ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" IAŞI
Tomul LII, s.I, Matematică, 2006, f.1

A CHARACTERIZATION OF NORMAL FRAMED φ-MANIFOLDS

BY

DOREL FETCU

Abstract. In this note we generalize some results concerning the adapted connections and the normality of the almost contact manifolds for the framed φ-manifolds.
Mathematics Subject Classification 2000: 53B05, 53D99.
Key words: Framed φ-manifold, adapted connection.

1. Preliminaries. Let M be an m-dimensional smooth manifold endowed with a tensor field φ of type (1,1), satisfying the algebraic condition

$$\varphi^3 + \varphi = 0. \tag{1.1}$$

The geometric structure on M defined by φ is called a φ-structure of rank r if the rank r of φ is constant on M and, in this case, M is called a φ-manifold. It follows easily that r is an even number.

If M is a φ-manifold and if there are $m-r$ vector fields ξ_a and $m-r$ differential 1-forms η_a satisfying

$$\varphi^2 = -I + \sum_{a=1}^{m-r} \eta_a \otimes \xi_a, \tag{1.2}$$

$$\eta_a(\xi_b) = \delta_{ab}, \tag{1.3}$$

where $a, b = \overline{1, m-r}$, M is said to be globally framed or to have a framed φ-structure. In this case M is called a globally framed φ-manifold or, simply, a framed φ-manifold. From 1.2 and 1.3, one obtains, by some algebraic computations

$$\varphi\xi_a = 0, \quad \eta_a \circ \varphi = 0, \quad \varphi^3 + \varphi = 0. \tag{1.4}$$

If $m = 2n+1$ and $rank\ \varphi = 2n$ one obtains an almost contact structure on M.

Let M be an m-dimensional globally framed φ-manifold with structure tensors (φ, ξ_a, η_a) with $rank\ \varphi = r$, and consider the manifold $M \times \mathbb{R}^{m-r}$. We denote a vector field on $M \times \mathbb{R}^{m-r}$ by $(X, \sum_{a=1}^{m-r} f_a \frac{\partial}{\partial t^a})$ where X is tangent to M, $\{t^1, ..., t^{m-r}\}$ are the coordinates on $\mathbb{R}^{m-r}$ and $\{f_1, ..., f_{m-r}\}$ are functions on $M \times \mathbb{R}^{m-r}$. Define an almost complex structure on $M \times \mathbb{R}^{m-r}$ by

$$J\Big(X, \sum_{a=1}^{m-r} f_a \frac{\partial}{\partial t^a}\Big) = \Big(\varphi X - \sum_{a=1}^{m-r} f_a \xi_a, \sum_{a=1}^{m-r} \eta_a(X) \frac{\partial}{\partial t^a}\Big). \tag{1.5}$$

It is easy to check that $J^2 = -I$. If J is integrable we say that the framed φ-structure is normal. One obtains that a framed φ-structure is normal if the tensor field S, of type (1,2), defined by

$$S = N_\varphi + \sum_{a=1}^{m-r} d\eta_a \otimes \xi_a, \tag{1.6}$$

vanishes, (see [3]), where N_φ, defined by

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y], \tag{1.7}$$

with $X, Y \in \chi(M)$, is the Nijenhuis tensor field of φ.

If g is a Riemannian metric on M such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{a=1}^{m-r} \eta_a(X)\eta_a(Y), \tag{1.8}$$

then we say that $(\varphi, \xi_a, \eta_a, g)$ is a metric framed φ-structure and M is called a metric framed φ-manifold. The metric g is called an associated Riemannian metric. The fundamental 2-form, Ω, of the considered metric framed φ-manifold, M, is defined just like in the case of the almost Hermitian and almost contact metric manifold, by $\Omega(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \chi(M)$.

Using the "Det" convention for calculus of $d\Omega$ and $d\eta_a$, $a=\overline{1,s}$, just like in [1] for a metric almost contact manifold, one obtains for the covariant derivative of φ

Lemma 1.1. *If $(M,\varphi,\xi_a,\eta_a,g)$ is a metric framed φ-manifold, where $a=\overline{1,s}$, then*

$$2g((\nabla_X\varphi)Y,Z)) = d\Omega(X,\varphi Y,\varphi Z) - d\Omega(X,Y,Z) + g(N_\varphi(Y,Z),\varphi X) + \sum_{a=1}^{s}[d\eta_a(\varphi Y,X)\eta_a(Z) - d\eta_a(\varphi Z,X)\eta_a(Y)] + \sum_{a=1}^{s}[d\eta_a(\varphi Y,Z) + d\eta_a(Y,\varphi Z)]\eta_a(X).$$

Finally, note that a normal metric framed φ-manifold M with structure tensors (φ,ξ_a,η_a,g), $a=\overline{1,s}$, is called an $\mathcal{S}$-manifold if $d\eta_a=\Omega$, for any $a=\overline{1,s}$, (see [8]).

