ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" IAŞI
Tomul LII, s.I, Matematică, 2006, f.1

CONVERGENCE OF SOLUTIONS OF NONLINEAR DIFFERENCE EQUATIONS

BY

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Abstract. There are established the sufficient conditions for the convergence of the solutions of equation (1).

Key words: Nonlinear difference equation, convergence.

We shall be concerned with the difference equation

(1) $\Delta^3 x(n)+a\Delta^2 x(n)+b\Delta x(n)+h(x(n))+p(n,x(n),\Delta x(n),\Delta^2 x(n))=0$

where $I=<n_0,\infty)\subset N_0=N\cup\{0\}$,
$D=\{(n,x,y,u): n\in N, |x|+|y|+|u|<\infty\}\subset R^4$,
$|\cdot|$ - euclidean norm,
$h:R\to R,\ p:D\to R$ and h,p are continuous for $x\in R$ and $(n,x,y,u)\in D$ respectively and differentiable as functions of x, respectively of x, y and u, a,b, are constants,
Δ - "difference operator" is defined by

$$\Delta x(n)=x(n+1)-x(n), n\in N_0,$$

and

$$h_x(x), p_x(n,x,y,u), p_y(n,x,y,u), p_u(n,x,y,u)$$

are derivative and partial derivatives of $h(x)$ and $p(n,x,y,u)$ respectively. By a solution of (1) we mean a real sequence denoted by $\{u_n\}\equiv u(n)$ satisfying equation (1) for $n=0,1,2,\ldots$.

Definition 1. *Solutions of* (1) *are said convergent if, for every pair of solutions* $x(n)$ *and* $y(n)$ *of* (1)

$$\lim_{n\to\infty}[x(n)-y(n)]=0. \tag{2}$$

Definition 2. *Solutions of* (1) *are said convergent strongly if, for every pair of solutions* $x(n), y(n)$ *of* (1)

$$\lim_{n\to\infty}[x(n)-y(n)]=0,$$

and

$$\lim_{n\to\infty}[\Delta x(n)-\Delta y(n)]=0, \tag{3}$$

and

$$\lim_{n\to\infty}[\Delta^2 x(n)-\Delta^2 y(n)]=0.$$

The present work is a continuation of an earlier investigation by the first author [2]. The main purpose of the present paper is to establish certain criteria for the convergence of all bounded solutions of (1). In particular, we have obtained various conditions for the convergence of all solutions of (1).

The problem similar to (2) for differential equations was discussed among other by the Lasota [1], Muntean [4], Swick [5] and Yoshizawa [6].

Theorem 1. *If all solutions of the equation*

$$\begin{gathered}\Delta^3 z(n)+(a+p_u(n,x,y,u))\Delta^2 z(n)+(b+p_y(n,x,y,u))\Delta z(n)+\\ +(h_x(x)+p_x(n,x,y,u))z(n)=0\end{gathered} \tag{4}$$

tend to zero as $n\to\infty$ *for arbitrary* $|x|+|y|+|u|<\infty$*, then all bounded solutions of* (1) *are convergent.*

Proof. Let $x_1(n), x_2(n)$ be a pair of defined on $I\subset N_0$ and bounded solutions of (1) and let V be a function

$$\begin{aligned} V(n,s) &= V(n,s,x_1(n),x_2(n)) \\ &= s\Delta^3 x_1(n) + (1-s)\Delta^3 x_2(n) + as\Delta^2 x_1(n) + a(1-s)\Delta^2 x_2(n) \\ &+bs\Delta x_1(n) + b(1-s)\Delta x_2(n) + h(sx_1(n)+(1-s)x_2(n)) \\ &+p(n, sx_1(n)+(1-s)x_2(n), s\Delta x_1(n) \\ &+(1-s)\Delta x_2(n), s\Delta^2 x_1(n) + (1-s)\Delta^2 x_2(n)). \end{aligned}$$

Let $n \in I$ is an arbitrary, then

$$\begin{aligned} V(n,0) &= \Delta^3 x_2(n) + a\Delta^2 x_2(n) + b\Delta x_2(n) + h(x_2(n)) \\ &+p(n, x_2(n), \Delta x_2(n), \Delta^2 x_2(n)) \equiv 0, \end{aligned}$$

and

$$\begin{aligned} V(n,1) &= \Delta^3 x_1(n) + a\Delta^2 x_1(n) + b\Delta x_1(n) + h(x_1(n)) \\ &+p(n, x_1(n), \Delta x_1(n), \Delta^2 x_1(n)) \equiv 0. \end{aligned}$$

