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ON SOME PROPERTIES OF THE GOULD TYPE INTEGRAL WITH RESPECT TO A MULTISUBMEASURE

BY

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Abstract. This paper continues our work [5] concerning a Gould type integral of a bounded, real valued function with respect to a multisubmeasure taking values in $\mathcal{P}_{bf}(X)$, X being a Banach space and $\mathcal{P}_{bf}(X)$ the family of all non-empty, bounded, closed subsets of X. We prove that the integral preserves the regularity as well as the o-continuity of the multisubmeasure. Also, some results concerning sequences of μ-integrable functions are obtained.

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1. Terminology and notations. Let T be an abstract, nonvoid set, $\mathcal{A}$ an algebra of subsets of T, X a real Banach space, $\mathcal{P}_f(X)$ the family of all nonempty, closed subsets of X, $\mathcal{P}_{bf}(X)$ the family of all nonvoid, closed, bounded subsets of X and $\mathcal{P}_{kc}(X)$ the family of all nonempty, compact, convex subsets of X.

Let also $f : T \to R$ be a real valued, bounded function.

By " $\overset{\bullet}{+}$ " we mean the Minkowski addition on $\mathcal{P}_f(X)$, that is:

$$M \overset{\bullet}{+} N = \overline{M + N}, \text{ for every } M, N \in \mathcal{P}_f(X). \tag{1}$$

Let h be the Hausdorff pseudometric on $\mathcal{P}_f(X)$. It is well-known that h becomes a metric on $\mathcal{P}_{bf}(X)$, and $(\mathcal{P}_{bf}(X), h), (\mathcal{P}_{kc}(X), h)$ are complete metric spaces. We know that $h(M,N) = \max\{e(M,N), e(N,M)\}$, where $e(M,N) = \sup\limits_{x\in M} d(x,N), d(x,N)$ being the distance from x to N with respect to the metric induced by the norm of X. We denote $|M| = h(M, \{0\})$, for every $M \in \mathcal{P}_f(X)$, where 0 is the origin of X.

Definition 1.1. I) *A partition of T is a finite family $P = \{A_i\}_{i=\overline{1,n}} \subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset, i \neq j$ and $\bigcup\limits_{i=1}^{n} A_i = T$.*

II) *Let $P = \{A_i\}_{i=\overline{1,n}}$ and $P' = \{B_j\}_{j=\overline{1,m}}$ be two partitions of T. P' is said to be finer than P (denoted by $P \leq P'$) if for every $j = \overline{1,m}$, there exists $i_j = \overline{1,n}$ so that $B_j \subseteq A_{i_j}$.*

III) *The common refinement of two partitions $P = \{A_i\}_{i=\overline{1,n}}$ and $P' = \{B_j\}_{j=\overline{1,m}}$ is the partition $P \wedge P' = \{A_i \cap B_j\}_{\substack{i=\overline{1,n}\\ j=\overline{1,m}}}$.*

Obviously, $P \wedge P' \geq P$ and $P \wedge P' \geq P'$.

Definition 1.2. *Let $\mu : \mathcal{A} \mapsto \mathcal{P}_f(X)$ be a multivalued set function.*

I) *μ is said to be exhaustive (with respect to h) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every disjoint sequence $(A_n)_n \subset \mathcal{A}$.*

II) *μ is said to be o-continuous (with respect to h) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$ for every sequence $(A_n)_n \subset \mathcal{A}$, with $A_n \searrow \emptyset$ (that is, $A_n \supseteq A_{n+1}$, for every $n \in N^*$ and $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$).*

III) *μ is said to be increasing convergent (with respect to h) if $\lim\limits_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every sequence $(A_n)_n \subset \mathcal{A}$, with $A_n \nearrow A \in \mathcal{A}$ (that is, $A_n \subseteq A_{n+1}$, for every $n \in N^*$ and $\bigcup\limits_{n=1}^{\infty} A_n = A$).*

IV) *μ is said to be $h-\sigma$-subadditive if $|\mu(A)| \leq \sum\limits_{n=1}^{\infty} |\mu(A_n)|$, for every (disjoint) sequence $(A_n)_n \subset \mathcal{A}$, with $A = \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{A}$.*

V) *μ is said to be absolutely continuous with respect to another multivalued set function $\nu : \mathcal{A} \mapsto \mathcal{P}_f(X)$, denoted by $\mu \ll \nu$, if $\nu(A) = \{0\}$ implies $\mu(A) = \{0\}$, for every $A \in \mathcal{A}$.*

If $\mu : \mathcal{A} \mapsto \mathcal{P}_f(X)$ is a multivalued set function, then we recall from [3] the following:

Definition 1.3. I) *μ is said to be a multimeasure if:*

a) $\mu(\emptyset) = \{0\}$ *and*

b) $\mu(A \cup B) = \mu(A) \overset{\bullet}{+} \mu(B)$, *for every* $A, B \in \mathcal{A}$, *with* $A \cap B = \emptyset$, *that is,* μ *is finite additive.*

II) μ *is said to be a multisubmeasure if:*

a) $\mu(\emptyset) = \{0\}$,

b) $\mu(A \cup B) \subseteq \mu(A) \overset{\bullet}{+} \mu(B)$, *for every* $A, B \in \mathcal{A}$, *with* $A \cap B = \emptyset$ *and*

c) $\mu(A) \subseteq \mu(B)$, *for every* $A, B \in \mathcal{A}$, *with* $A \subseteq B$ (*that is,* μ *is monotone increasing on* $\mathcal{A}$).

It is easy to observe that the last condition b) is equivalent to the condition $b')$ $\mu(A \cup B) \subseteq \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{A}$.