2. Adapted connections on a framed φ-manifold. It seems to be interesting to get some new results concerning the properties of framed φ-manifolds, independent of the choice of an adapted metric, in the same way as in the case of almost contact manifolds (see [2], [4], [5]).

In [4] and in [5] was introduced and studied a class of connections on an almost contact manifold, called the adapted connections. In the following we extend some of the results obtained in [4] to framed φ-manifolds.

Let (M,φ,ξ_a,η_a), $a=\overline{1,s}$, be a $(2n+s)$-dimensional framed φ-manifold and let us denote by $v=\sum_{a=1}^{s}\eta_a\otimes\xi_a$ and $h=I-\sum_{a=1}^{s}\eta_a\otimes\xi_a$ the projectors on the distributions $\mathcal{V}=span\{\xi_a\}_{a=1}^{s}$ and $\mathcal{H}=\mathcal{V}^{\perp}$, respectively. It is easy to see that

$$\begin{cases} h^2=h, & v^2=v, & hv=vh=0,\\ \varphi^2=-h, & h\varphi=\varphi h=\varphi, & v\varphi=\varphi v=0. \end{cases} \tag{2.1}$$

Definition 2.1. *We call a linear connection, ∇, on the framed φ-manifold (M,φ,ξ_a,η_a) an adapted connection if, for any $X,Y\in\chi(M)$,*

$$\begin{cases} (\nabla_X\varphi)Y = -\frac{1}{2}hX\sum\limits_{a=1}^{s}\eta_a(Y) + \frac{1}{4}\sum\limits_{a=1}^{s}[d\eta_a(\varphi X,hY) - d\eta_a(X,\varphi Y)]\xi_a,\\ (\nabla_X\eta_a)(Y) = \frac{1}{4}[d\eta_a(X,Y)+d\eta_a(\varphi X,\varphi Y)],\\ \nabla_X\xi_a = -\frac{1}{2}\varphi X - \frac{1}{4}\sum\limits_{b=1}^{s}d\eta_b(X,\xi_a)\xi_b, \quad a=\overline{1,s}. \end{cases} \tag{2.2}$$

It is easy to see, from Lemma 1.1, that on an $\mathcal{S}$-manifold the Levi-Civita connection is an adapted connection.

In order to prove the existence of the adapted connections let us consider the following tensor fields of type (2,2) on the framed φ-manifold $(M, \varphi, \xi_a, \eta_a)$ (as in [4] for an almost contact manifold).

$$\begin{cases} \Phi = \frac{1}{2}(I \otimes I - \varphi \otimes \varphi), \\ \Theta = \frac{1}{2}(I \otimes I - h \otimes h), \\ \Psi = \frac{1}{2}(I \otimes I + \varphi \otimes \varphi). \end{cases} \tag{2.3}$$

It is easy to verify, working in local coordinates, that

$$\begin{cases} \Phi + \Psi = I \otimes I, \quad \Phi^2 = \Phi - \frac{1}{2}\Theta, \quad \Psi^2 = \Psi - \frac{1}{2}\Theta, \\ \Phi\Psi = \Psi\Phi = \Phi\Theta = \Theta\Phi = \Theta\Psi = \Psi\Theta = \Theta^2 = \frac{1}{2}\Theta, \\ (\Psi + \Theta) + (\Phi - \Theta) = I \otimes I, \\ (\Psi + \Theta)(\Phi - \Theta) = (\Phi - \Theta)(\Psi + \Theta) = 0, \\ (\Psi + \Theta)(\Psi + \Theta) = \Psi + \Theta, \quad (\Phi - \Theta)(\Phi - \Theta) = \Phi - \Theta. \end{cases} \tag{2.4}$$

