

## ON THE SPATIAL BEHAVIOUR OF HARMONIC VIBRATIONS IN AN ELASTIC CYLINDER

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**Abstract.** This paper studies the spatial behaviour of the steady-state vibrations in a right elastic cylinder, under mild conditions upon the elastic coefficients. In fact, we establish some exponential decay estimates of Saint-Venant type for two cross-sectional area measures associated with the amplitude of the steady-state vibrations, provided that the elasticity tensor is assumed to be strongly elliptic and the prescribed frequency is lower than a certain critical value.

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**Key words:** spatial behaviour, harmonic vibrations, strongly elliptic elasticity tensor, isotropic materials.

**1. Introduction.** Saint-Venant's principle in linear elasticity was established by TOUPIN [1], who gave an inequality describing exponential spatial decay of effects with distance from the excited end of the right cylinder. This result was given by assuming that the elasticity tensor satisfies a positive definiteness condition. FLAVIN, KNOPS and PAYNE [2] established the same kind of results, for an isotropic cylinder with null displacement on the lateral boundary, under weak assumptions on the elasticity tensor. In fact, they established some decay estimates for the static problem when the Lamé moduli range so that  $\mu > 0$  and  $\lambda + \mu > 0$ . For linear elastodynamics some results on the spatial behaviour of solutions were given by CHIRIȚĂ [3] for an isotropic and homogeneous elastic solid whose elasticity tensor is strongly elliptic, that is  $\mu > 0$ ,  $\lambda + 2\mu > 0$ .

On the other hand, the study of spatial behaviour for the harmonic elastic vibrations has been initiated by FLAVIN and KNOPS [4]. They have

established the exponential decay of an appropriate measure associated with the amplitude of vibration by supposing the positive definiteness of the elasticity tensor, provided that the prescribed exciting frequency is lower than a certain critical frequency.

Information on the Saint–Venant principle can be found in the survey articles written by HORGAN and KNOWLES [5] and HORGAN [6,7].

In the present paper we study the spatial behaviour of the amplitude of the harmonic vibrations in an elastic, homogeneous and isotropic cylinder assuming that the elasticity tensor is strongly elliptic, that is a condition which characterizes an extended class of materials that one uses in [4]. We consider a cylinder made of an elastic, homogeneous and isotropic material constrained on the lateral boundary and one of the ends, and the other end is assumed to be excited by prescribed harmonic vibrations. For the study of spatial behaviour of the amplitude of vibration that takes place in the cylinder, we introduce two appropriate measures and then we establish, for each of them, a first–order differential inequality, again under the condition that the exciting frequency does not exceed a certain critical frequency. The integration of these differential inequalities makes possible the obtaining of some spatial estimates that describes the exponential decay of effects with distance from the excited end.

The plan of the paper is the following one. In Section 2 we recall the basic equations and formulate the problem of steady–state vibrations. In Section 3 we study the spatial behaviour of the amplitude of harmonic vibrations in the class of isotropic elastic materials for which the Lamé moduli obey the inequalities  $\mu > 0$ ,  $3\lambda + 4\mu > 0$ . A similar result we obtain in Section 4 for the isotropic elastic materials characterized by  $\mu > 0$ ,  $\lambda < 0$ ,  $\lambda + 2\mu > 0$ . In this manner we obtain the results for the elastic materials whose elasticity tensor is strongly elliptic.

**2. Preliminaries.** Throughout this paper we consider a cylinder of uniform cross–section  $D$  whose boundary is sufficiently smooth to allow the application of the divergence theorem.

We choose a rectangular cartesian system  $Ox_1x_2x_3$  so that the  $Ox_3$  axis is parallel with the generator of the cylinder and one of the ends is in the  $x_1Ox_2$  plane. The Latin subscripts are understood to range over the integers 1, 2, 3, whereas Greek subscripts are confined to the range 1, 2. Summation over repeated subscripts and other typical conventions for differential operations are implied such as superposed dot or comma followed by a subscript

to denote partial derivative with respect to time or the corresponding cartesian coordinate.