All over this paper we assume that $\mu : \mathcal{A} \mapsto \mathcal{P}_{bf}(X)$ is a multisubmeasure. Let us also consider the following set functions associated to μ:

$\overline{\mu}$ defined by

$$\overline{\mu}(A) = \sup\{\sum_{i=1}^{n} |\mu(A_i)|\}, \text{ for every } A \in \mathcal{A}, \tag{2}$$

where supremum is extended over all finite partitions $\{A_i\}_{i=\overline{1,n}}$ of A. $\overline{\mu}$ is said to be *the variation* of μ;

$\widehat{\mu}$ defined by

$$\widehat{\mu}(A) = |\mu(A)|, \text{ for every } A \in \mathcal{A} \tag{3}$$

and $\widetilde{\mu}$ defined by

$$\widetilde{\mu}(A) = \inf\{\overline{\mu}(B); A \subseteq B, B \in \mathcal{A}\}, \text{ for every } A \subseteq T. \tag{4}$$

We have observed in [3] that $\overline{\mu}$ is a finite additive set function on $\mathcal{A}$ and $\widehat{\mu}$ is a submeasure in Drewnowski's sense [2] on $\mathcal{A}$. We also note that $\widetilde{\mu}(A) = \overline{\mu}(A)$, for every $A \in \mathcal{A}$.

Definition 1.4. *We say that a property* (P) *holds* μ*-almost everywhere if the property* (P) *is valid on* $T \backslash A$, *with* $\widetilde{\mu}(A) = 0$.

Definition 1.5. *We say that a sequence of functions* $(f_n)_n$, *where* $f_n : T \to R$ *for every* $n \in N$, *is convergent in submeasure to* f (*denoted by* $f_n \xrightarrow{\mu} f$) *if for every* $\delta > 0$, $\lim_{n\to\infty} \widetilde{\mu}(B_n(\delta)) = 0$, *where*

$$B_n(\delta) = \{t \in T; |f_n(t) - f(t)| \geq \delta\}.$$

Definition 1.6. *A multisubmeasure* $\mu : \mathcal{A} \mapsto \mathcal{P}_{bf}(X)$ *is said to be of finite variation if* $\overline{\mu}(T) < \infty$.

We also recall the following (see [1], [3] and [7]):

If, particularly, T is a locally compact Hausdorff space, let $\mathcal{B}$ (respectively, $\mathcal{B}'$) be the borelian δ-ring (respectively, σ-ring) generated by the compact subsets of T and $\mathcal{B}_0$ (respectively $\mathcal{B}'_0$) be the Baire δ-ring (respectively, σ-ring) generated by the compact subsets of T which are G_δ (that is, countable intersection of open sets).

If T is a compact space, then $\mathcal{B}'$ becomes a σ-algebra, and if T is a metrizable compact space, then T is G_δ, so, $\mathcal{B}'_0$ becomes a σ-algebra, too.

In the assumption for T to be a locally compact space, we have introduced in [3] the following notions:

Definition 1.7. i) *A set* $A \in \mathcal{A}$ *is said to be* R'_l*-regular* (*with respect to* μ) *if for every* $\varepsilon > 0$, *there exists a compact set* $K \in \mathcal{A}, K \subset A$ *such that* $|\mu(B)| < \varepsilon$, *for every* $B \in \mathcal{A}, B \subset A \backslash K$.

ii) *A set* $A \in \mathcal{A}$ *is said to be* R'_r*-regular* (*with respect to* μ) *if for every* $\varepsilon > 0$, *there exists an open set* $D \in \mathcal{A}, A \subset D$ *such that* $|\mu(B)| < \varepsilon$, *for every* $B \in \mathcal{A}, B \subset D \backslash A$.

iii) *A set* $A \in \mathcal{A}$ *is said to be* R_l*-regular* (*with respect to* μ) *if for every* $\varepsilon > 0$, *there exists a compact set* $K \in \mathcal{A}, K \subset A$ *such that* $h(\mu(B), \mu(A)) < \varepsilon$, *for every* $B \in \mathcal{A}, K \subset B \subset A$.

iv) *A set* $A \in \mathcal{A}$ *is said to be* R_r*-regular* (*with respect to* μ) *if for every* $\varepsilon > 0$, *there exists an open set* $D \in \mathcal{A}, A \subset D$ *such that* $h(\mu(B), \mu(A)) < \varepsilon$, *for every* $B \in \mathcal{A}, A \subset B \subset D$.

Definition 1.8. μ *is said to be* R'_l*-regular* (R'_r*-regular,* R_l*-regular,* R_r*-regular, respectively*) *on* $\mathcal{A}$ *if every set* $A \in \mathcal{A}$ *is* R'_l*-regular* (R'_r*-regular,* R_l*-regular,* R_r*-regular, respectively*).

We remind from [3] and [4] the following:

Theorem 1.9. i) R'_l*-regularity is equivalent to* R_l*-regularity on* $\mathcal{A}$ (*for multisubmeasures of finite variation*);

ii) R'_l*-regularity is equivalent to o-continuity on* $\mathcal{B}'_0$*;*

iii) *R'_l-regularity implies o-continuity on $\mathcal{B}'$;*

iv) *R'_l-regularity is equivalent to R'_r-regularity on $\mathcal{B}$.*

2. The Gould type integral with respect to a multisubmeasure. In the sequel, without any special assumptions, T will be an abstract nonvoid set, $\mu : \mathcal{A} \mapsto \mathcal{P}_{bf}(X)$ a multisubmeasure of finite variation and $f : T \to R$ a real valued, bounded function.

Definition 2.1. I) *f is said to be $\widetilde{\mu}$-totally-measurable on $(T, \mathcal{A}, \mu)$ if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of T such that:*

i) *$\widetilde{\mu}(A_0) < \varepsilon$ and*

ii) *$\sup\limits_{t,s\in A_i} |f(t) - f(s)| = osc(f, A_i) < \varepsilon$, for every $i = \overline{1,n}$.*

II) *f is said to be $\widetilde{\mu}$-totally-measurable on $B \in \mathcal{A}$ if its restriction $f|_B$ of f to B is $\widetilde{\mu}$-totally measurable on $(B, \mathcal{A}_B, \mu_B)$, where $\mu_B = \mu|_{\mathcal{A}_B}$ and $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$.*

Let $\sigma(P) = {}^\bullet\sum\limits_{i=1}^{n} f(t_i)\mu(A_i)$, for every partition $P = \{A_i\}_{i=\overline{1,n}}$ of T and every $t_i \in A_i, i = \overline{1,n}$.

