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VITALI AND METRICALLY PERFECT SETS

BY

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Abstract. In this paper we continue our studies on the concepts of Vitali sets and ψ-derived sets. Another concept ”metrically perfect set” is found to be extremely helpful to finding out more properties of these sets and of a topological space whose open sets are ψ-open where ψ is an outer measure in a metric space.

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1. Introduction. We introduced [7] the ideas of Vitali sets and ψ-derived sets in a metric space with the help of an outer measure ψ defined on the power set of the metric space. Several properties of these sets have been found including a necessary and sufficient condition for a set to be a Vitali set. In this paper we consider another concept called ”metrically perfect set”, which appears to contribute fruitfully to finding out more significant properties of ψ-derived sets and Vitali sets. We observe further that this concept also becomes helpful to finding out several topological properties of the topological space (X, T_ψ) constructed in [7] with the help of ψ-open sets, where ψ is an outer measure defined on the power set of the metric space.

The use of the term *Vitali set* in [7] and here appears to be justified because our proceedings were formulated from the concept of classical Vitali covers followed by Vitali theorem [9]. The results of the present paper have no connection with those of [3] where the title involved sets after Vitali's name.

Section 2 outlines some definitions and results from [7] relevant to the present paper. Section 3 provides further results on ψ-derived sets and Vitali sets. Section 4 introduces the concept of metrically perfect sets where several properties connecting ψ-derived sets are proved. Section 5 deals with some topological consequences in the topological space (X, T_ψ).

2. Known definitions and results. Let X be a metric space and ψ be an outer measure defined on the power set of X with $0 < \psi(X) < \infty$. Let $\mathcal{C}$ denote the family of all closed balls in X (closed intervals, closed rectangles etc. in the case of Euclidian spaces).

Let $A \subset X$ be a non-void set. Let V be subfamily of $\mathcal{C}$ each of positive outer measure.

Definition I. *The subfamily V is to be a Vitali cover of A if for each $x \in A$ and $\varepsilon > 0$, there exists an $I \in V$ such that $x \in int(I)$ and $\psi(I) < \varepsilon$.*

Definition II. *Let $A \subset X$. Then A is said to be a Vitali set if every Vitali cover V of A contains a countable pairwise disjoint subcollection $\{I_k\}$ such that $\psi[A \backslash \bigcup_k I_k] = 0$.*

Definition III. *The metric space X is said to be smooth with respect to ψ if the collection of closed spheres of positive outer measure is a Vitali cover of X.*

Examples are available in [7] showing the existence of Vitali sets as well as sets which are not Vitali sets. Also examples of smooth and non-smooth metric spaces have been exhibited in [7].

We assume throughout that X is a smooth space with respect to ψ. In a smooth space, clearly any subset of X has a Vitali cover and that each singleton and so any countable set has outer measure zero. Throughout sets A, B etc are subsets of X and unless otherwise stated, sets are always assumed to be non-void.

Definition IV. *A point $\xi \in X$ is called a ψ-accumulation point of a set A if for every sequence $\{F_n\}$ from $\mathcal{C}$ with $\psi(F_n) > 0, \xi \in int(F_n) (n = 1, 2, \ldots), \psi(F_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{F_{n_k}\}$ such that $\psi(A \cap F_{n_k}) > 0$ for $k = 1, 2 \ldots$*

The collection of all ψ-accumulation point of A is called the *ψ-derived set* of A and is denoted by $D_\psi(A)$.

