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SINE TRANSFORMATION FOR REACTION DIFFUSION CONTROL PROBLEM

BY

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Abstract. The Reaction Diffusion Equations system is transformed into an optimization problem and the result control problem has opened us to the possibility that will allow us to study Reaction Diffusion Equations and their numerically applications. Specifically, the modified version of Conjugate Gradient Method (CGM) algorithm can be used as a method of solving for some of its applications.

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1. Introduction. This is an first attempt to bring the concept of optimization theory and application into Reaction Diffusion Equations. The work concerns with the transformation of linear reaction diffusion equations into a control problem so that optimization techniques can be used for solving the resulting problem. This new area of research is designated by the control of reaction diffusion equations.

Basically, this work considers the transformation of reaction diffusion system into the quadratic cost functional with differential equations as constraints. This type of problem can be solved through penalty function method, specifically, the E.C.G.M. algorithm due to Ibiejugba[7] can be used as a method of solution. Once the existence and uniqueness of solution is established a control operator based on the formalism of the construction of operator A in the conventional Conjugate Gradient Method (CGM) algorithm [6] can be constructed explicitly. Our result is stated in section 2.

For the reaction diffusion equations considered in [5] there are no explicit or analytic form of solution. However, HILL [5] outlined a general procedure for obtaining a closed form representations of the solutions $u(x,t)$ and $v(x,t)$ for the linear reaction diffusion equations:

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_1 \nabla^2 u - au + bv \\ \frac{\partial v}{\partial t} &= D_1 \nabla^2 v + cu - dv \end{aligned} \tag{1.1}$$

where D_1, D_2, a, b, c and d are all non-negative constants. HILL [5] showed that closed form solutions of (1.1) can be given in term of integral arbitrary heat functions $h_1(x,t)$ and $h_2(x,t)$.

These functions satisfy the classical heat equation

$$\frac{\partial h}{\partial t} = \Delta h, \tag{1.2}$$

In particular he established that the formal solutions of (1.1) are

$$\begin{aligned} u(x,t) &= e^{-at} h_1(x, D_1 t) \\ &+ \frac{b^{\frac{1}{2}} e^{-\lambda t}}{(D_1 - D_2)} \int_{D_2 t}^{D_1 t} e^{-\mu\xi} \left\{ e^{\frac{1}{2}} \frac{(\xi - D_2 t)^{\frac{1}{2}}}{D_1 t - \xi} I(n) h_1(x,\xi) \right. \\ &\left. + b^{\frac{1}{2}} I_0(n) h_2(x,\xi) \right\} d\xi \end{aligned} \tag{1.3}$$

$$\begin{aligned} v(x,t) &= e^{-at} h_2(x, D_2 t) \\ &+ \frac{c^{\frac{1}{2}} e^{\lambda t}}{(D_1 - D_2)} \int_{D_2 t}^{D_1 t} e^{-\mu\xi} \left\{ e^{\frac{1}{2}} \frac{(D_1^{t-\xi})^{\frac{1}{2}}}{\xi - D_2 t} I(\tau) h_2(x,t) \right. \\ &\left. + c^{\frac{1}{2}} I_0(\tau) h_2(x,\xi) \right\} d\xi \end{aligned} \tag{1.4}$$

where the constants λ and μ are given by

$$\lambda = \frac{(aD_2 - dD_1)}{D_1 - D_2} \tag{1.5}$$

$$\mu = \frac{(a-d)}{D_1 - D_2} \tag{1.6}$$

I_o, I_1 are the usual modified Bessel functions and τ is given by

$$\tau = \frac{2(bc)^{\frac{1}{2}}}{(D_1 - D_2)}[(D_1 t - \xi)(\xi - D_2 t)]^{\frac{1}{2}}. \tag{1.7}$$

Hill [5] considered the application of his general formulae to the stability problem arising from a model of an arms race which incorporates the features of deteriorating armaments.