Definition 2.2. I) *f is said to be μ-integrable on T if the net $(\sigma(P))_{P\in(\mathcal{P},\leq)}$ is convergent in $(\mathcal{P}_{bf}(X), h)$ (where $\mathcal{P}$ is the set of all partitions of T and "$\leq$" the order relation on $\mathcal{P}$ given in the definition 1.1 II), for every choice of the points $t_i \in A_i$; its limit is called the integral of f on T with respect to the multisubmeasure μ, denoted by $\int_T f d\mu$.*

Hence, f is μ-integrable on T if there exists a set $I \in \mathcal{P}_{bf}(X)$ such that for every $\varepsilon > 0$ there exists a partition P_ε of T so that for every other partition $P = \{A_i\}_{i=\overline{1,n}}$, with $P \geq P_\varepsilon$ and every choice of points $t_i \in A_i, i = \overline{1,n}$, we have

$$h(\sigma(P), I) < \varepsilon. \tag{5}$$

II) *If $B \in \mathcal{A}, f$ is said to be μ-integrable on B if the restriction $f|_B$ of f to B is μ-integrable on $(B, \mathcal{A}_B, \mu_B)$.*

Obviously, if there exists, the integral is unique.

Let us note that, all over this paper, if we deal with a $\mathcal{P}_{kc}(X)$-valued multisubmeasure, then the Minkowski addition " $\overset{\bullet}{+}$ " changes in fact into " $+$ ".

In the sequel, we shall use the following results that we have established in [5]:

Theorem 2.3. *If f is μ-integrable on T and $\alpha \in R$, then αf is μ-integrable on T and*

$$\int_T \alpha f d\mu = \alpha \int_T f d\mu. \tag{6}$$

Theorem 2.4. *Suppose that $\mu : \mathcal{A} \mapsto \mathcal{P}_{kc}(X)$ and $f, g : T \mapsto R$ are two bounded, μ-integrable functions on T so that $f(t) \cdot g(t) \geq 0$, for every $t \in T$. Then $f + g$ is μ-integrable on T and*

$$\int_T (f+g) d\mu = \int_T f d\mu + \int_T g d\mu. \tag{7}$$

Theorem 2.5. *Let $f, g : T \mapsto R$ be two bounded, μ-integrable functions. Then:*
i)

$$h\left(\int_T f d\mu, \int_T g d\mu\right) \leq \sup_{t \in T} |f(t) - g(t)| \cdot \overline{\mu}(T) \text{ and} \tag{8}$$

ii)

$$\left| \int_T f d\mu \right| \leq \sup_{t \in T} |f(t)| \cdot \overline{\mu}(T). \tag{9}$$

In the sequel, we remind from [5] an important class of functions which are μ-integrable, where $\mu : \mathcal{A} \to P_{kc}(R)$ is the multisubmeasure induced by a submeasure $\nu : \mathcal{A} \to R_+$ of finite variation, defined by:
$\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{A}$.

Theorem 2.6. *Let $\mu : \mathcal{A} \to P_{kc}(R)$ be the multisubmeasure induced by a submeasure $\nu : \mathcal{A} \to R_+$ of finite variation and $f : T \to R_+$ be a* *bounded, $\widetilde{\mu}$-totally-measurable function on T. Then f is μ-integrable on T and, moreover,*

$$\int_T f\,d\mu = [0, \int_T fd\overline{\nu}]$$

($\int_T fd\overline{\nu}$ represents the Gould integral [6] of f with respect to the variation $\overline{\nu}$ of ν, which is a finite additive set function on $\mathcal{A}$).

Theorem 2.7. *Let $B, C \in \mathcal{A}$, with $B \cap C = \emptyset$. If f is μ-integrable on B and C, then f is μ-integrable on $B \cup C$, and, moreover,*

$$\int_{B\cup C} fd\mu = \int_B fd\mu \stackrel{\bullet}{+} \int_C fd\mu. \tag{10}$$

Theorem 2.8. *Let $B, C \in \mathcal{A}$, with $B \subseteq C$. Then*

$$\int_B fd\mu \subseteq \int_C fd\mu \tag{11}$$

supposing the existence of both integrals.

Theorem 2.9. *If $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$, $f : T \to R$ is μ-integrable on T and $B \in \mathcal{A}$ is an arbitrarily set, then f is μ-integrable on B.*

Corollary 2.10. *If $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$ and f is μ-integrable on T, then:*

i) *$M : \mathcal{A} \to \mathcal{P}_{kc}(X)$, defined by $M(A) = \int_A fd\mu$, for every $A \in \mathcal{A}$, is a monotone multimeasure;*

ii) *$M \ll \mu$.*

We establish in the following some properties of the integral.

Theorem 2.11. *Let $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$. Then:*

i) *If μ is o-continuous (increasing convergent, respectively) on $\mathcal{A}$, then M is o-continuous (increasing convergent, respectively) on $\mathcal{A}$;*

ii) *If μ is h-σ-subadditive, then M is a h-multimeasure, that is, $M(\emptyset) = 0$ and $\lim\limits_{n\to\infty} h(M(A), \sum\limits_{k=1}^{n} M(A_k)) = 0$, for every disjoint sequence of sets $(A_n)_{n\in N^*} \subset \mathcal{A}$, with $A = \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{A}$.*

Proof. i) Let us suppose that μ is o-continuous on $\mathcal{A}$. Then (see [3]), $\overline{\mu}$ is o-continuous on $\mathcal{A}$. Let $(A_n)_n \subset \mathcal{A}$, $A_n \searrow \emptyset$. From the o-continuity of $\overline{\mu}$ on $\mathcal{A}$, we get that for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in N$ such that $\overline{\mu}(A_n) < \frac{\varepsilon}{\alpha}$, for every $n \geq n_0$, where $\alpha = \sup\limits_{t\in T} |f(t)|$. The inequality $|M(A_n)| \leq \alpha \cdot \overline{\mu}(A_n)$, for every $n \in N^*$ yields $|M(A_n)| < \varepsilon$, for every $n \geq n_0$, that is, M is o-continuous on $\mathcal{A}$.