Theorem I. *$A \backslash D_\psi(A)$ is a Vitali set if and only if $\psi[A\backslash D_\psi(A)] = 0$.*

Lemma I. *Let $\{A_\alpha : \alpha \in \Lambda\}$ be a collection of subsets of X, where Λ is an index set. Then* i) $D_\psi(\bigcap_\alpha A_\alpha) \subset \bigcap_\alpha D_\psi(A_\alpha)$, ii) $D_\psi(\bigcup_{n=1}^{N} A_n) = \bigcup_{n=1}^{N} D_\psi(A_n)$, *where $A_n, n = 1, 2, \ldots, N$ belong to the collection.*

Definition V. *A subset A is called ψ-closed if and only if $D_\psi(A) \subset A$.*

Definition VI. *A subset A is called ψ-open if and only if $X\backslash A$ is ψ-closed.*

Clearly X and the empty set φ are ψ-closed and ψ-open.

Theorem II. *The collection of all ψ-open sets forms a topology, denoted by (X, T_ψ).*

Proposition I. *If ψ is finitely additive then a subset A is ψ-open if and only if for every $\xi \in A$ there exists a sequence $\{F_k\}$ from $\boldsymbol{\mathcal{C}}$ with $\psi(F_k) > 0, \psi(F_k) \to 0$ as $k \to \infty$, $\xi \in int(F_k)(k = 1, 2, \ldots)$ such that $\psi(A \cap F_k) = \psi(F_k)$ for $k = 1, 2, \ldots$.*

Corollary I. *If A is a non-empty ψ-open set then $\psi(A) > 0$.*

Proposition II. *Let ψ be finitely additive, $A \subset X$ and $\xi \in X$. If there exists a ψ-one set U containing ξ such that $\psi(U \cap A) = 0$ then ξ is not a ψ-accumulation point of A.*

3. ψ-derived sets and Vitali sets. We first prove the following theorem on ψ-derived sets.

Theorem 1. *If for $A, \psi[D_\psi(A)\backslash A] = 0$ and ψ is finitely additive then $D_\psi(A)$ is ψ-closed.*

Proof. We shall show that $B = X\backslash D_\psi(A)$ is ψ-open. Let $\xi \in B$. Then there exists a sequence $\{F_k\}$ from $\boldsymbol{\mathcal{C}}$ such that $\xi \in int(F_k), \psi(F_k) > 0, \psi(F_k) \to 0$ as $k \to \infty$ and $\psi(A \cap F_k) = 0$ for $k = 1, 2, \ldots$ Now

(1) $\psi(F_k) = \psi[(F_k \cap B) \cup (F_k \cap D_\psi(A)] = \psi(F_k \cap B) + \psi[F_k \cap D_\psi(A)].$

Since $F_k \cap D_\psi(A) \subset [F_k \cap (D_\psi(A) \backslash A)] \cup [F_k \cap A]$, it follows that

$$\psi[F_k \cap D_\psi(A)] \leq \psi[F_k \cap (D_\psi(A) \backslash A)] + \psi(F_k \cap A) = 0$$

and so from (1), $\psi(F_k) = \psi(F_k \cap B)$ for $k = 1, 2, \ldots$. Hence by Proposition I B is ψ-open and the theorem is proved.

Corollary 1. *If A is ψ-closed and ψ is finitely additive then $D_\psi(A)$ is also ψ-closed.*

Proof. If A is ψ-closed then $D_\psi(A) \subset A$ and the proof follows from Theorem 1.

We now prove some theorems on Vitali sets in relation to ψ-derived sets.

Theorem 2. *If $A \backslash D_\psi(A)$ is a Vitali set then A can be expressed as $A = B \cup C$ where $\psi(B) = 0, C \subset D_\psi(C)$ and $B \cap C = \emptyset$.*

Proof. Let $B = A \backslash D_\psi(A)$ and $C = A \cap D_\psi(A)$. By Theorem I, $\psi(B) = 0$. Let $\xi \in C$. Then ξ is a ψ-accumulation point of a A. Let $\{F_n\}$ be an arbitrary sequences of sets from $\mathcal{C}$ such that $\psi(F_n) > 0, \psi(F_n) \to 0$ as $n \to \infty$ and $\xi \in int(F_n), n = 1, 2, \ldots$ So there exists a subsequence $\{F_{n_k}\}$ such that $\psi(A \cap F_{n_k}) > 0$ for $k = 1, 2, \ldots$

So for $k = 1, 2, \ldots$ we have

$$0 < \psi(A \cap F_{n_k}) = \psi[(B \cup C) \cap F_{n_k}] \leq \psi(C \cap F_{n_k}).$$

So, $\xi \in D_\psi(C)$ and the theorem is proved.