Let us now consider an isotropic and homogeneous body that occupies the region  $B = D \times [0, L]$  with the lateral surfaces  $\Sigma = \partial D \times [0, L]$ , where  $L$  is the length of the cylinder. We denote by  $D(x_3)$  the bounded cross-section of the cylinder situated at the distance  $x_3$  from the  $x_1Ox_2$  plane and we impose that  $\partial D(x_3)$  is a smooth simple closed curve.

According to the linear theory of elastodynamics, in the absence of external body force, the fundamental system of field equations consists [8] of the strain-displacement relations

$$(2.1) \quad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } \bar{B} \times [0, \infty),$$

the stress-strain relation

$$(2.2) \quad t_{ij} = \lambda e_{rr} \delta_{ij} + 2\mu e_{ij} \quad \text{in } \bar{B} \times [0, \infty)$$

and the equations of motion

$$(2.3) \quad t_{ji,j} = \rho \ddot{u}_i \quad \text{in } B \times (0, \infty).$$

Here  $u_i$  are the components of the displacement vector,  $e_{ij}$  are the components of the strain tensor,  $t_{ij}$  are the components of the stress tensor. Further,  $\rho$  is the constant mass density,  $\lambda$  and  $\mu$  are the Lamé moduli and  $\delta_{ij}$  is the Kronecker symbol.

To the basic equations (2.1)–(2.3) we must adjoin the initial conditions

$$(2.4) \quad u_i = a_i, \quad \dot{u}_i = b_i \quad \text{in } B \times \{0\}$$

and the boundary conditions

$$(2.5) \quad u_i = 0 \quad \text{on } \partial D \times [0, L],$$

$$(2.6) \quad u_i = 0 \quad \text{on } D(L),$$

$$(2.7) \quad u_j = \tilde{u}_j(x_1, x_2) e^{i\omega t} \quad \text{on } D(0),$$

where  $\omega = \text{constant}$ ,  $\omega > 0$ .

It is easy to see that

$$(2.8) \quad u_j(x_1, x_2, x_3, t) = U_j(x_1, x_2, x_3, t) + v_j(x_1, x_2, x_3) e^{i\omega t},$$

where  $U_j$  absorbs the initial conditions and satisfies the null boundary conditions and the equations (2.1)–(2.3), meanwhile  $v_j$  satisfies the boundary value problem consisting of

$$(2.9) \quad \sigma_{ji,j} = -\rho\omega^2 v_i,$$

$$(2.10) \quad \sigma_{ij} = \lambda\varepsilon_{rr}\delta_{ij} + 2\mu\varepsilon_{ij},$$

$$(2.11) \quad 2\varepsilon_{ij} = v_{i,j} + v_{j,i},$$

subjected to the lateral boundary conditions

$$(2.12) \quad v_i = 0 \quad \text{on} \quad \partial D \times [0, L]$$

and the end boundary conditions

$$(2.13) \quad v_i = \tilde{u}_i \quad \text{on} \quad D(0), \quad v_i = 0 \quad \text{on} \quad D(L).$$

Next, we will study the spatial behaviour of the amplitude  $\mathbf{v} = (v_1, v_2, v_3)$  of the stationary vibrations satisfying the equations (2.9)–(2.11), which can be written in the form

$$(2.14) \quad \mu v_{i,jj} + (\lambda + \mu)v_{r,ri} = -\rho\omega^2 v_i$$

and boundary conditions (2.12) and (2.13), under the hypothesis that the elasticity tensor is strongly elliptic.

**3. First estimate.** Throughout this section we will assume that

$$(3.1) \quad \mu > 0, \quad 3\lambda + 4\mu > 0.$$

We will use this assumption for the study of spatial behaviour of solutions for the boundary value problem  $\mathcal{P}$  defined by the equation (2.14) with the boundary conditions (2.12) and (2.13). To this end we write the basic equations (2.14) in the following form

$$(3.2) \quad s_{ji,j} = -\rho\omega^2 v_i,$$

where

$$(3.3) \quad s_{ij} = \mu v_{j,i} + (\lambda + \mu)v_{r,r}\delta_{ij}.$$

Furthermore, we define the function

$$(3.4) \quad K(x_3) = \int_{D(x_3)} (s_{3i}\bar{v}_i + \bar{s}_{3i}v_i)da, \quad x_3 \in [0, L],$$

where the superposed bar denotes complex conjugate.