We suppose now that μ is increasing convergent on $\mathcal{A}$. Let $(A_n)_n \subset \mathcal{A}$ be such that $A_n \nearrow A \in \mathcal{A}$.

We shall prove that for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in N$ so that $h(M(A_n), M(A)) < \varepsilon$, for every $n \geq n_0$. From theorem 2.5 ii) we get that:

$$\begin{aligned} h(M(A_n), M(A)) &= h(\int_{A_n} f d\mu, \int_A f d\mu) \\ &= h(\int_{A_n} f d\mu, \int_{A_n} f d\mu + \int_{A\setminus A_n} f d\mu) \\ &= |\int_{A\setminus A_n} f d\mu| \leq \alpha \cdot \overline{\mu}(A\setminus A_n) = \alpha \cdot (\overline{\mu}(A) - \overline{\mu}(A_n)). \end{aligned}$$

We prove in the sequel that there exists $n_0(\varepsilon) \in N$ so that $\overline{\mu}(A) - \overline{\mu}(A_n) < \frac{\varepsilon}{\alpha}$, for every $n \geq n_0$.

Indeed, if $\{B_i\}_{i=\overline{1,m}} \subset \mathcal{A}$ is an arbitrary partition of A, then $\sum\limits_{i=1}^{m} |\mu(B_i)| \leq \sum\limits_{i=1}^{m} |\mu(B_i \cap A_n)| + \sum\limits_{i=1}^{m} h(\mu(B_i), \mu(B_i \cap A_n))$, for every $n \in N^*$.

Since $\{B_i \cap A_n\}_{i=\overline{1,m}}$ is a partition of A_n, for every $n \in N^*$, we have

$$\sum_{i=1}^{m} |\mu(B_i)| \leq \overline{\mu}(A_n) + \sum_{i=1}^{m} h(\mu(B_i), \mu(B_i \cap A_n)).$$

But μ is increasing convergent and $B_i \cap A_n \nearrow B_i \cap A = B_i$, for every $i = \overline{1,m}$; therefore, for every $i = \overline{1,m}$, there exists $n_0^i(\varepsilon) \in N$ so that for every $n \geq n_0^i$, $h(\mu(B_i), \mu(B_i \cap A_n)) < \frac{\varepsilon}{2^i \cdot \alpha}$. Consequently,

$$\sum_{i=1}^{m} |\mu(B_i)| \leq \overline{\mu}(A_n) + \sum_{i=1}^{m} \frac{\varepsilon}{2^i \cdot \alpha} < \overline{\mu}(A_n) + \frac{\varepsilon}{\alpha},$$

$$\text{for every } n \geq n_0 = \max\{n_0^i\}_{i=\overline{1,m}}.$$

Taking the supremum on the left side we get that $\overline{\mu}(A) \leq \overline{\mu}(A_n) + \frac{\varepsilon}{\alpha}$ and, finally, $h(M(A_n), M(A)) < \alpha \cdot \frac{\varepsilon}{\alpha} = \varepsilon$, for every $n \geq n_0$, as claimed.

ii) Let $(A_n)_n \subset \mathcal{A}$ be a disjoint sequence of sets, with $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. Because μ is $h-\sigma$-subadditive, then, equivalently, it is o-continuous (see [3]), hence M is o-continuous, too. Since $B_n = \bigcup_{k=n+1}^{\infty} A_n \searrow \emptyset$ and $(B_n)_{n\in N^*} \subset \mathcal{A}$, then for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in N$ such that $|M(B_n)| < \varepsilon$, for every $n \geq n_0$.

Because M is finite additive we get that $M(A)=\sum_{k=1}^{n} M(A_k)+M(B_n)$, which implies

$h(M(A), \sum_{k=1}^{n} M(A_k))=h(\sum_{k=1}^{n} M(A_k)+M(B_n), \sum_{k=1}^{n} M(A_k)) =$
$|M(B_n)| < \varepsilon$, for every $n \geq n_0$.

This means that M is indeed a h-multimeasure, since, obviously, $M(\emptyset) = \{0\}$. □

Theorem 2.12. *Let T be a locally compact Hausdorff space and $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$. Then:*

i) *If μ is R_l'-regular on $\mathcal{A}$, then M is R_l'-regular on $\mathcal{A}$. Also, if $\inf_{t\in T} f(t) = m > 0$ and M is R_l'-regular on $\mathcal{A}$, then μ is R_l'-regular on $\mathcal{A}$;*

ii) *If $\inf_{t\in T} f(t) = m > 0$, then μ is R_l-regular on $\mathcal{A}$ if and only if M is R_l-regular on $\mathcal{A}$;*

iii) *If T is a compact space and $\inf_{t\in T} f(t) = m > 0$, then μ is R_r'-regular on $\mathcal{B}$ if and only if M is R_r'-regular on $\mathcal{B}$;*

iv) *If T is a compact space, $\inf_{t\in T} f(t) = m > 0$ and M is R_r-regular on $\mathcal{B}$, then μ is R_r-regular on $\mathcal{B}$.*

Proof. i) If μ is R_l'-regular on $\mathcal{A}$ then, equivalently, $\overline{\mu}$ is R_l'-regular on $\mathcal{A}$ (see [3]). One can easily get then that M is R_l'-regular on $\mathcal{A}$, using the inequality $|M(A)| \leq \alpha \cdot \overline{\mu}(A)$, which holds for every $A \in \mathcal{A}$.

In what concerns the converse, we suppose that, additionally, $\inf_{t\in T} f(t) = m > 0$. Let $A \in \mathcal{A}$. Because M is R_l'-regular in A, then for every $\varepsilon > 0$, there exists a compact set $K \in \mathcal{A}$ so that $K \subset A$ and $|M(B)| < \frac{\varepsilon}{2}$, for every $B \in \mathcal{A}$, with $B \subset A \backslash K$.