The situation is as follows:

In [5], Richardson proposed that the military spending of two nations locked in an arms race can be modelled by the following linear system

$$\frac{dp}{dt}(t) = -ap(t) + bq(t) + g \tag{1.8}$$

$$\frac{dq}{dt}(t) = cp(t) - dq(t) + h \tag{1.9}$$

where $p(t)$ and $q(t)$ denote armament levels of the two nations at time t and a, b, c, d, g and h denote positive constants. The constants b and c are called "Threat coefficients" and they signify the degree to which a nation is stimulated by another nation's weapon stock to increase her own stocks. The constants a and d are called "fatigue coefficients" and are a measure of the prevailing economic circumstances which inhibit armament build-up. The constants g and h denote a measure of the circumstances which prevents a complete disarmament in the situation when both nations have zero armaments. A "balance of power" situation results when the armament level remains constant over a long period of time and these levels are given in [5] by the following equations

$$p_0 = \frac{gd + hb}{(ad - bc)} \tag{1.10}$$

$$q_0 = \frac{gc + ha}{(ad - bc)} \tag{1.11}$$

$(ad - bc) > 0$

In [5], Gopalsamy developed Richardson model and proposed that the armament levels $p(x, t)$ and $q(x, t)$ satisfy

$$\frac{\partial p}{\partial t} + e_1 \frac{\partial p}{\partial x} = \frac{\delta_1^2}{2} \frac{\delta^2 p}{\partial x^2} - ap + bq + g \tag{1.12}$$

and

$$\frac{\partial q}{\partial t} + e_2 \frac{\partial q}{\partial x} = \frac{\delta_2^2}{2} \frac{\partial^2 q}{\partial x^2} - cq + dp + h \tag{1.13}$$

where e_1, e_2, δ_1 and δ_2 denote positive constants and the remaining constants are as previously defined. Hill [5] further developed the model and asserted that in order to investigate the stability of power situation (p_0, q_0) given by (1.10) and (1.11) he sets

$$p(x,t) = p_0 + u(x,t) \tag{1.14}$$

and

$$q(x,t) = q_0 + v(x,t) \tag{1.15}$$

so that from (1.12) and (1.13)

$$\frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} - e_1 \frac{\partial u}{\partial x} - au + bv, \quad 0 \le t \le 1$$
$$\frac{\partial v}{\partial t} = D_2 \frac{\partial^2 v}{\partial x^2} - e_2 \frac{\partial v}{\partial x} - cu + dv, \quad 0 \le t \le 1$$

$$D_i = \frac{\delta_i^2}{2} (i = 1, 2) \tag{1.16}$$

$u(x,0) = 0, \quad v(x,0) = 0$
$u(0,t) = u_0, \quad v(0,t) = v_0$
$u(x,t), v(x,t) \to 0$ as $x \to \infty$.

We shall now transform the whole of reaction diffusion system (1.16) into a control problem in our main result.

2. Main result

Theorem 2.1. *The reaction diffusion system (1.6) can be transformed into a quadratic cost functional of the form:*

Minimize $\int_0^t \{v_1^2(t) + v_2^2(t) + \ldots + v_n^2(t) + u_1^2(t) + u_2^2(t) + \ldots u_n^2(t)\} dt$ *subject to* $\dot{u}_i(t) - \dot{v}_i(t) = Cv_i(t) + Du_i(t)$, *where*

$$C = D_2 \pi^2 i^2 + d + b, \quad D = -D_1 \pi^2 i^2 - a - c, \quad i = 1, \ldots, n.$$

Proof. We proceed as follows. Consider the problem:

Minimize $\int_0^1 \int_0^1 [u^2(x,t) + v^2(x,t)]dxdt$ subject to:

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_1 \frac{\partial^2 u}{\partial x^2} - e_1 \frac{\partial u}{\partial x} - au + bv = 0 \\ \frac{\partial v}{\partial t} &= D_2 \frac{\partial^2 v}{\partial x^2} - e_2 \frac{\partial v}{\partial x} - cu + dv = 0, \quad 0 \leq t \leq 1 \end{aligned} \tag{1.17}$$

with the following initial and boundary conditions:

$$\begin{aligned} &u(x,0) = 0, \quad v(x,0) = 0 \\ &u(0,t) = u_0, \quad v(0,t) = v_0 \\ &u(x,t), v(x,t) \to 0 \text{ as } x \to \infty \\ &\frac{\partial u(x,1)}{\partial t} = \frac{\partial v(x,1)}{\partial t} = 1 \end{aligned}$$

where the boundary condition at x equals zero represents the fact that both nations looked in the arms race are maintaining a constant level of perfect undeteriorated strategic weapon system and the integral given by (1.17) is a measure of the cost.

To obtain explicit solutions of these boundary value problems (1.16) Gopalsamy in ref. [5] assumed that $e_1 = e_2, D_1 = D_2$ and $a = d$. We also adopt in this study these values for simplicity and consistency.

Let

$$\begin{aligned} v(x,t) &= \sum_{i=1}^{\infty} v_i(t) \frac{\sin \pi i x}{1}, \quad 0 \leq x \leq 1 \equiv 1 \\ u(x,t) &= \sum_{i=1}^{\infty} u_i(t) \frac{\sin \pi i x}{1}, \quad 0 \leq x \leq 1 \equiv 1 \end{aligned}$$

$0 \leq t \leq 1 \equiv 1$.