By direct differentiation, we get

$$(3.5) \quad K'(x_3) = \int_{D(x_3)} (s_{3i,3}\bar{v}_i + \bar{s}_{3i,3}v_i + s_{3i}\bar{v}_{i,3} + \bar{s}_{3i}v_{i,3})da.$$

From (3.2), we deduce that

$$(3.6) \quad s_{3i,3}\bar{v}_i + \bar{s}_{3i,3}v_i = -s_{\rho i, \rho}\bar{v}_i - \bar{s}_{\rho i, \rho}v_i - 2\rho\omega^2v_i\bar{v}_i,$$

and hence, the relation (3.5) can be written in the form

$$(3.7) \quad \begin{aligned} K'(x_3) = & - \int_{D(x_3)} (s_{\rho i, \rho}\bar{v}_i + \bar{s}_{\rho i, \rho}v_i)da - 2\omega^2 \int_{D(x_3)} \rho v_i\bar{v}_i da \\ & + \int_{D(x_3)} (s_{3i}\bar{v}_{i,3} + \bar{s}_{3i}v_{i,3})da. \end{aligned}$$

By using an integration by parts and the boundary conditions (2.12), we can deduce that

$$(3.8) \quad K'(x_3) = \int_{D(x_3)} (s_{ji}\bar{v}_{i,j} + \bar{s}_{ji}v_{i,j})da - 2\omega^2 \int_{D(x_3)} \rho v_i\bar{v}_i da$$

and hence, by means of the relation (3.3), we obtain

$$(3.9) \quad K'(x_3) = 2 \int_{D(x_3)} [\mu v_{i,j}\bar{v}_{i,j} + (\lambda + \mu)v_{r,r}\bar{v}_{s,s} - \rho\omega^2v_i\bar{v}_i] da.$$

Now, we introduce the following bilinear form

$$(3.10) \quad W(\xi, \eta) = \frac{1}{2} \{ \mu(\xi_{ij}\bar{\eta}_{ij} + \bar{\xi}_{ij}\eta_{ij}) + (\lambda + \mu)(\xi_{rr}\bar{\eta}_{ss} + \bar{\xi}_{rr}\eta_{ss}) \}, \quad \text{for all } \xi_{ij}, \eta_{ij}$$

and then we set

$$(3.11) \quad W(\xi) = W(\xi, \xi) = \mu\xi_{ij}\bar{\xi}_{ij} + (\lambda + \mu)\xi_{rr}\bar{\xi}_{ss}, \quad \text{for all } \xi_{ij}.$$

Further, we write

$$(3.12) \quad W(\xi) = W_1(\xi) + W_2(\xi),$$

where

$$(3.13) \quad \begin{aligned} W_1(\xi) &= (\lambda + 2\mu)(\xi_{11}\bar{\xi}_{11} + \xi_{22}\bar{\xi}_{22} + \xi_{33}\bar{\xi}_{33}) \\ &+ (\lambda + \mu)(\xi_{11}\bar{\xi}_{22} + \bar{\xi}_{11}\xi_{22} + \xi_{22}\bar{\xi}_{33} + \bar{\xi}_{22}\xi_{33} + \xi_{33}\bar{\xi}_{11} + \bar{\xi}_{33}\xi_{11}), \end{aligned}$$

$$(3.14) \quad W_2(\xi) = \mu(\xi_{12}\bar{\xi}_{12} + \xi_{21}\bar{\xi}_{21} + \xi_{23}\bar{\xi}_{23} + \xi_{32}\bar{\xi}_{32} + \xi_{13}\bar{\xi}_{13} + \xi_{31}\bar{\xi}_{31}).$$