But f is μ-integrable on T, so it is μ-integrable on an arbitrary, fixed set $B \in \mathcal{A}$, with $B \subset A \backslash K$. Consequently, there is a partition $P_\varepsilon = \{B_i\}_{i=\overline{1,p}} \in \mathcal{P}_B$ such that

$$h(\int_B f d\mu, \sum_{i=1}^{p} f(t_i)\mu(B_i)) < \frac{\varepsilon}{2}, \text{ for every } t_i \in B_i,\ i = \overline{1,p}.$$

Therefore, $|\sum\limits_{i=1}^{p} f(t_i)\mu(B_i)| \leq h(\int_B f d\mu, \sum\limits_{i=1}^{p} f(t_i)\mu(B_i)) + |M(B)| < \varepsilon$.

But, on the other hand, $m\,\mu(B) \subseteq m \sum\limits_{i=1}^{p} \mu(B_i) \subseteq \sum\limits_{i=1}^{p} f(t_i)\mu(B_i)$, so, $|m\,\mu(B)| \leq |\sum\limits_{i=1}^{p} f(t_i)\mu(B_i)|$, which implies that $|m\,\mu(B)| < \varepsilon$, that is, $|\mu(B)| < \frac{\varepsilon}{m}$. This means μ is R'_l-regular in A.

So, if $\inf\limits_{t \in T} f(t) = m > 0$, then μ is R'_l-regular on $\mathcal{A}$ if and only if M is R'_l-regular on $\mathcal{A}$.

ii) We use the fact that M is also a multisubmeasure on $\mathcal{A}$ and R'_l-regularity is equivalent to R_l-regularity for multisubmeasures of finite variation.

iii) Since T is a compact space, then $\mathcal{A} = \mathcal{B}$ is an algebra. Because on $\mathcal{B}$ R'_l-regularity is equivalent to R'_r-regularity (see [3]), we have the conclusion.

iv) Let $A \in \mathcal{B}$ and $\varepsilon > 0$.

Because M is R_r-regular in A, then there exists an open set $D \in \mathcal{B}$, $A \subset D$, so that $h(M(A), M(B)) < \varepsilon$, for every $B \in \mathcal{B}$, with $A \subset B \subset D$.

But $h(M(A), M(B)) = h(M(A), M(A) + M(B \backslash A)) = |M(B \backslash A)|$ since $M : \mathcal{B} \to \mathcal{P}_{kc}(X)$. Therefore, $|M(B \backslash A)| < \varepsilon$, for every $B \in \mathcal{B}$, with $A \subset B \subset D$.

Now, let $B \in \mathcal{B}$, with $B \subset D \backslash A$. Then $B = B \backslash A = (A \cup B) \backslash A$ and, since $A \subset A \cup B \subset D$, it follows that $|M(B)| < \varepsilon$, that is, M is R'_r-regular on $\mathcal{B}$. From iii) we get that μ is R'_r-regular, hence it is R_r-regular on $\mathcal{B}$ (see [3]), as claimed. □

We point out now other properties of the integral.

Theorem 2.13. *Let $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$ and $f, g : T \mapsto R$ be two bounded functions on T, so that:*

i) *f is μ-integrable on T and*

ii) $f = g$ *μ-a.p.t.*

Then g is μ-integrable on T and $\int_T f d\mu = \int_T g d\mu$.

Proof. Let $\varepsilon > 0$ be arbitrarily. Since f is μ-integrable on T, there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}_T$ so that

$$h(\sigma(P), \int_T f d\mu) < \frac{\varepsilon}{2}, \text{ for every } P \in \mathcal{P}_T, \text{ with } P \geq P_\varepsilon.$$

Let $E \subset T$ be such that $f = g$ on $T \backslash E$ and $\widetilde{\mu}(E) = 0$. From the definition of $\widetilde{\mu}$, there exists a set $A \in \mathcal{A}$ so that $E \subseteq A$ and $\overline{\mu}(A) < \frac{\varepsilon}{4M}$, where $M = \max\{\sup\limits_{t \in T} |f(t)|, \sup\limits_{t \in T} |g(t)|\}$.

We consider the partitions $P_0 = \{A, T \backslash A\} \in \mathcal{P}_T$ and $P_0 \wedge P_\varepsilon = \{A \cap A_i, A_i \backslash A\}_{i=\overline{1,n}} \in \mathcal{P}_T$. Let also $P = \{B_j\}_{j=\overline{1,m}} \in \mathcal{P}_T$, with $P \geq P_0 \wedge P_\varepsilon$ and $t_j \in B_j, j = \overline{1,m}$ be arbitrarily.

Because $P \geq P_0 \wedge P_\varepsilon$, then for every $j = \overline{1,m}$, there exists $i_j = \overline{1,n}$ with $B_j \subset A \cap A_{i_j}$ or $B_j \subset A_{i_j} \backslash A$. Without any loss of generality, we suppose that $B_j \subset A \cap A_{i_j}$, for every $j = \overline{1,p}$ and $B_j \subset A_{i_j} \backslash A$, for every $j = \overline{p+1,m}$ (we may have only one of these situations).

We shall prove that $h(\int_T f d\mu, {}^\bullet\sum\limits_{j=1}^{m} g(t_j)\mu(B_j)) < \varepsilon$ (consequently, g is μ-integrable on T and $\int_T f d\mu = \int_T g d\mu$).

Indeed,

$$\begin{aligned} h(\int_T f d\mu, {}^\bullet\sum_{j=1}^{m} g(t_j)\mu(B_j)) &\leq h(\int_T f d\mu, {}^\bullet\sum_{j=1}^{m} f(t_j)\mu(B_j)) \\ &+ h({}^\bullet\sum_{j=1}^{m} f(t_j)\mu(B_j), {}^\bullet\sum_{j=1}^{m} g(t_j)\mu(B_j)) \\ &< \frac{\varepsilon}{2} + \sum_{j=1}^{m} |f(t_j) - g(t_j)| \cdot |\mu(B_j)|. \end{aligned}$$

Because $B_j \subset A \cap A_{i_j}$ it follows that $|\mu(B_j)| \leq |\mu(A \cap A_{i_j})|$, for every $j = \overline{1,p}$.

