ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" IAŞI
Tomul LII, s.I, Matematică, 2006, f.1

SOME LARGE DEVIATION FOR ONE SAMPLE

BY

BOGDAN-GHE. MUNTEANU

Abstract. This article present the problem of large deviation for stochastic processes with application in secvential statistics. This for difference of clasic statistics which interpretation the results of one special sample search the dimension of sample decrease in this way the errors. In other way it's trying to take decision at every moment.

Mathematics Subject Classification 2000: 60E10, 60F10, 60F15, 60G50, 60J50.
Key words: Cramer transformation, the sample of one law, limit laws.

1. The Cramer transformation of one law. Let $\Psi : I \to R$ be one convex function, $I \subset R$.

Definition 1. *Can call Young transformation h of Ψ function, the definition of $\Re$, $h : J \to \Re$, $J \subset \Re$, $a \in J$*

$$h(a) = \sup_{u \in I}[ua - \Psi(u)]$$

It's possible to see h it is convex too (it's the superior bound of one degree function).

Can suppose Ψ it's strictly increasing (the graphic not contain a right segment) and derived inside of set of I (not. $\overset{*}{I}$); derivative it's strictly increasing.

Let $a \in \Psi'(\overset{*}{I})$ be. Then derivative of function $u \mapsto ua - \Psi(u)$ is $a - \Psi'(u)$. Cancel in $u = \Psi'^{-1}(a) = g(a)$. Then $h(a) = ag(a) - \Psi(g(a))$. If Ψ is derived twice, then $h'(a) = g(a)$.

So $h'(a) = \Psi'^{-1}(a)$, in this way can see the functions h and Ψ, are define on $\Psi'(\overset{*}{I})$ and $\overset{*}{I}$, have the same derivatives. The h is convex, then on $\overset{*}{I}$ is:

$\Psi(u) = \sup_{a\in\Psi'(\overset{*}{I})} [ua - h(a)]$.

The relation is duality relation.

Let X be a random variable of law F and $I_F = \{t, \quad E_F(e^{tX}) < \infty\}$ where $E(e^{tX}) = \int e^{tx} dF(x)$. After [1,p.73], I_F is a interval (α_F, β_F) (the bounds can be known or not) and the function is definite on I_F: $t \mapsto \Psi_F(t) = lnE(e^{tX})$ is convex.

Definition 2. *The Cramer transformation h_F of law F on $\Re$ is Young transformation of Ψ_F function:*

$$h_F(a) = \sup_{t\in I_F} [at - \Psi_F(t)]$$

After the propotition [1,p.73], results F has moments of any order and can derive Ψ_F in 0.

Proposition 1. *The function $t \mapsto \Psi_F(t) = lnE(e^{tX})$ is definite on $I_F = \{t, \quad E_F(e^{tX}) < \infty\}$ with Cramer transformation has next conditions:*

i. $\Psi_F(0) = 0$, $\Psi'_F(0) = m$;

ii. $h'_F(m) = 0$; $h_F(m) = -\Psi_F(0) = 0$.

2. The large deviation for one sample. The construction geometry. The asymptotic theory met in probability theory answer to the law of large numbers $Z_n = \frac{X_1+...+X_n}{n} \overset{a.s}{\to} m$ and a central limit theorem which study the convergence $\sqrt{n}(Z_n - m)$.

In the same way can study the convergence to zero of $P[\mid Z_n - m \mid > a]$ for an $a > 0$; a theorem of this type is **the large deviation theorem** [1, theorem 4.4.22]. The usual formulate of those results is the type:

$$\overline{\lim}_{n\to\infty} \frac{1}{n} \cdot lnP[\mid Z_n - m \mid > a] = -h(a) < 0$$

or for any $\varepsilon > 0$ and n very large :

$$P[\mid Z_n - m \mid > a] \le e^{-n[h(a)-\varepsilon]}$$

Can say Z_n **tend to** m **with exponential speed**.

Same time can get a decrease

$$\underline{\lim}_{n\to\infty}\frac{1}{n}\cdot lnP[|\ Z_n-m\ |>a]=-h(a)<0$$

or

$$P[|\ Z_n-m\ |>a]\geq e^{-n[h(a)+\varepsilon]}$$

In these kind of condition can say the row Z_n **converge to** m **with a** $e^{-nh(a)}$ speed.

In first section we definited **Cramer transformation** h_F of F from:

$$h_F(a)=\sup_t\{at-\Psi(t);\ t\in(\alpha_F,\beta_F)\}$$

Concomitantly Ψ' function is strictly increasing on (α_F,β_F) and Ψ is convex in concordance with [1, p.73].

Prolong the functions Ψ' and Ψ in α_F (and β_F too), and take the limits to the left (respectively to the right), finished or not.