Theorem 2.1. *If $\dot{\nabla}$ is a linear connection on a framed φ-manifold, $(M, \varphi, \xi_a, \eta_a)$, $a = \overline{1, s}$, then the family of adapted connections on M is given by*

$$\nabla = \dot{\nabla} + P, \tag{2.5}$$

where P is a tensor field of type (1,2) on M, defined by $P(X, Y) = P_X(Y)$, $X, Y \in \chi(M)$, where P_X is a tensor field of type (1,1) given by

$$P_X = \frac{1}{2}(\dot{\nabla}_X \varphi)\varphi - \frac{1}{2}\sum_{a=1}^{s}(\dot{\nabla}_X \xi_a) \otimes \eta_a - \frac{1}{4}\sum_{a=1}^{s} \xi_a \otimes [i_{\varphi X} d\eta_a \circ \varphi + i_X d\eta_a]$$
$$-\frac{1}{2}\varphi X \otimes \sum_{a=1}^{s} \eta_a + \frac{1}{2}\sum_{a=1}^{s}(\dot{\nabla}_X \eta_a) \otimes \xi_a + \frac{1}{2}\sum_{a,b=1}^{s} \eta_a(\dot{\nabla}_X \xi_b)\xi_a \otimes \eta_b + (\Phi - \Theta)Q_X,$$

where Q is an arbitrary tensor field of type (1,2).

Proof. Let ∇ be an adapted connection on M, given by $\nabla_X = \dot{\nabla}_X + P_X$, $X \in \chi(M)$. From (2.2) one obtains

$$(2.6)\qquad \begin{cases} P_X(\xi_a) = -\frac{1}{2}\varphi X - \frac{1}{4}\sum\limits_{b=1}^{s} i_X d\eta_b(\xi_a)\xi_b - \dot{\nabla}_X\xi_a, \quad a = \overline{1,s}, \\ P_X\varphi - \varphi P_X = -\dot{\nabla}_X\varphi - \dfrac{1}{2}hX \otimes \sum\limits_{a=1}^{s}\eta_a \\ +\dfrac{1}{4}\sum\limits_{a=1}^{s}\xi_a \otimes [i_{\varphi X}d\eta_a - i_X d\eta_a \circ \varphi - \sum\limits_{b=1}^{s}(i_{\varphi X}d\eta_a(\xi_b))\eta_b]. \end{cases}$$

From this equations one obtains, after a straightforward computation, it follows

$$(2.7)\qquad P_X + \varphi P_X\varphi = (\dot{\nabla}_X\varphi)\varphi - \sum_{a=1}^{s}(\dot{\nabla}_X\xi_a) \otimes \eta_a - \frac{1}{2}\varphi X \otimes \sum_{a=1}^{s}\eta_a$$
$$-\frac{1}{4}\sum_{a=1}^{s}\xi_a \otimes [i_{\varphi X}d\eta_a \circ \varphi + i_X d\eta_a].$$

This can be written

$$(2.8)\qquad \Psi P_X = \frac{1}{2}A_X,$$

where A_X denotes the right side of (2.7).

Using (2.4) we have

$$(2.9)\qquad 2\Theta\Psi P_X = \Theta P_X = \Theta A_X.$$

From (2.8) and (2.9) one obtains

$$(2.10)\qquad (\Psi + \Theta)P_X = \frac{1}{2}A_X + \Theta A_X = B_X.$$

Using again (2.4) the expression of P_X arises, as follows,

$$P_X = B_X + (\Phi - \Theta)Q_X,$$

where Q is an arbitrary tensor field of type (1,2) and

$$B_X = A_X - \frac{1}{2}hA_X \circ h$$

$$= \frac{1}{2}(\dot{\nabla}_X\varphi)\varphi - \frac{1}{2}\sum_{a=1}^{s}(\dot{\nabla}_X\xi_a)\otimes\eta_a - \frac{1}{4}\sum_{a=1}^{s}\xi_a\otimes[i_{\varphi X}d\eta_a\circ\varphi + i_X d\eta_a]$$

$$-\frac{1}{2}\varphi X\otimes\sum_{a=1}^{s}\eta_a + \frac{1}{2}\sum_{a=1}^{s}(\dot{\nabla}_X\eta_a)\otimes\xi_a + \frac{1}{2}\sum_{a,b=1}^{s}\eta_a(\dot{\nabla}_X\xi_b)\xi_a\otimes\eta_b.$$

□

Remark 2.2. If $\dot{\nabla}$ is an adapted connection on M then the family of adapted connections on M is given by

$$\nabla_X = \dot{\nabla}_X + (\Phi - \Theta)Q_X,$$

where Q is an arbitrary tensor field of type (1,2).

3. Adapted connections and normal framed φ-manifolds. In [4] it is proved the following result

Theorem 3.1. *An almost contact manifold (M, φ, ξ, η) is normal if and only if there exists a torsion free adapted connection on M.*

The main result in this section is a similar statement for a framed φ-manifold.

In the following, let ∇ be an adapted connection on the framed φ-manifold, $(M, \varphi, \xi_a, \eta_a)$, $a = \overline{1, s}$, with the torsion tensor

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \tag{3.1}$$

for any $X, Y \in \chi(M)$.