The quadratic form  $W_1(\xi)$  is positive definite if and only if

$$\mu > 0, \quad 3\lambda + 4\mu > 0,$$

and  $W_2(\xi)$  is positive definite if and only if

$$\mu > 0.$$

The eigenvalues of the quadratic form  $W_1(\xi)$  are

$$(3.15) \quad k_1 = 3\lambda + 4\mu, \quad k_2 = \mu.$$

If we set

$$(3.16) \quad k_m = \min\{k_1, k_2\}, \quad k_M = \max\{k_1, k_2\},$$

then we have

$$(3.17) \quad k_m(\xi_{11}\bar{\xi}_{11} + \xi_{22}\bar{\xi}_{22} + \xi_{33}\bar{\xi}_{33}) \leq W_1(\xi) \leq k_M(\xi_{11}\bar{\xi}_{11} + \xi_{22}\bar{\xi}_{22} + \xi_{33}\bar{\xi}_{33}).$$

Thus, by the relations (3.11) to (3.14) we get

$$(3.18) \quad k_m \xi_{ij} \bar{\xi}_{ij} \leq W(\xi) \leq k_M \xi_{ij} \bar{\xi}_{ij}, \quad \text{for all } \xi_{ij}.$$

On the basis of this inequality, the relation (3.9) gives

$$(3.19) \quad K'(x_3) \geq 2 \int_{D(x_3)} (k_m v_{i,j} \bar{v}_{i,j} - \omega^2 \rho v_i \bar{v}_i) da.$$

In view of the boundary conditions (2.12) and (2.13), we have the following inequality

$$(3.20) \quad \lambda_1 \int_{D(x_3)} v_i \bar{v}_i da \leq \int_{D(x_3)} v_{i,\rho} \bar{v}_{i,\rho} da,$$

where  $\lambda_1$  is the lowest eigenvalue of the corresponding membrane problem. Then, by using the estimate (3.20) into relation (3.19), we find

$$(3.21) \quad K'(x_3) \geq 2k_m \left(1 - \frac{\omega^2 \rho}{\lambda_1 k_m}\right) \int_{D(x_3)} v_{i,j} \bar{v}_{i,j} da.$$

In what follows we will suppose that  $\omega$  satisfies the inequality

$$(3.22) \quad \omega^2 \leq \omega_1^2,$$

where

$$(3.23) \quad \omega_1 = \sqrt{\frac{\lambda_1 k_m}{\rho}}.$$

We proceed now to obtain an estimate for the  $K(x_3)$  in terms of  $K'(x_3)$ . By using (3.3), the Schwarz inequality for the bilinear form  $W$  and the relation (3.18), we obtain

$$(3.24) \quad s_{33} \bar{s}_{33} \leq k_M^2 (v_{1,1} \bar{v}_{1,1} + v_{2,2} \bar{v}_{2,2} + v_{3,3} \bar{v}_{3,3}).$$

By using the relations (3.3), (3.20), (3.24), the Schwarz inequality and the arithmetic–geometric inequality, for any  $\varepsilon > 0$ , we deduce

$$(3.25) \quad \begin{aligned} |K(x_3)| &\leq \int_{D(x_3)} \left( \varepsilon s_{3i} \bar{s}_{3i} + \frac{1}{\varepsilon} v_i \bar{v}_i \right) da \\ &\leq \int_{D(x_3)} \left( \varepsilon \mu^2 v_{\rho,3} \bar{v}_{\rho,3} + \frac{1}{\lambda_1 \varepsilon} v_{\rho,\alpha} \bar{v}_{\rho,\alpha} + \varepsilon s_{33} \bar{s}_{33} + \frac{1}{\lambda_1 \varepsilon} v_{3,\rho} \bar{v}_{3,\rho} \right) da \\ &\leq \int_{D(x_3)} \left[ \varepsilon k_M^2 v_{\rho,3} \bar{v}_{\rho,3} + \frac{1}{\lambda_1 \varepsilon} v_{\rho,\alpha} \bar{v}_{\rho,\alpha} + \varepsilon k_M^2 (v_{1,1} \bar{v}_{1,1} + v_{2,2} \bar{v}_{2,2} + \right. \\ &\quad \left. + v_{3,3} \bar{v}_{3,3}) + \frac{1}{\lambda_1 \varepsilon} v_{3,\rho} \bar{v}_{3,\rho} \right] da. \end{aligned}$$