In these conditions, can have next analytic present for Cramer transformation:

$$h_F(a)=\begin{cases} a\beta_F-\Psi(\beta_F) & \text{for}\quad a\geq\Psi'(\beta_F)\\ a\alpha_F-\Psi(\alpha_F) & \text{for}\quad a\leq\Psi'(\alpha_F)\\ a\Psi'^{-1}(a)-\Psi[\Psi'^{-1}(a)] & \text{for}\quad \Psi'(\alpha_F)<a<\Psi'(\beta_F)\end{cases}$$

On $(\Psi'(\alpha_F),\Psi'(\beta_F))$, h'_F is the mutual function of Ψ' function. The zero point is always in $[\alpha_F,\beta_F]$, and $\Psi'(0)=m$, $h_F(m)=0$ and $\Psi'^{-1}(m)=0$ in accordance with proposition 1.

Either $a\in(\Psi'(\alpha_F),\Psi'(\beta_F))$, $\Psi'(h'_F(a))=a$ (for $h'_F(a)=\Psi'^{-1}(a)$) is the fact like whrite equation:

$$(d):\quad y=a(x-h'_F(a))+\Psi(h'_F(a))=ax-h_F(a)$$

is tangent to the graphic function Ψ (see fig. 1).
From here next construction: draw the graphic Ψ function, can see the evidently tangent to the Ψ graphic; the intersection of these tangents with y axle wich is $-h_F(a)$.

Theorem 1. (Chernov) *Let (X_n) be one sample of law F on $\Re$. Supose F-integrable, mean m and Cramer transformation h_F.*

We consider $S_n=X_1+...+X_n$ and $I=\{t;\ \int e^{tx}\mathrm{d}F(x)<\infty\}$ next to 0.

i.

$$For \quad a > m, \quad P[S_n \geq na] \leq e^{-nh_F(a)}$$

$$For \quad a < m, \quad P[S_n \leq na] \leq e^{-nh_F(a)}$$

where $h_F(a) > 0$*;*

ii. *Let us consider* $\Psi(t) = ln \int e^{tx} \mathrm{d}F(x)$; $\beta_F = \sup\{\Psi'(t),\, t \in I\}$ *and* $\alpha_F = \inf\{\Psi'(t),\, t \in I\}$.

$$For \quad m < a < \beta_F \quad \lim_{n\to\infty} \frac{1}{n} \cdot lnP[S_n \geq na] = -h_F(a)$$

$$For \quad \alpha_F < a < m \quad \lim_{n\to\infty} \frac{1}{n} \cdot lnP[S_n \leq na] = -h_F(a)$$

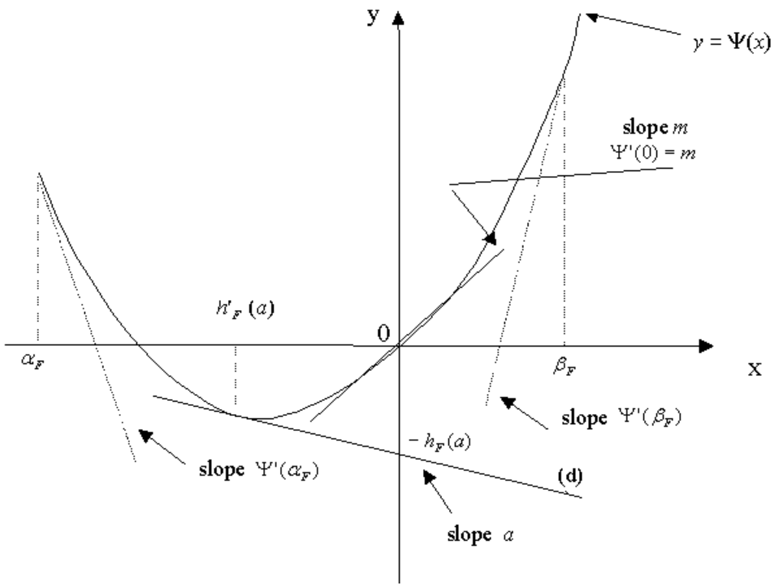

Figure 1: geometry construction

Proof. Suppose next $a > m$. The case $a < m$ results if change (X_n) in $(-X_n)$.

i.

$$\begin{aligned} P[S_n \geq na] &= P\left[e^{u(S_n - na)} \geq 1\right] \leq \inf E\left[e^{-una + uS_n}\right] \\ &= \inf \left\{ e^{-una} \underbrace{E(e^{uS_n})}_{e^{\Psi_F(u)}} \right\} = e^{-n \sup\{ua - \Psi_F(u)\}} = e^{-nh_F(a)} \end{aligned}$$

with $u \in (0, \beta_F)$.