Let us recall that the framed φ-manifold M is called normal if the tensor field S, defined by (1.6), vanishes.

According to (2.2) we have that

$$S_X = T_X + \varphi T_X\circ\varphi + \varphi T_{\varphi X} - T_{\varphi X}\circ\varphi \tag{3.2}$$

$$+\frac{1}{2}\sum_{a=1}^{s}[i_{\varphi^2 X}d\eta_a\circ\varphi^2 - i_X d\eta_a]\otimes\xi_a,$$

for any $X \in \chi(M)$, where $S_X(Y) = S(X, Y)$ and $T_X(Y) = T(X, Y)$, for any $Y \in \chi(M)$.

Proposition 3.2. *If ∇ is an adapted connection on the framed φ-manifold, $(M, \varphi, \xi_a, \eta_a)$, $a = \overline{1, s}$, we have $vS_X = 2vT_X$, for any $X \in \chi(M)$.*

Proof. From (3.1) and (2.2) one obtains

$$vT(X,Y) = \sum_{a=1}^{s} \eta_a(T(X,Y))\xi_a = \sum_{a=1}^{s} \eta_a(\nabla_X Y - \nabla_Y X - [X,Y])\xi_a$$
$$= \sum_{a=1}^{s} [d\eta_a(X,Y) - (\nabla_X \eta_a)(Y) + (\nabla_Y \eta_a)(X)]\xi_a$$
$$= \sum_{a=1}^{s} [d\eta_a(X,Y) - \frac{1}{2} d\eta_a(X,Y) - \frac{1}{2} d\eta_a(\varphi X, \varphi Y)]\xi_a$$
$$= \frac{1}{2} \sum_{a=1}^{s} [d\eta_a(X,Y) - d\eta_a(\varphi X, \varphi Y)]\xi_a,$$

for any $X, Y \in \chi(M)$.

On the other hand

$$vS(X,Y) = \sum_{a=1}^{s} \eta_a(S(X,Y))\xi_a = \sum_{a=1}^{s} \{\eta_a([\varphi X, \varphi Y]) + d\eta_a(X,Y)\}\xi_a$$
$$= \sum_{a=1}^{s} [d\eta_a(X,Y) - d\eta_a(\varphi X, \varphi Y)]\xi_a = 2vT(X,Y),$$

for any $X, Y \in \chi(M)$. □

Now, using (3.2), it is easy to see that

$$hS_X = hT_X + \varphi T_X \circ \varphi + \varphi T_{\varphi X} - hT_{\varphi X} \circ \varphi. \tag{3.3}$$

From (3.3) and Proposition 3.2 one obtains

Proposition 3.3. *If on a framed φ-manifold, $(M, \varphi, \xi_a, \eta_a)$, there exists a torsion free adapted connection, then M is normal.*

Now, in order to obtain the expression of hS_X, we use that $\varphi \otimes \varphi = h \otimes h - 2(\Phi - \Theta)$, and we have

$$(\varphi T_X \circ \varphi)(Y) = (hT_X - \sum_{a=1}^{s} hT_X(\xi_a) \otimes \eta_a)(Y) - 2(\Phi - \Theta)_Y T_X, \tag{3.4}$$

for any $X, Y \in \chi(M)$, where $(\Phi-\Theta)_Y$ is a tensor field of type (2,1), defined by $(\Phi-\Theta)_Y(X) = (\Phi-\Theta)(Y, X)$.

Using the properties of the projector h, from (3.4), it follows, after a straightforward computation,

$$(hT_{\varphi X} \circ \varphi)(Y) = (-hT_X + \sum_{a=1}^{s} hT_X(\xi_a) \otimes \eta_a)(Y) \tag{3.5}$$

$$-(\sum_{a=1}^{s} hT_Y(\xi_a) \otimes \eta_a)(X) + 2(\Phi-\Theta)_Y(\varphi T_{\varphi X}) - 2(\Phi-\Theta)_X T_{hY} + hT_{vY}(vX).$$

Since T is skew-symmetric we can write

$$2(hT_{\varphi X} \circ \varphi)(Y) = (hT_{\varphi X} \circ \varphi)(Y) - (hT_{\varphi Y} \circ \varphi)(X). \tag{3.6}$$

