$  that satisfies the equations in (3.7) and (3.8), where  $AC^1(J, \mathbb{R})$  is the space of all continuous real-valued functions on  $J$  whose first derivative exists and is absolutely continuous on  $J$ .

**Theorem 3.3.** *Assume that the hypotheses  $(H_3)$  and  $(H_5)$ - $(H_6)$  hold. Further suppose that there exists a real number  $r > 0$  such that*

$$(3.9) \quad r > \frac{F + \|\phi\|_{L^1} \Omega(r)}{4 - \|\alpha\|_{L^1}},$$

where  $\|\alpha\|_{L^1} < 4$  and  $F = \int_0^1 |f(t, 0)| ds$ . Then the FBVP (3.7) has a solution on  $J$ .

**Proof.** The functional BVP (3.7) is equivalent to the FIE

$$(3.10) \quad x(t) = \int_0^1 G(t, s) f(s, x(\theta(s))) ds + \int_0^1 G(t, s) g(s, x(\eta(s))) ds$$

for all  $t \in J$ , where  $G(t, s)$  is a Green's function associated with the linear homogeneous BVP

$$(3.11) \quad \begin{cases} -x''(t) = 0, & \text{a.e. } t \in J, \\ x(0) = 0 = x(1) \end{cases}$$

and is given by

$$(3.12) \quad G(t, s) = \begin{cases} s(1-t) & \text{if } 0 \leq s \leq t \leq 1, \\ t(1-s) & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

It is clear that the function  $G(t, s)$  is continuous and nonnegative on  $J \times J$  and satisfies the inequality

$$(3.13) \quad |G(t, s)| = G(t, s) \leq \frac{1}{4}$$

for all  $t, s \in J \times J$ . See Bailey et. al. [1].

Now the functions involved in (3.10) satisfy all the conditions of Theorem 2.1 with  $q(t) = 0$  on  $J$ ,  $\mu(t) = 1 = \sigma(t)$  for all  $t \in J$  and  $k(t, s) = G(t, s) = v(t, s)$  for all  $t, s \in J$ . Hence an application of it yields that FBVP (3.7) has a solution on  $J$ .  $\square$

**Theorem 3.4.** *Assume that the hypotheses  $(H_3)$ , and  $(H_5)$ - $(H_6)$  hold. Further suppose that there exists a real number  $r > 0$  such that*

$$(3.14) \quad r > \frac{F + \|\phi\|_{L^1} \Omega(r)}{1 - \|\alpha\|_{L^1}},$$

where  $\|\alpha\|_{L^1} < 1$  and  $F = \int_0^1 |f(t, 0)| ds$ . Then the FBVP (3.8) has a solution on  $J$ .

**Proof.** The functional BVP (3.8) is equivalent to the FIE

$$(3.15) \quad x(t) = \int_0^1 H(t, s) f(s, x(\theta(s))) ds + \int_0^1 H(t, s) g(s, x(\eta(s))) ds$$

for all  $t \in J$ , where  $H(t, s)$  is a Green's function associated with the linear homogeneous BVP

$$(3.16) \quad \begin{cases} -x''(t) = 0, & t \in J, \\ x(0) = 0 = x'(1) \end{cases}$$

and is given by

$$(3.17) \quad H(t, s) = \begin{cases} s & \text{if } 0 \leq s \leq t \leq 1, \\ t & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

It is known that the Green's function  $H(t, s)$  is continuous and nonnegative on  $J \times J$  and satisfies the inequality

$$(3.18) \quad |H(t, s)| = H(t, s) \leq 1.$$

Now an application of Theorem 2.1 with  $q(t) = 0$ ,  $k(t, s) = H(t, s) = v(t, s)$  for all  $t, s \in J$  and  $\mu(t) = 1 = \sigma(t)$  for all  $t \in J$  yields that the FIE (3.15) has a solution on  $J$ . Clearly this solution belongs to the space  $AC^1(J, \mathbb{R})$ . Consequently the FBVP (3.8) has a solution on  $J$ . This completes the proof.  $\square$

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