Theorem 3. *Let A be a Vitali set. Then A is ψ-closed and ψ-discrete (i.e. $A \cap D_\psi(A) = \emptyset$) if and only if $\psi(A) = 0$.*

Proof. If $\psi(A) = 0$ then A has no ψ-accumulation point and so A is ψ-closed and ψ-discrete. Conversely, if A is ψ-closed and ψ-discrete then $D_\psi(A) = \emptyset$ and so by Theorem I, $\psi(A) = \psi[A \backslash D_\psi(A)] = 0$ because $A \backslash D_\psi(A) = A$ is a Vitali set. This proves the theorem.

Theorem 4. *Let A and $A \backslash D_\psi(A)$ be Vitali sets. If $\psi[D_\psi(A)] = 0$ then $D_\psi(A) = \emptyset$, the void set.*

Proof. We see that

$$\psi(A) \leq \psi[A \backslash D_\psi(A)] + [A \cap D_\psi(A)] = 0$$

by Theorem I. So by Theorem 3, A is ψ-closed and ψ-discrete. Therefore $D_\psi(A) = \emptyset$ and this proves the theorem.

4. ψ-derived sets and metrically perfect sets. If $A \triangle D_\psi(A)$ is void, where $\triangle$ is the symmetric difference of sets, then A is in some sense metrically dense-in-itself (cf. [2], [5], [6], [8]). In this section we consider a more general situation which helps to study more closely the ψ-derived set of a set, Vitali sets and some topological consequences.

Definition 1. *A set $A \subset X$ with $\psi(A) > 0$ is called metrically perfect if $\psi[A \triangle D_\psi(A)] = 0$.*

Note 1. If $\psi(A) = 0$ then $D_\psi(A) = \emptyset$ and thus A is metrically perfect. We omit this trivial situation.

Note 2. If A is metrically perfect then $\psi[A \backslash D_\psi(A)] = 0$ and thus $A \backslash D_\psi(A)$ is a Vitali set by Theorem 1.

Example 1. Let $X = R$, the real line with the usual metric and ψ be the Lebesgue outer measure. Let $A = [0,1]$ then $D_\psi(A) = [0,1]$ and so A is metrically perfect.

Example 2. Let $X = R$, the real line with the usual metric and ψ be the Lebesgue outer measure. Let $I = (0,1)$ and A denote an open everywhere dense subset of I such that $\psi(A) = 1/2$ (see [1], p. 291 and also [4]). Then $D_\psi(A) = [0,1]$ and so A is not metrically perfect.

In the following theorem we decompose a metrically perfect set into two disjoint sets, one of which has ψ-measure zero.

Theorem 5. *If A is metrically perfect then $(\alpha) A = G \cup H$ where $\psi(G) = 0, H \subset D_\psi(H)$ and $G \cap H = \emptyset$. Conversely, if for a set $A, (\alpha)$ holds then $\psi[A \backslash D_\psi(A)] = 0$, i.e. $A \backslash D_\psi(A)$ is a Vitali set.*

Proof. If A is metrically perfect then from Note 2 it follows that $A \backslash D_\psi(A)$ is a Vitali set and so the first part follows from Theorem 2. Conversely, using Lemma I (ii), $A \backslash D_\psi(A) = A \backslash D_\psi(G \cup H) = A \backslash \{D_\psi(G) \cup D_\psi(H)\} = A \backslash D_\psi(H) \subset A \backslash H = G$, and the theorem is proved.