Further, we set

$$(3.26) \quad \varepsilon = \frac{1}{k_M \sqrt{\lambda_1}},$$

so that, we deduce

$$(3.27) \quad |K(x_3)| \leq \frac{2k_M}{\sqrt{\lambda_1}} \int_{D(x_3)} v_{i,j} \bar{v}_{i,j} da.$$

From the relations (3.21), (3.22) and (3.27), we deduce the following first-order differential inequality

$$(3.28) \quad |K(x_3)| \leq \frac{k_M}{\sqrt{\lambda_1}} \frac{1}{k_m \left(1 - \frac{\omega^2}{\omega_1^2}\right)} K'(x_3).$$

We now proceed to integrate this differential inequality. In this aim we note that the function  $K$  is nondecreasing on  $[0, L]$  and moreover, from the boundary conditions we have  $K(L) = 0$  and hence we deduce that

$$(3.29) \quad K(x_3) \leq 0 \quad \text{for all } x_3 \in [0, L].$$

If we set  $c = \frac{k_m}{k_M} \sqrt{\lambda_1} \left(1 - \frac{\omega^2}{\omega_1^2}\right)$ , then the first-order differential inequality (3.28) becomes

$$cK(x_3) + K'(x_3) \geq 0,$$

which, by an integration, leads to the following inequality

$$(3.30) \quad -K(x_3) \leq -K(0)e^{-cx_3} \quad \text{for all } x_3 \in [0, L].$$

So, provided that the exciting frequency is lower than the critical frequency  $\omega_1$  defined by the relation (3.23), we have established the exponential spatial decay estimate (3.30). It describes the spatial behaviour of the amplitudes  $\mathbf{v} = (v_1, v_2, v_3)$ , in terms of the cross-sectional measure  $-K$ , for the class of elastic materials characterized by  $\mu > 0$ ,  $3\lambda + 4\mu > 0$ .

**4. Second estimate.** In this section we describe a second method for discussing the spatial behaviour of the amplitudes which allows us to extend the class of materials by relaxing the range of elastic moduli.

Throughout this section we will assume the following inequalities

$$(4.1) \quad \mu > 0, \quad \lambda < 0, \quad \lambda + 2\mu > 0.$$

In order to treat the spatial behaviour under the above hypotheses upon the characteristic constants of the material we write the basic equations (2.14) in the following form

$$(4.2) \quad S_{ji,j} = -\rho\omega^2 v_i,$$

where

$$(4.3) \quad S_{ji} = \mu v_{i,j} + (\lambda + \mu)v_{j,i}.$$



Furthermore, we define the following cross-section integral

$$(4.4) \quad H(x_3) = \int_{D(x_3)} (S_{3i}\bar{v}_i + \bar{S}_{3i}v_i)da, \quad x_3 \in [0, L].$$

By direct differentiation, we obtain

$$(4.5) \quad H'(x_3) = \int_{D(x_3)} (S_{3i,3}\bar{v}_i + \bar{S}_{3i,3}v_i + S_{3i}\bar{v}_{i,3} + \bar{S}_{3i}v_{i,3})da.$$

In view of the relation (4.2), we have

$$(4.6) \quad S_{3i,3}\bar{v}_i + \bar{S}_{3i,3}v_i = -S_{\rho i, \rho}\bar{v}_i - \bar{S}_{\rho i, \rho}v_i - 2\rho\omega^2v_i\bar{v}_i$$

and hence, the relation (4.5) can be written in the form

$$\begin{aligned} H'(x_3) &= - \int_{D(x_3)} (S_{\rho i, \rho}\bar{v}_i + \bar{S}_{\rho i, \rho}v_i)da - 2\omega^2 \int_{D(x_3)} \rho v_i\bar{v}_i da \\ &\quad + \int_{D(x_3)} (S_{3i}\bar{v}_{i,3} + \bar{S}_{3i}v_{i,3})da. \end{aligned}$$