Similarly, $|\mu(B_j)| \leq |\mu(A_{i_j} \backslash A)|$, for every $j = \overline{p+1, m}$; therefore,

$$h(\int_T f d\mu, {}^{\bullet}\sum_{j=1}^{m} g(t_j)\mu(B_j)) < \frac{\varepsilon}{2} + \sum_{j=1}^{p} |f(t_j) - g(t_j)| \cdot |\mu(A \cap A_{i_j})|$$
$$+ \sum_{j=p+1}^{m} |f(t_j) - g(t_j)| \cdot |\mu(A_{i_j} \backslash A)|.$$

But $f = g$ on $T \backslash A$, so $f(t_j) = g(t_j)$, for every $t_j \in A_{i_j} \backslash A$, with $j = \overline{p+1, m}$. This implies that

$$h(\int_T f d\mu, {}^{\bullet}\sum_{j=1}^{m} g(t_j)\mu(B_j)) < \frac{\varepsilon}{2} + \sum_{j=1}^{p} |f(t_j) - g(t_j)| \cdot |\mu(A \cap A_{i_j})|$$
$$\leq \frac{\varepsilon}{2} + 2M \cdot \sum_{j=1}^{p} |\mu(A \cap A_{i_j})| \leq \frac{\varepsilon}{2} + 2M \cdot \overline{\mu}(A) < \varepsilon.$$

The proof is thus finished. □

Remark 2.14. i) If $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$, then $\mu(A) \subseteq \int_A d\mu$, supposing the existence of the integral. If $\mu : \mathcal{A} \mapsto \mathcal{P}_{kc}(X)$ and $m = \inf_{t \in T} f(t) > 0$, then $m \cdot \mu(A) \subseteq \int_A f d\mu$.

Indeed, for every $\varepsilon > 0$, there exists $P_\varepsilon = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}_A$ so that

$$h(\sigma(P), \int_A d\mu) < \varepsilon, \text{ for every } P \in \mathcal{P}_A, \text{ with } P \geq P_\varepsilon.$$

Particularly, $h({}^{\bullet}\sum_{i=1}^{n} \mu(A_i), \int_A d\mu) < \varepsilon$.

Therefore,

$$e(\mu(A), \int_A d\mu) \leq e(\mu(A), {}^{\bullet}\sum_{i=1}^{n} \mu(A_i)) + e({}^{\bullet}\sum_{i=1}^{n} \mu(A_i), \int_A d\mu)$$
$$= e({}^{\bullet}\sum_{i=1}^{n} \mu(A_i), \int_A d\mu) < \varepsilon,$$

for every $\varepsilon > 0$, hence $\mu(A) \subseteq \int_A d\mu$.

In what concerns the second statement, we have

$$e(m \cdot \mu(A), \int_A f d\mu)$$
$$\leq e(m \cdot \mu(A), \sum_{i=1}^{n} f(t_i)\mu(A_i)) + e(\sum_{i=1}^{n} f(t_i)\mu(A_i), \int_A f d\mu)$$
$$< \varepsilon + e(m \cdot \mu(A), m\sum_{i=1}^{n} \mu(A_i)) + e(m\sum_{i=1}^{n} \mu(A_i), \sum_{i=1}^{n} f(t_i)\mu(A_i)) = \varepsilon,$$
$$\text{for every } \varepsilon > 0,$$

which completes the proof.

ii) On the other hand, if we deal with the Gould integral [6] with respect to a finite additive set function $m : \mathcal{A} \to X$, X being a Banach space, we have $\int_A dm = m(A)$.

iii) Although $\mu(A) \subseteq \int_A d\mu := M(A)$, we observe that $\overline{\mu}(A) = \overline{M}(A)$. Indeed, because $\mu(A) \subseteq \int_A d\mu$, then $|\mu(A)| \leq |\int_A d\mu|$, hence $\overline{\mu}(A) \leq \overline{M}(A)$. In order to prove " $\geq$ ", let $\{B_i\}_{i=\overline{1,n}} \in \mathcal{P}_A$ be an arbitrary partition. We shall demonstrate that $\sum\limits_{i=1}^{n} |M(B_i)| \leq \overline{\mu}(A)$.

Since for every $i = \overline{1,n}$ and every $\varepsilon > 0$ there exists $P_\varepsilon^i = \{B_j^i\}_{j=\overline{1,q_i}} \in \mathcal{P}_{B_i}$ so that for every $P \geq P_\varepsilon^i$, with $P \in \mathcal{P}_{B_i}$, we have $h(\sigma(P), \int_{B_i} d\mu) < \frac{\varepsilon}{2^i}$, we get that, particularly, $h(\sum\limits_{j=1}^{q_i} \mu(B_j^i), \int_{B_i} d\mu) < \frac{\varepsilon}{2^i}$, for every $i = \overline{1,n}$. Therefore, $\sum\limits_{i=1}^{n} |\int_{B_i} d\mu| \leq \sum\limits_{i=1}^{n} h(\sum\limits_{j=1}^{q_i} \mu(B_j^i), \int_{B_i} d\mu) + \sum\limits_{i=1}^{n} |\sum\limits_{j=1}^{q_i} \mu(B_j^i)| < \sum\limits_{i=1}^{n} \frac{\varepsilon}{2^i} + \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{q_i} |\mu(B_j^i)| < \varepsilon + \sum\limits_{i=1}^{n} \overline{\mu}(B_i) = \varepsilon + \overline{\mu}(A)$, for every $\varepsilon > 0$, which implies that $\sum\limits_{i=1}^{n} |M(B_i)| \leq \overline{\mu}(A)$, hence $\overline{\mu}(A) \geq \overline{M}(A)$, as claimed.

Of course, the existence of all the integrals that appear here can be assured if we suppose that, for instance, $\mu : \mathcal{A} \mapsto \mathcal{P}_{kc}(X)$ and f is μ-integrable on T.