Applying the mean value theorem there exists $\{q_n\}, (0 < q_n < 1)$ such that

$$\frac{\partial}{\partial s} V(n, q_n) = 0 \quad for \quad n \in I,$$

i.e.

$$\begin{aligned} &\Delta^3 x_1(n) - \Delta^3 x_2(n) + a(\Delta^2 x_1(n) - \Delta^2 x_2(n)) \\ &+b(\Delta x_1(n) - \Delta x_2(n)) + h_x(q_n x_1(n) + (1-q_n)x_2(n))(x_1(n) - x_2(n)) \\ &+p_x(n, q_n x_1(n) + (1-q_n)x_2(n), q_n\Delta x_1(n) + (1-q_n)\Delta x_2(n), q_n\Delta^2 x_1(n) \\ &+(1-q_n)\Delta^2 x_2(n))(x_1(n) - x_2(n)) \\ &+p_y(n, q_n x_1(n) + (1-q_n)x_2(n), q_n\Delta x_1(n) + (1-q_n)\Delta x_2(n), q_n\Delta^2 x_1(n) \\ &+(1-q_n)\Delta^2 x_2(n))(\Delta x_1(n) - \Delta x_2(n)) \\ &+p_u(n, q_n x_1(n) + (1-q_n)x_2(n), q_n\Delta x_1(n) + (1-q_n)\Delta x_2(n), q_n\Delta^2 x_1(n) \\ &+(1-q_n)\Delta^2 x_2(n))(\Delta^2 x_1(n) - \Delta^2 x_2(n)) = 0. \end{aligned}$$

It is clear that $z(n) = x_1(n) - x_2(n)$, is a solution of (4) for $n \in I$ if

$$\begin{aligned} x &= q_n x_1(n) + (1-q_n)x_2(n), \\ y &= q_n \Delta x_1(n) + (1-q_n)\Delta x_2(n), \\ u &= q_n \Delta^2 x_1(n) + (1-q_n)\Delta^2 x_2(n). \end{aligned}$$

Since all solutions of (4) tend to zero as $n \to \infty$, it follows that all solutions of (1) converge since $x_1(n)$ and $x_2(n)$ were arbitrary solutions of (1).

In the next theorems we display different type conditions under which all solutions of the equation (4) tend to zero as $n \to \infty$. Then, according to theorem (1), all bounded solutions of (1) are convergent.

Theorem 2. *If there exists constants* $\alpha, \beta, a_i, b_i, c_i, (i = 1, 2, 3)$ *such that in* D :

1^o $\alpha > \beta^2 \geq 0, \quad \alpha(\alpha - \beta^2) > 1,$

2^o $2A - \alpha A^2 \geq a_1 + b_1 + c_1, \quad a_1 > 0, b_1 > 0, c_1 > 0,$

$2B - \alpha - 2\beta - \alpha\beta^2 \geq a_2 + b_2 + c_2, \quad a_2 > 0, b_2 > 0, c_2 > 0,$

$\alpha - 1 - 2C^2 \geq a_3 + b_3 + c_3, \quad a_3 > 0, b_3 > 0, c_3 > 0,$

3^o $(2B + 2A - 2\alpha - 2\alpha AB)^2 \leq 4b_1a_2,$

$(2\alpha BC - 2 - 2C - 2\beta)^2 \leq 4a_3b_2,$

$(2\beta + 2C - 2\alpha AC - 2)^2 \leq 4c_3c_1,$

where

$$\begin{aligned} A &= A(n,x,y) = h_x + p_x(n,x,y,u), \\ B &= B(n,x,y,u) = b + p_y(n,x,y,u), \\ C &= C(n,x,y,u) = 1 - a - p_u(n,x,y,u), \end{aligned} \tag{5}$$

then all bounded solutions of (1) *are convergent strongly.*

Proof. By setting $z(n) = w_1(n), \Delta z(n) = w_2(n), \Delta^2 z(n) = w_3(n)$, the equation (4) may be replaced with the system

$$
\begin{aligned}
w_1(n+1) &= w_1(n) + w_2(n),\\
w_2(n+1) &= w_2(n) + w_3(n),\\
w_3(n+1) &= -Aw_1(n) - Bw_2(n) + Cw_3(n)
\end{aligned}
\tag{6}
$$

or

$$w(n+1) = M(n)w(n) \tag{6'}$$

where

$$M(n) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -A & -B & C \end{bmatrix} \quad \text{and} \quad w(n) = [w_1(n), w_2(n), w_3(n)]^T.$$