Moreover, using an integration by parts, the boundary conditions (2.12) and the relations (4.3), we deduce that

$$(4.7) \quad H'(x_3) = 2 \int_{D(x_3)} [\mu v_{i,j}\bar{v}_{i,j} + (\lambda + \mu)v_{j,i}\bar{v}_{i,j} - \rho\omega^2v_i\bar{v}_i] da.$$

Let us consider the following bilinear form

$$(4.8) \quad \Omega(\xi, \eta) = \frac{1}{2}[\mu(\xi_{ij}\bar{\eta}_{ij} + \bar{\xi}_{ij}\eta_{ij}) + (\lambda + \mu)(\xi_{ij}\bar{\eta}_{ji} + \bar{\xi}_{ij}\eta_{ji})], \quad \text{for all } \xi_{ij}, \eta_{ij}$$

and then we set

$$(4.9) \quad \Omega(\xi) = \Omega(\xi, \xi) = \Omega_1(\xi) + \Omega_2(\xi),$$

where

$$(4.10) \quad \Omega_1(\xi) = (\lambda + 2\mu)(\xi_{11}\bar{\xi}_{11} + \xi_{22}\bar{\xi}_{22} + \xi_{33}\bar{\xi}_{33}),$$

$$(4.11) \quad \begin{aligned} \Omega_2(\xi) &= \mu(\xi_{12}\bar{\xi}_{12} + \xi_{21}\bar{\xi}_{21} + \xi_{23}\bar{\xi}_{23} + \xi_{32}\bar{\xi}_{32} + \xi_{13}\bar{\xi}_{13} + \xi_{31}\bar{\xi}_{31}) \\ &\quad + (\lambda + \mu)(\xi_{12}\bar{\xi}_{21} + \xi_{21}\bar{\xi}_{12} + \xi_{23}\bar{\xi}_{32} + \xi_{32}\bar{\xi}_{23} + \xi_{13}\bar{\xi}_{31} + \xi_{31}\bar{\xi}_{13}). \end{aligned}$$

The quadratic form  $\Omega_1(\xi)$  is positive definite if and only if

$$\lambda + 2\mu > 0,$$

while  $\Omega_2(\xi)$  is positive definite if and only if

$$\mu > 0, \quad \lambda < 0, \quad \lambda + 2\mu > 0.$$

The eigenvalues of the quadratic form  $\Omega_2(\xi)$  are

$$(4.12) \quad h_1 = \lambda + 2\mu, \quad h_2 = -\lambda.$$

If we set

$$(4.13) \quad h_m = \min\{h_1, h_2\}, \quad h_M = \max\{h_1, h_2\},$$

then from (4.7), we get

$$(4.14) \quad H'(x_3) \geq 2h_m \left(1 - \frac{\omega^2 \rho}{\lambda_1 h_m}\right) \int_{D(x_3)} v_{i,j} \bar{v}_{i,j} da.$$

We proceed now to obtain an estimate for  $H(x_3)$  in terms of  $H'(x_3)$ . On the basis of the Schwarz inequality and the arithmetic-geometric inequality, from (4.4), we deduce

$$(4.15) \quad |H(x_3)| \leq \int_{D(x_3)} \left[ \varepsilon S_{3i} \bar{S}_{3i} + \frac{1}{\varepsilon} v_i \bar{v}_i \right] da \quad \text{for any } \varepsilon > 0.$$

By using (4.3), we get

$$(4.16) \quad \begin{aligned} S_{33} \bar{S}_{33} &\leq S_{11} \bar{S}_{11} + S_{22} \bar{S}_{22} + S_{33} \bar{S}_{33} \\ &\leq (\lambda + 2\mu)^2 [v_{1,1} \bar{v}_{1,1} + v_{2,2} \bar{v}_{2,2} + v_{3,3} \bar{v}_{3,3}]. \end{aligned}$$

Moreover, using the Schwarz inequality for the bilinear form  $\Omega$  we obtain

$$(4.17) \quad S_{31} \bar{S}_{31} + S_{32} \bar{S}_{32} \leq h_M^2 [v_{1,3} \bar{v}_{1,3} + v_{3,1} \bar{v}_{3,1} + v_{2,3} \bar{v}_{2,3} + v_{3,2} \bar{v}_{3,2}].$$