3. Sequences of μ-integrable functions. It is easy to prove the following:

Theorem 3.1. *Let $f, g : T \mapsto R$ be two bounded and $\widetilde{\mu}$-totally-measurable functions. Then:*

i) *$f + g$ is $\widetilde{\mu}$-totally-measurable;*

ii) *λf is $\widetilde{\mu}$-totally-measurable, for every $\lambda \in R$;*

iii) *f^2 and fg are $\widetilde{\mu}$-totally-measurable.*

Theorem 3.2. *We suppose that $(f_n)_n : T \mapsto R$ is a sequence of $\widetilde{\mu}$-totally-measurable, bounded functions so that $f_n \xrightarrow{\mu} f$, where $f : T \mapsto R$ is a bounded function. Then f is $\widetilde{\mu}$-totally-measurable.*

Proof. Let $\varepsilon > 0$. For every $n \in N$, f_n is $\widetilde{\mu}$-totally-measurable, hence, for every $n \in N$, there exists a partition of T, $P_\varepsilon^n = \{A_i^n\}_{i=\overline{0,m_n}}$, so that $\widetilde{\mu}(A_0^n) < \frac{\varepsilon}{2^n}$ and $\sup\limits_{t,s\in A_i^n} |f_n(t) - f_n(s)| < \frac{\varepsilon}{3\cdot 2^n}$, for every $i = \overline{1, m_n}$.

Since $f_n \xrightarrow{\mu} f$, we get that for every $\delta > 0$, $\lim \widetilde{\mu}(B_n(\delta)) = 0$, where $B_n(\delta) = \{t \in T, |f_n(t) - f(t)| \geq \delta\}$. Then for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in N$ so that $\widetilde{\mu}(B_{n_0}\left(\frac{\varepsilon}{3}\right)) < \frac{\varepsilon}{2}$. From the definition of $\widetilde{\mu}$, there is a set $C_{n_0} \in \mathcal{A}$, so that $B_{n_0}(\frac{\varepsilon}{3}) \subseteq C_{n_0}$ and $\overline{\mu}(C_{n_0}) = \widetilde{\mu}(C_{n_0}) < \frac{\varepsilon}{2}$.

Let then be the partition

$$P_\varepsilon = \{C_{n_0} \cup A_0^{n_0}, A_1^{n_0} \cap cC_{n_0}, A_2^{n_0} \cap cC_{n_0}, \ldots, A_{m_{n_0}}^{n_0} \cap cC_{n_0}\}$$

of T.

Since $\widetilde{\mu}(C_{n_0} \cup A_0^{n_0}) = \overline{\mu}(C_{n_0} \cup A_0^{n_0}) \leq \overline{\mu}(C_{n_0}) + \overline{\mu}(A_0^{n_0}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2^{n_0}} \leq \varepsilon$, it only remains to prove that $\sup\limits_{t,s\in cC_{n_0}\cap A_i^{n_0}} |f(t) - f(s)| < \varepsilon$, for every $i = \overline{1, m_{n_0}}$.

Indeed,

$$\sup_{t,s\in cC_{n_0}\cap A_i^{n_0}} |f(t) - f(s)| \leq \sup_{t\in cC_{n_0}} |f(t) - f_{n_0}(t)| + \sup_{t,s\in A_i^{n_0}} |f_{n_0}(t) - f_{n_0}(s)|$$
$$+ \sup_{s\in cC_{n_0}} |f_{n_0}(s) - f(s)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3\cdot 2^{n_0}} + \frac{\varepsilon}{3} \leq \varepsilon,$$

for every $i = \overline{1, m_{n_0}}$, as claimed. □

Theorem 3.3. *Let $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$ and $(f_n)_n : T \to \mathbb{R}$ be a uniformly bounded sequence of μ-integrable functions so that $f_n \xrightarrow{\mu} f$, where $f : T \mapsto$*

R is a bounded function. Then f is μ-integrable on A and

$$\lim_{n\to\infty}\int_A f_n d\mu = \int_A f d\mu \text{ (with respect to } h), \text{ for every } A\in\mathcal{A}. \tag{12}$$

Proof. Let $M' = \overline{\mu}(T), M_1 = \sup_{t\in T}|f(t)|, M_2 = \sup_{t\in T, n\in N}|f_n(t)|$ and $M = \max(M_1, M_2)$.

Since $f_n \xrightarrow{\mu} f$, we get that for every $\varepsilon > 0$, there exists $n_0(\varepsilon)\in N$ so that $\widetilde{\mu}(B_n\left(\frac{\varepsilon}{6M'}\right)) < \frac{\varepsilon}{4M}$, for every $n\geq n_0$. Particularly, $\widetilde{\mu}(B_{n_0}\left(\frac{\varepsilon}{6M'}\right)) < \frac{\varepsilon}{4M}$. Consequently, from the definition of $\widetilde{\mu}$, there is a set $C_{n_0}\in\mathcal{A}$ such that $B_{n_0}(\frac{\varepsilon}{6M'})\subseteq C_{n_0}$ and $\widetilde{\mu}(C_{n_0}) = \overline{\mu}(C_{n_0}) < \frac{\varepsilon}{4M}$.

Let $A\in\mathcal{A}$ be arbitrarily, but fixed. We prove first that f is μ-integrable on C_{n_0} (it will also be μ-integrable on $C_{n_0}\cap A$, see theorem 2.9). Indeed, for every $\varepsilon > 0$, there exists the partition $P_\varepsilon = \{C_{n_0}\}\in\mathcal{P}_{C_{n_0}}$ so that, for every other partition $P' = \{D_l\}_{l=\overline{1,p}}\in\mathcal{P}_{C_{n_0}}$ with $P'\geq P_\varepsilon$ and for every $t_l\in D_l$, $l=\overline{1,p}$ and every $c\in C_{n_0}$, we have:

$$\begin{aligned}
&h(\sigma(P'), f(c)\mu(C_{n_0}))\\
&= h(\sum_{l=1}^{p} f(t_l)\mu(D_l),\ f(c)\mu(C_{n_0})) \leq |\sum_{l=1}^{p} f(t_l)\mu(D_l)| + |f(c)\mu(C_{n_0})|\\
&\leq M_1\sum_{l=1}^{p}|\mu(D_l)| + M_1|\mu(C_{n_0})| \leq 2M_1\overline{\mu}(C_{n_0}) < 2M_1\frac{\varepsilon}{4M}\leq\varepsilon/2.
\end{aligned}$$

Similarly, for every partition $P''\in\mathcal{P}_{C_{n_0}}$, with $P''\geq P_\varepsilon$, we get that $h(\sigma(P''), f(c)\mu(C_{n_0})) < \frac{\varepsilon}{2}$, hence $h(\sigma(P'),\sigma(P'')) < \varepsilon$.
This means $(\sigma(P))_{P\in\mathcal{P}_{C_{n_0}}}$ is a Cauchy, hence a convergent net in the complete metric space $\mathcal{P}_{kc}(X)$.