Theorem 6. *If A is ψ-measurable, metrically perfect and ψ is finitely additive then $D_\psi(A)$ is also ψ-measurable and metrically perfect.*

Proof. We show first that $D_\psi(A)$ is ψ-measurable. From definition we have

$$\psi[A\backslash D_\psi(A)] = 0 = \psi[D_\psi(A)\backslash A].$$

So both $A\backslash D_\psi(A)$ and $D_\psi(A)\backslash A$ are ψ-measurable. Then $A \cap D_\psi(A) = A\backslash[A\backslash D_\psi(A)]$ is also ψ-measurable. Now, ψ-measurability of $D_\psi(A)$ follows from the relation

$$D_\psi(A) = [A \cap D_\psi(A)] \cup [D_\psi(A)\backslash A].$$

Next, using Lemma I, we see that

$$D_\psi(A) \subset D_\psi[A\backslash D_\psi(A)] \cup D_\psi(D_\psi(A)).$$

We have clearly $D_\psi[A\backslash D_\psi(A)] = \emptyset$ and so $D_\psi(A) \subset D_\psi(D_\psi(A))$. The reverse inclusion follows from Theorem 1. Thus $D_\psi(A) = D_\psi(D_\psi(A))$ and this gives that $D_\psi(A)$ is metrically perfect.

Theorem 7. *If ψ is finitely additive and A is metrically perfect and ψ-open then $A \cap D_\psi(A)$ is also metrically perfect and ψ-open.*

Proof. We first show that $A \cap D_\psi(A)$ is metrically perfect. Since

$$D_\psi[A \cap D_\psi(A)]\backslash A \cap D_\psi(A) \subset D_\psi(A)\backslash A \cap D_\psi(A) = D_\psi(A)\backslash A,$$

we obtain

$$\psi[D_\psi(A \cap D_\psi(A))\backslash A \cap D_\psi(A)] = 0. \tag{2}$$

As ψ is finitely additive, for $F \in \boldsymbol{\mathcal{C}}$ we have

$$\psi(F \cap A) = \psi[F \cap A \cap D_\psi(A)] + \psi[F \cap (A\backslash D_\psi(A)] = \psi[F \cap A \cap D_\psi(A)]. \tag{3}$$

If $\xi \in A \cap D_\psi(A)$ then for $F_k \in \boldsymbol{\mathcal{C}}$ with $\psi(F_k) > 0, \xi \in int(F_k), k = 1, 2, \ldots, \psi(F_k) \to 0$ as $k \to \infty$, there exists a subsequence $\{F_{k_i}\}$ such that $\psi(F_{k_i} \cap A) > 0$, $i = 1, 2, \ldots$ So, from (3), $\psi[F_{k_i} \cap A \cap D_\psi(A)] > 0$ for $i = 1, 2, \ldots$ which implies that $\xi \in D_\psi[A \cap D_\psi(A)]$. Hence

$$\psi[A \cap D_\psi(A) - D_\psi(A \cap D_\psi(A))] = 0. \tag{4}$$

(2) and (4) shows that $A \cap D_\psi(A)$ is metrically perfect.

We now show that $A \cap D_\psi(A)$ is ψ-open. Since A is ψ-open, if $\xi \in A \cap D_\psi(A)$, we obtain by Proposition I, a sequence $\{F_k\}$ from $\boldsymbol{\mathcal{C}}$ such that $\xi \in int(F_k), \psi(F_k) > 0$ for $k = 1, 2, \ldots$ and $\psi(F_k) \to 0$ as $k \to \infty$ such that $\psi(A \cap F_k) = \psi(F_k) > 0$ for $k = 1, 2 \ldots$ Clearly from above

$$\psi(F_k) = \psi[F_k \cap A \cap D_\psi(A)] + \psi(F_k \cap A) - \psi[F_k \cap A \cap D_\psi(A)] \tag{5}$$

for $k = 1, 2, \ldots$ As in (3) we obtain

$$\psi(F_k \cap A) = \psi[F_k \cap A \cap D_\psi(A)] + \psi[F_k \cap (A \backslash D_\psi(A)] = \psi[F_k \cap A \cap D_\psi(A)].$$

So (5) gives $\psi[F_k \cap A \cap D_\psi(A)] = \psi(F_k)$ and by Proposition I, $A \cap D_\psi(A)$ is ψ-open. This proves the theorem.