From (3.5) and (3.6) one obtains

$$(hT_{\varphi X} \circ \varphi)(Y) = (-hT_X + \sum_{a=1}^{s} hT_X(\xi_a) \otimes \eta_a)(Y) - hT_{vX}(vY) \tag{3.7}$$

$$-(\sum_{a=1}^{s} hT_Y(\xi_a) \otimes \eta_a)(X) - (\Phi-\Theta)_X(T_{hY} + \varphi T_{\varphi Y}) + (\Phi-\Theta)_Y(T_{hX} + \varphi T_{\varphi X}).$$

Finally, from (3.3), (3.4) and (3.7), we have

$$hS_X(Y) = (4hT_X - 2\sum_{a=1}^{s} hT_X(\xi_a) \otimes \eta_a)(Y) \tag{3.8}$$

$$+2(\sum_{a=1}^{s} hT_Y(\xi_a) \otimes \eta_a)(X) + (\Phi-\Theta)_X(2T_Y + T_{hY} + \varphi T_{\varphi Y})$$

$$-(\Phi-\Theta)_Y(2T_X + T_{hX} + \varphi T_{\varphi X}) + hT_{vX}(vY).$$

Also, one obtains

$$hS_X(\xi_a) = 2hT_X(\xi_a) - hT_{vX}(\xi_a) + 2(\Phi-\Theta)_X T_{\xi_a}, \tag{3.9}$$

for any $a = \overline{1, s}$.

From (3.8) and (3.9) it follows

$$(hS_X + hS_X \circ v + hS_{vX} + hS_{vX} \circ v)(Y) = 4hT_X(Y) \tag{3.10}$$

$$-(\Phi-\Theta)_Y(3T_X+\varphi T_{\varphi X}+T_{vX})+(\Phi-\Theta)_X(3T_Y+\varphi T_{\varphi Y}+T_{vY}),$$

for any $X, Y \in \chi(M)$.

If the framed φ-manifold M is normal we obtain, from Proposition 3.2 and (3.10)

$$\begin{cases} vT_X(Y)=0, \\ hT_X(Y)-\frac{1}{4}(\Phi-\Theta)_Y(3T_X+\varphi T_{\varphi X}+T_{vX}) \\ +\frac{1}{4}(\Phi-\Theta)_X(3T_Y+\varphi T_{\varphi Y}+T_{vY})=0, \end{cases}$$

for any $X, Y \in \chi(M)$.

Let us consider the connection $\widetilde{\nabla}$ on M, given by

$$\widetilde{\nabla}_X Y=\nabla_X Y-\frac{1}{4}(\Phi-\Theta)_Y(3T_X+\varphi T_{\varphi X}+T_{vX}),$$

for any $X, Y \in \chi(M)$. It is easy to verify that the torsion of $\widetilde{\nabla}$ vanishes and, from Remark 2.2, $\widetilde{\nabla}$ is an adapted connection on M.

Taking account of the properties of the connection $\widetilde{\nabla}$ and the Proposition 3.3 we can state the following

Theorem 3.4. *A framed φ-manifold, $(M, \varphi, \xi_a, \eta_a)$, $a=\overline{1,s}$, is normal if and only if there exists a torsion free adapted connection on M.*

REFERENCES

1. Blair, D.E. – *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhäuser Boston, Progress in Mathematics, Vol. 203(2002), pp. x, 260.
2. Bokan, N. – *C-holomorphically projective transformations in a normal almost contact manifold*, Tensor, NS 33(1979), 189-199.
3. Goldberg, S.I.; Yano, K. – *On normal globally framed $f-$manifold*, Tôhoku Math. Journ. 22(1970), 362-370.
4. Matzeu, P. – *Su una caratterizzazione delle varieta' normali*, Boll. U.M.I., Algebra e Geometria, Serie VI, Vol. IV-D, 1(1985), 153-160.
5. Matzeu, P.; Oproiu, V. – *C-projective curvature of normal almost-contact manifolds*, Rend. Sem. Mat. Univers. Politecn. Torino, Vol. 45, 2(1987), 41-58.

6. OPROIU, V. – *Some remarks on the almost cocomplex connections*, Rev. Roum. Math. Pures Appl. 16(1971), 383-393.

7. STOJKOVIC-BOKAN, N. – *The connection between H-flat and CH-flat curves and their corresponding transformations*, Tensor, NS 39(1982), 67-72.

8. DI TERLIZZI, L.; PASTORE, A.M. – *Some results on $\mathcal{K}$-manifolds*, Balkan Journal of Geometry and Its Applications, 7(1)(2002), 43-62.

Received: 25.XI.2005

Department of Mathematics,
"Gh. Asachi" Technical University,
Carol I Blvd., 11, 700506 Iaşi,
ROMÂNIA,
dfetcu@math.tuiasi.ro