Therefore, f is μ-integrable on C_{n_0}.

We prove now that f is μ-integrable on A. Since f is μ-integrable on $A\cap C_{n_0}$, then, according to theorem 2.7, it is sufficient to prove that f is μ-integrable on $A\backslash C_{n_0}$.

Because f_n is μ-integrable on T, for every $n\in N$, we get that, particularly, f_{n_0} is μ-integrable on $A\backslash C_{n_0}$; there exists then a partition $P_\varepsilon^{n_0} = \{A_i\}_{i=\overline{1,m_{n_0}}}\in\mathcal{P}_{A\backslash C_{n_0}}$ so that for every $P\in\mathcal{P}_{A\backslash C_{n_0}}$, with $P\geq P_\varepsilon^{n_0}$, $h(\sigma(P),\sigma(P_\varepsilon^{n_0})) < \frac{\varepsilon}{3}$.

Let $P = \{D_j\}_{j=\overline{1,l}} \in \mathcal{P}_{A\setminus C_{n_0}}$ with $P \geq P_\varepsilon^{n_0}$ be arbitrarily, but fixed. For every $t_j \in D_j$, $j = \overline{1,l}$ and every $c_i \in A_i$, $i = \overline{1, m_{n_0}}$, we have:

$$\begin{aligned} & h(\sum_{j=1}^{l} f(t_j)\mu(D_j), \sum_{i=1}^{m_{n_0}} f(c_i)\mu(A_i)) \\ & \quad \leq h(\sum_{j=1}^{l} f(t_j)\mu(D_j), \sum_{j=1}^{l} f_{n_0}(t_j)\mu(D_j)) \\ & \quad + h(\sum_{j=1}^{l} f_{n_0}(t_j)\mu(D_j), \sum_{i=1}^{m_{n_0}} f_{n_0}(c_i)\mu(A_i)) \\ & \quad + h(\sum_{i=1}^{m_{n_0}} f_{n_0}(c_i)\mu(A_i), \sum_{i=1}^{m_{n_0}} f(c_i)\mu(A_i)) \\ & \quad \leq \sum_{j=1}^{l} |\mu(D_j)| \sup_{t_j \in D_j \subset A\setminus C_{n_0}} |f(t_j) - f_{n_0}(t_j)| \\ & \quad + \frac{\varepsilon}{3} + \sum_{i=1}^{m_{n_0}} |\mu(A_i)| \sup_{c_i \in A_i \subset A\setminus C_{n_0}} |f(c_i) - f_{n_0}(c_i)| \\ & \quad \leq M' \frac{\varepsilon}{6M'} + \frac{\varepsilon}{3} + M' \frac{\varepsilon}{6M'} < \varepsilon, \end{aligned}$$

hence f is indeed μ-integrable on $A\setminus C_{n_0}$.

In the sequel, we shall prove the equality (12). Since f_n is μ-integrable on T for every $n \in N$, from theorem 2.9 we get that f_n is μ-integrable on A, for every $n \in N$. Hence there exists $\int_A f_n d\mu$ for every $n \in N$ and $\int_A f d\mu$.

Let us use, the same as before, the sets $B_n\left(\frac{\varepsilon}{6M'}\right)$, with $n \geq n_0$. From the definition of $\widetilde{\mu}$ we get that for every $n \geq n_0$ there is a set $C_n \in \mathcal{A}$ such that $B_n(\frac{\varepsilon}{6M'}) \subseteq C_n$ and $\widetilde{\mu}(C_n) = \overline{\mu}(C_n) < \frac{\varepsilon}{4M}$. Consequently, for every $n \geq n_0$, we have:

$$\begin{aligned} h(\int_A f_n d\mu, \int_A f d\mu) &= h(\int_{A\setminus C_n} f_n d\mu + \int_{A\cap C_n} f_n d\mu, \int_{A\setminus C_n} f d\mu + \int_{A\cap C_n} f d\mu) \\ & \leq h(\int_{A\setminus C_n} f_n d\mu, \int_{A\setminus C_n} f d\mu) + h(\int_{A\cap C_n} f_n d\mu, \int_{A\cap C_n} f d\mu) \end{aligned}$$

$$\leq \sup_{t\in A\setminus C_n} |f_n(t) - f(t)| \cdot \overline{\mu}(A\setminus C_n) + \sup_{t\in A\cap C_n} |f_n(t) - f(t)| \cdot \overline{\mu}(A\cap C_n)$$

$$< \frac{\varepsilon}{6M'} \cdot M' + 2M \cdot \overline{\mu}(C_n) < \frac{\varepsilon}{2} + 2M \cdot \frac{\varepsilon}{4M} = \varepsilon.$$

This completes the proof. □

Remark 3.4. Particularly, if $\mu : \mathcal{A} \to \mathcal{P}_{kc}(R)$ is the multisubmeasure induced by a submeasure of finite variation $\nu : \mathcal{A} \to R_+$ and if $(f_n)_n : T \to R_+$ is a uniformly bounded sequence of $\widetilde{\mu}$-totally-measurable functions such that $f_n \xrightarrow{\mu} f$, where $f : T \mapsto R$ is a bounded function then

$$\lim_{n\to\infty} \int_A f_n d\mu = \int_A f d\mu \ \text{ (with respect to } h\text{), for every } A \in \mathcal{A}. \tag{13}$$

Indeed, from theorem 2.6 and theorem 2.9 we get that for every $n \in N^*$, f_n is μ-integrable on A, for every $A \in \mathcal{A}$.

By applying theorem 3.3, the proof finishes.

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