Theorem 8. *Let ψ be finitely additive and A be metrically perfect. If A is both ψ-open and ψ-closed then $D_\psi(A)$ is also both ψ-open and ψ-closed.*

Prof. That $D_\psi(A)$ is ψ-closed follows from Theorem 1. Since $D_\psi(A) = A \cap D_\psi(A)$, it follows from Theorem 7 that $D_\psi(A)$ is ψ-open.

5. Some topological consequences. We know [7] that the collection of all ψ-open sets form a topology, which has been denoted by (X, T_ψ).

Definition 2. *For a subset A, the set $A \cup D_\psi(A)$ is called the ψ-closure of A and is denoted by $cl_\psi(A)$.*

Note 3. A is ψ-closed if and only if $A = cl_\psi(A)$.

Theorem 9. *If ψ is finitely additive and A is ψ-measurable and metrically perfect then $cl_\psi(A)$ is ψ-measurable, ψ-closed and metrically perfect.*

Proof. From Theorem 6 it follows that $cl_\psi(A)$ is ψ-measurable. Since A is metrically perfect, using Theorem 1 and Lemma 1 we can see that

$$D_\psi[cl_\psi(A)] = D_\psi(A) \cup D_\psi(D_\psi(A)) \subset D_\psi(A) \subset cl_\psi(A)$$

and thus $cl_\psi(A)$ is ψ-closed.

By Lemma I and Theorems 1 and 6 we obtain

$$D_\psi[cl_\psi(A)] = D_\psi(A) \cup D_\psi(D_\psi(A)) = D_\psi(A)$$

and so

$$\psi[cl_\psi(A)\triangle D_\psi(cl_\psi(A))] \leq \psi[A\backslash D_\psi(A)] = 0.$$

This proves the theorem.

Corollary 2. *If $A\backslash D_\psi(A)$ is a Vitali set and ψ is finitely additive then $cl_\psi(A)\backslash D_\psi[cl_\psi(A)]$ is also a Vitali set.*

Proof. Since $\psi[A\backslash D_\psi(A)] = 0$, the proof follows from the proof of the last part of Theorem 9.

We now obtain the usual characterization of $cl_\psi(A)$, provided A is metrically perfect.

Theorem 10. *If A is metrically perfect and ψ is finitely additive then*

$$cl_\psi(A) = \cap[F : F \supset A, F \text{ is } \psi - \text{closed}].$$

Proof. If $\overline{A}$ stands for the right hand expression then from Lemma I it follows that $\overline{A}$ is ψ-closed an so by Note 3 $cl_\psi(A) \subset cl_\psi(\overline{A}) = \overline{A}$. Also $A \subset cl_\psi(A)$ and from the proof of Theorem 9 it follows that $cl_\psi(A)$ is ψ-closed. So $\overline{A} \subset cl_\psi(A)$. Thus $cl_\psi(A) = \overline{A}$ and the theorem is proved.

Note 4. We see from Theorem 10 that $cl_\psi(A)$ coincides with the closure of A in the topological space (X, T_ψ) if A is metrically perfect and ψ is finitely additive.

Theorem 11. *If ψ is finitely additive then (X, T_ψ) is not separable.*

Proof. If possible, suppose the contrary. Then there exists a countable set A such that $cl_{T_\psi}(A) = X$ where $cl_{T_\psi}(A)$ is the closure of A in (X, T_ψ). Since $\psi(A) = 0$ and so $D_\psi(A) = \emptyset$, it follows that A is metrically perfect and so by Theorem 10, $cl_\psi(A) = X$. Hence

$$\psi(X) = \psi[cl_\psi(A)] = \psi(A) = 0,$$

a contradiction. This proves the theorem.