Consider the quadratic form

$$
\begin{aligned}
W(w) = W(w_1, w_2, w_3) &= [w_1, w_2, w_3] \begin{bmatrix} \alpha & \beta & 1 \\ \beta & 1 & 0 \\ 1 & 0 & \alpha \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\
&= \alpha {w_1}^2 + 2\beta w_1 w_2 + 2 w_1 w_3 + {w_2}^2 + \alpha {w_3}^2,
\end{aligned}
\tag{7}
$$

which is positive defined according to 1^o. Consider the variation of W along the solutions of (6)

$$
\begin{aligned}
\Delta W(w(n)) &= W(w(n+1)) - W(w(n)) \\
&= \alpha {w_1}^2(n+1) + 2\beta w_1(n+1) w_2(n+1) + 2 w_1(n+1) w_3(n+1) \\
&\quad + {w_2}^2(n+1) + \alpha {w_3}^2(n+1) - \alpha {w_1}^2(n) - 2\beta w_1(n) w_2(n) \\
&\quad - 2 w_1(n) w_3(n) - {w_2}^2(n) - \alpha {w_3}^2(n) \\
&\le -a_1 {w_1}^2 - c_2 {w_2}^2 - b_3 {w_3}^2
\end{aligned}
$$

which is negative defined according to 2^o and 3^o.

We conclude from this that, the equation (6') is asymptotically stable, and all its solutions tend to zero as n tends to infinity.

According to Theorem 1 all bounded solutions of (1) are convergent or strongly convergent.

The relation (6) may be written as a system

$$w(n+1) = Pw(n) + Q(n)w(n), \tag{8}$$

where

$$P = \begin{bmatrix} 1-\alpha & 1-\beta & 0 \\ 0 & 1-\alpha & 1-\beta \\ 0 & -b & 1-a-\alpha \end{bmatrix}, Q(n) = \begin{bmatrix} \alpha & \beta & 0 \\ 0 & \alpha & \beta \\ -h_x - g_x & -g_y & -g_u + \alpha \end{bmatrix}$$

$\alpha, \beta = \text{const}.$

Theorem 3. *Suppose there are a constants $\alpha, \beta \in R$ and the function $\varphi : N_0 \to R^+ =< 0, \infty)$ with $\Delta\varphi(n) < 0$, $\sum\limits_{n=0}^{\infty} \varphi(n) < \infty$ and $(|P| + |Q(n)| + \varphi(0) - 1) < 0$ for $n \in N_0$, then all bounded solutions of equation (1) are convergent strongly.*

Proof. Define

$$\overline{V}(n, w(\cdot)) = |w(n)| + \sum_{s=0}^{n-1} \varphi(n-s-1)|w(s)|.$$

Along solutions of (8) we have

$$\begin{aligned} &\Delta\overline{V}(n, w(\cdot)) = |w(n+1)| - |w(n)| \\ &+ \sum_{s=0}^{n} \varphi(n-s)|w(s)| - \sum_{s=0}^{n-1} \varphi(n-s-1)|w(s)| \\ &\leq (|P| + |Q(n)| + \varphi(0) - 1)|w(n)| + \sum_{s=0}^{n-1} \Delta\varphi(n-s-1)|w(s)| < 0. \end{aligned}$$

Therefore under the conditions $(|P| + |Q(n)| + \varphi(0) - 1) < 0, \Delta\varphi(n) < 0$ it follows that the equation (6) is asymptotically stable, and all its solutions tend to zero as $n \to \infty$. Then by Theorem 1 all bounded solutions of (1) are convergent strongly.