By using the relations (4.3), (4.16) and (4.17), from (4.15) we deduce that

$$\begin{aligned} |H(x_3)| &\leq \int_{D(x_3)} \{ \varepsilon [h_M^2 (v_{1,1} \bar{v}_{1,1} + v_{2,2} \bar{v}_{2,2} + v_{3,3} \bar{v}_{3,3}) \\ &\quad + h_M^2 (v_{1,3} \bar{v}_{1,3} + v_{3,1} \bar{v}_{3,1} + v_{2,3} \bar{v}_{2,3} + v_{3,2} \bar{v}_{3,2})] + \frac{1}{\varepsilon \lambda_1} v_{i,\rho} \bar{v}_{i,\rho} \} da \\ &\leq \int_{D(x_3)} \left[ \varepsilon h_M^2 v_{i,j} \bar{v}_{i,j} + \frac{1}{\varepsilon \lambda_1} v_{i,j} \bar{v}_{i,j} \right] da. \end{aligned}$$

Further, we set  $\varepsilon = \frac{1}{h_M \sqrt{\lambda_1}}$ , so that, we deduce

$$(4.18) \quad |H(x_3)| \leq \frac{2h_M}{\sqrt{\lambda_1}} \int_{D(x_3)} v_{i,j} \bar{v}_{i,j}.$$

In what follows we will suppose that the frequency of vibration  $\omega$  satisfies the inequality

$$\omega^2 \leq \omega_2^2,$$

where

$$\omega_2 = \sqrt{\frac{\lambda_1 h_m}{\rho}}.$$

Thus, using the relations (4.14) and (4.18) we get the following first-order differential inequality

$$(4.19) \quad |H(x_3)| \leq \frac{h_M}{\sqrt{\lambda_1} h_m} \frac{1}{\left(1 - \frac{\omega^2}{\omega_2^2}\right)} H'(x_3)$$

and similar as in the above section, we obtain

$$(4.20) \quad -H(x_3) \leq -H(0)e^{-\tilde{c}x_3}, \quad \text{for all } x_3 \in [0, L],$$

where  $\tilde{c} = \frac{h_m}{h_M} \sqrt{\lambda_1} \left(1 - \frac{\omega^2}{\omega_2^2}\right)$ .

The exponential decay estimate (4.20) describes the spatial behaviour of the amplitudes  $\mathbf{v} = (v_1, v_2, v_3)$ , in term of the cross-sectional measure  $-H$ , for the class of elastic materials characterized by  $\mu > 0$ ,  $\lambda < 0$ ,  $\lambda + 2\mu > 0$ , when the frequency of the vibration is lower than the critical frequency  $\omega_2$ .

**5. Conclusion.** The main purpose of this paper was to study the spatial behaviour of the amplitude of harmonic vibrations in a right cylinder filled with a homogeneous and isotropic elastic material, under assumptions that the elasticity tensor is strongly elliptic and the prescribed exciting frequency is lower than a certain critical value.

For the class of materials characterized by  $\mu > 0$ ,  $3\lambda + 4\mu > 0$ , provided  $\omega \leq \sqrt{\frac{\lambda_1 k_m}{\rho}}$ , we have established the exponential decay estimate (3.30) while for materials characterized by  $\mu > 0$ ,  $\lambda < 0$ ,  $\lambda + 2\mu > 0$ , supposing that  $\omega \leq \sqrt{\frac{\lambda_1 h_m}{\rho}}$ , we derived the exponential decay estimate (4.20). These

estimates improve the results obtained by FLAVIN and KNOPS [4] under more restrictive assumptions upon constitutive coefficients, that is  $\mu > 0$  and  $3\lambda + 2\mu > 0$ .

For various types of elastic materials, to obtain a complete description of spatial behaviour of harmonic vibration we can combine the above two type results and, on the other hand we can combine the present results with those already obtained by FLAVIN and KNOPS [4].

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