Theorem 12. *(X, T_ψ) is not first countable.*

Proof. Let $\xi \in X$. If possible, suppose that $\{B_i\}$ is a countable ψ-open base at ξ. By Corollary I each $\psi(B_i) > 0$ and so each B_i is uncountable.

We choose $\xi_1 \neq \xi$ from $B_1, \xi_2 \neq \xi,\ \xi_1$ from $B_2, \xi_3 \neq \xi, \xi_1, \xi_2$ from B_3, and so on. Let $A = \{\xi_1, \xi_2, \ldots\}$. Then A is ψ-closed because $\psi(A) = 0$. Let U be an arbitrary ψ-open set containing ξ. Then $U\backslash A = X \cap (U\backslash A)$ is ψ-open and contains ξ. Clearly no B_i is contained in $U\backslash A$ and so $\{B_i\}$ is not a base at ξ. This contradiction proves the theorem.

In the following we like to show that the topological space (X, T_ψ) can also be generated without using explicitly the concept of ψ-accumulation points.

Definition 3. *A point $\xi \in X$ is called a ψ^*-accumulation point of a set A if for every ψ-open set U containing ξ we have $\psi(U \cap A) > 0$. The collection of all ψ^*-accumulation points of A is called the ψ^*-derived set of A and is denoted by $D_{\psi^*}(A)$.*

We can show, using Proposition II that if ψ is finitely additive then for any A, $D_\psi(A) \subset D_{\psi^*}(A)$. Also as in [6], we say a subset A to be ψ^*-closed if $D_{\psi^*}(A) \subset A$ and ψ^*-open if $X\backslash A$ is ψ^*-closed.

Clearly the void set and X are ψ^*-open and ψ^*-closed. Proving a lemma on $D_{\psi^*}(A)$ similar to Lemma I we can state the following theorem.

Theorem 13. *The collection of all ψ^*-open sets forms a topology denoted by (X, T_{ψ^*}).*

Theorem 14. *If ψ is finitely additive then $(X, T_\psi) = (X, T_{\psi^*})$.*

Proof. We shall show that any ψ^*-open set is ψ-open and conversely. Let 0 be a ψ^*-open set. Then $D_{\psi^*}(X\backslash 0) \subset X\backslash 0$ and because for any A, $D_\psi(A) \subset D_{\psi^*}(A)$, we obtain

$$D_\psi(X\backslash 0) \subset D_{\psi^*}(X\backslash 0) \subset X\backslash 0.$$

Thus $X\backslash 0$ is ψ-closed and so 0 is ψ-open. Let now 0 be ψ-open. So no ψ^*-accumulation point of $X\backslash 0$ can lie in 0, because otherwise $\psi[0 \cap (X\backslash 0)] > 0$ which is not possible. So 0 is ψ^*-open. This proves the theorem.

Definition 4. *The ψ-interior of a set A, denoted by $int_\psi(A)$ is*

$$int_\psi(A) = \cup\{G : G \subset A \text{ and } G \text{ is } \psi - open\}.$$

Theorem 15. *Suppose that A is metrically perfect and ψ is finitely additive. If $\psi(A) = 0$ then A is nowhere dense in (X, T_ψ).*

Proof. If $\psi(A) = 0$ then A is ψ-closed and so $A = cl_\psi(A)$. We note that $int_\psi(A) = \emptyset$, because otherwise by, Corollary I,

$$0 < \psi[int_\psi(A)] \leq \psi(A)$$

which is a contradiction. Using Note 4, we see that

$$int_\psi[cl_{T_\psi}(A)] = int_\psi[cl_\psi(A)] = int_\psi(A) = \emptyset$$

and so A is nowhere dense in (X, T_ψ).

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