System (6) is equivalent to the system

$$w(n+1) = Pw(n) + Q(n)w(n) \tag{9}$$

where

$$P = \begin{bmatrix} 1-\alpha & 1 & 0 \\ 0 & 1-\alpha & 1 \\ 0 & -b & 1-a \end{bmatrix}, Q(n) = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ -h_x - g_x & -g_y & -g_u \end{bmatrix},$$

$\alpha = const.$

Theorem 4. *Suppose that*

1^o $\alpha \in R,\ \alpha > 1,\ \alpha > 2 - a$ *and* $(\alpha - 1)(1 - a) < b,$

2^o $\sum_{n=0}^{\infty} |Q(n)| \leq c < \infty,$

then all bounded solution of (1) *are convergent.*

Proof. Let ϕ denote fundamental matrix of system

$$w(n+1) = Pw(n),$$

then, by the variation of constants formula the solution of (9) is given by

$$w(n) = \phi(n,0)w(0) + \sum_{s=0}^{n-1} \phi(n, s+1)Q(s)w(s). \tag{10}$$

From assumption 1^o it follows that $|\phi(n,s)| \leq M\eta^{(n-s)},\ n \geq s \geq 0$ for some $M \geq 1$ and $\eta \in (0,1)$. Thus

$$w(n)\eta^{-n} \leq M|w(0)| + M\eta^{-1} \sum_{s=0}^{n-1} |Q(s)|\eta^{-s}|w(s)|.$$

Then applying the Gronwall inequality one obtain

$$|w(n)| \leq \eta^n M|w(0)| \exp[M\eta^{-1} \sum_{s=0}^{n-1} |Q(s)|].$$

This proves $w(n) \to 0$ as $n \to \infty$.
Consequently all solutions of the equation (1) are convergent strongly.

Consider the equation corresponding to equation (6')

$$\Delta w(n) = f(n, w(n)), \tag{11}$$

where $f(n, w(n)) = (M(n) - I)w(n)$.
Let Z denote the set

$$Z = N_0 \times D_r,\ D_r = \{w = [w_1, w_2, w_3] : |w| < r\},$$

$$|w| = (w, u)^{\frac{1}{2}}, (w, u) = \sum_{i=1}^{3} w_i u_i, D_r \subset R^3, r > 0$$

and in particular $r = +\infty$ is possible.

Theorem 5. *Assume that*

1^o $2w \cdot f(n, w) + f(n, w) \cdot f(n, w) =$

$$= \sum_{i=1}^{3} [2w_i f_i(n, w) + f_i(n, w) \cdot f_i(n, w)] \leq 0$$

for each $(n, w) \in Z$,

2^o $\limsup_{\substack{n\to\infty \\ w\to\bar{w}}} [2w \cdot f(n, w) + f(n, w) \cdot f(n, w)] = \delta < 0$

for any $\bar{w} \in D_r,\ |\bar{w}| > 0$.
Then all bounded solutions of equation (1) *are convergent strongly.*

Proof. It follows from Lemma 1 [3] that the solution $w = 0$ of (11) is stable.
Now we shall show that the solution $w = \varphi(n)$ of system (11) satisfies the condition

$$\lim_{n\to\infty} |\varphi(n)| = 0.$$

The function

$$u(n) = |\varphi(n)|^2 = \sum_{i=1}^{3} (\varphi_i(n))^2 \tag{12}$$

is non-increasing, because

(13)

$$\Delta u(n) = \sum_{i=1}^{3}[(\varphi_i(n+1))^2 - (\varphi_i(n))^2] = \sum_{i=1}^{3} \Delta\varphi_i(n)[\varphi_i(n+1) - \varphi_i(n)] \leq 0.$$

We will show that $\lim_{n\to\infty} u(n) = 0$.

For this purpose we assume that the least condition does not hold. Then $u(n) > 0$ and $\Delta u(n) \leq 0$ for any $n \in N(0)$ implies that

$$\lim_{n\to\infty} u(n) = \vartheta,\ \vartheta > 0 \tag{14}$$

From (12), (13), and (14) we conclude that there exists the limit

$$\limsup_{n\to\infty} \varphi(n) = \bar{\bar{w}} \neq 0$$

and from assumption 2^o we infer that

$$\limsup_{n\to\infty} \Delta u(n) = \limsup_{n\to\infty} \sum_{i=1}^{3} \Delta\varphi_i(n)[\varphi_i(n+1) + \varphi_i(n)] = \delta < 0.$$

Now from Lemma 2 [3] we have

$$\lim_{n\to\infty} u(n) = -\infty,$$

what contradicts (14) and all solutions tend to zero as $n \to \infty$.
According to Theorem 1 all bounded solutions of (1) are convergent strongly.

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Received: 11.VII.2005
Revised: 20.XII.2005

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