ii. Remaining to stable $\underline{\lim}_{n\to\infty} \frac{1}{n} \cdot lnP[S_n \geq na] \geq -h_F(a)$.
We'll make for this a change of probabilities.
Either $t \in (0, \beta_F)$. We consider $F_t = f_t \cdot F$ where $f_t(x) = (\Phi(t))^{-1} e^{tx}$; is a probability on $\Re$.
In space $(\Omega, \mathcal{A}) = (\Re, \mathcal{B}_{\Re})^N$, notify: $P = F^{*N}$ and $P_t = F_t^{*N}$. On generate σ-algebra of X, that is on $\sigma(X_1, ..., X_n)$ we have: $P_t = (f_t \cdot F)^{*N} = (\Phi(t))^{-n} e^{tS_n} P$ and $P = \Phi^n(t) e^{-tS_n} P_t$.
Where:

$$\begin{aligned} P[S_n \geq na] &= E_t\left[\Phi^n(t) e^{-tS_n} \mathbf{1}_{(S_n \geq na)}\right] \\ &= \Phi^n(t) e^{-nta} E_t\left[e^{nta} e^{-tS_n} \mathbf{1}_{(S_n \geq na)}\right] \\ &= \Phi^n(t) e^{-nta} E_t\left[e^{-nt(\frac{S_n}{n} - a)} \mathbf{1}_{(S_n \geq na)}\right] \end{aligned}$$

Let $\varepsilon > 0$ be. If it's logarithming the last relation, we obtain:

$$\begin{aligned} \frac{1}{n} \ln P[S_n \geq na] &= \ln \Phi(t) - at + \frac{1}{n} \ln E_t\left[e^{-nt(\frac{S_n}{n} - a)} \mathbf{1}_{(S_n \geq na)}\right] \\ &\geq -at + \ln \Phi(t) + \frac{1}{n} \ln E\left[e^{-nt\varepsilon} \mathbf{1}_{na \leq S_n \leq na + \varepsilon}\right] \\ &= -at + \Psi(t) + \frac{1}{n}\left[\ln e^{-nt\varepsilon} + \ln E\left(\mathbf{1}_{na \leq S_n \leq na + \varepsilon}\right)\right] \\ &= -at + \Psi(t) - t\varepsilon + \frac{1}{n} \ln E\left(\mathbf{1}_{na \leq S_n \leq na + \varepsilon}\right) \end{aligned}$$

The mean of F_T is $\Psi(t)$. If $a \in (m, \Psi'(\beta_F))$ and $t \in (0, \beta_F) \Rightarrow 0 < \Psi'(t) < \Psi'(\beta_F)$, then can see $\Psi'(t)$ is situate close to a, that is $a < \Psi'(t) < a + \varepsilon$.
For any n, $P_t\left(na \leq S_n \leq na + \varepsilon\right) \leq A$, with $A \in [0, 1]$. Where:

$$\frac{1}{n} \ln P[S_n \geq na] \geq -at + \Psi(t) - t\varepsilon + \frac{1}{n} \ln A$$

Other way:

$$\underline{\lim}_{n\to\infty}\frac{1}{n}\ln P[S_n \geq na] \geq -at + \Psi(t) - t\varepsilon$$

How ε is arbitrary, we obtain get what was expectation, anyone like $\Psi'^{-1}(a)$ $\leq t \leq \Psi'^{-1}(a+\varepsilon)$ □

REFERENCES

1. Dacunha-Castelle, D.; Duflo, M. – *Probabilités et statistique-problémes a temps fixe*, Masson Paris Milan Barcelone, 1994.
2. Dacunha-Castelle, D.; Duflo, M. – *Probabilités et statistique-problémes a temps mobile*, Masson Paris Milan Barcelone, 1993.
3. Janssen, A. – *Conditions for local asymptotic normality of exponential families,* Statistic Decisions, 10, 1992, 173-182
4. Nussbaum, M. – *Asymptotic equivalence of density estimation and gaussian withe noise,* Ann.Statist. 24, 2399-2430, 1996.
5. Orman, G.V. – *Masura si procese aleatoare*, Editura Universitatii Transilvania, 2000.
6. Orman, G.V. – *Handbook of limit theorems and stochastic approximation*, Transilvania University Press, 2003.
7. Petrov, V.V. – *Limit Theorems of Probability Theory; Sequences of Independent Random Variables*, Clarendon Press, Oxford, 1995.
8. Wentzell, A.D. – *Limit Theorems on Large Deviations for Markov Stochastic Processes*, Kulwer Academic Publishers, Dordrecht/Boston/London,1990.

Received: 11.IV.2005

"Transilvania" University of Braşov,
Departament of Mathematical Analysis and Probability,
Iuliu Maniu 50 Street, 2200 Braşov,,
ROMÂNIA
munteanu77@yahoo.com