Thus, we have

$$\begin{aligned} v_t(x,t) &= \sum_{i=1}^{\infty} \dot{v}_i(t) \sin \pi i x \\ u_t(x,t) &= \sum_{i=1}^{\infty} \dot{u}_i(t) \sin \pi i x \end{aligned}$$

$$\begin{aligned}
v_{xx}(x,t) &= -\pi^2 i^2 \sum_{i=1}^{\infty} v_i(t)\sin \pi i x \\
u_{xx}(x,t) &= -\pi^2 i^2 \sum_{i=1}^{\infty} u_i(t)\sin \pi i x \\
v^2(x,t) &= \sum_{i=1}^{\infty} v_i^2(t)\sin^2 \pi i x \\
u^2(x,t) &= \sum_{i=1}^{\infty} u_i^2(t)\sin^2 \pi i x
\end{aligned}$$

Substituting the values of $v^2(x,t), u^2(x,t)$in the integral of (1.17). We obtain

$$\int_0^1\int_0^1 [v^2(x,t)+u^2(x,t)]dxdt$$

$$= \int_0^1\int_0^1 \left[\sum_{i=1}^{\infty} v_i^2(t)\sin^2 \pi i x + \sum_{i=1}^{\infty} u_i^2(t)\sin^2 \pi i x\right]dxdt$$

$$= \frac{1}{2}\int_0^1\int_0^1 \left[\sum_{i=1}^{\infty} v_i^2(t)[1-\cos 2\pi i x] + \sum_{i=1}^{\infty} u_i^2(t)[1-\cos \pi i x]\right]dxdt$$

$$= \frac{1}{2}\int_0^1 \left\{\sum_{i=1}^{\infty} v_i^2(t)\left[\frac{\sin \pi i x}{2\pi i}\right]_0^1 + \sum_{i=1}^{\infty} u_i^2(t)\left[\frac{\sin 2\pi i x}{2\pi i}\right]_0^1\right\}dt$$

$$= \frac{1}{2}\int_0^1 \left\{\sum_{i=1}^{\infty} v_i^2(t)[1-0-(+0-0)] + \sum_{i=1}^{\infty} u_i^2[1-0-(+0-0)](t)\right\}dt$$

$$= \frac{1}{2}\int_0^1 \left\{\sum_{i=1}^{\infty} v_i^2(t)\sum_{i=1}^{\infty} u_i^2\right\}dt$$

$$= \frac{1}{2}\int_0^1 \left\{v_1^2(t)+v_2^2(t)+\ldots+v_n^2+u_1^2(t)+u_2^2(t)+\ldots+u_n^2\right\}dt.$$

Since it is a minimization problem the $\frac{1}{2}$ factor in the right hand side can be omitted.

Thus, we have

$$\int_0^1\int_0^1 [v^2(x,t)+u^2(x,t)]dxdt$$

$$= \int_0^1 \{v_1^2(t) + v_2^2(t) + \ldots + v_n^2(t) + u_1^2(t) + u_2^2(t) + \ldots + u_n^2(t)\}dt.$$

Following the idea of Gopalsamy in ref.[5] we set

$$e_1 = e_2 = 1.$$

Equating the constraints in (1.17) we obtain

$$\frac{\delta u}{\delta t} - D_1 \frac{\delta^2 u}{\delta x^2} + \frac{\delta u}{\delta x} + au - bv = \frac{\delta v}{\delta t} - D_2 \frac{\delta^2 v}{\delta x^2} + \frac{\delta v}{\delta x} - cu + dv.$$

Next substituting values for $u_t, v_t, u_{xx}, v_{xx}, u, v$ in the last equation, we obtain

$$\sum_{i=1}^{\infty} u_i(t) \sin \pi i x + D_1 \pi_1^2 i^2 \sum_{i=1}^{\infty} \sin \pi i x + a \sum_{i=1}^{\infty} u_i \sin \pi i x - b \sum_{i=1}^{\infty} v_i(t) \sin \pi i x$$
$$= \sum_{i=1}^{\infty} v_i(t) \sin \pi i x + D_2 \pi^2 i^2 \sum_{i=1}^{\infty} \sin \pi i x - c \sum_{i=1}^{\infty} u_i \sin \pi i x + d \sum_{i=1}^{\infty} v_i(t) \sin \pi i x$$

Therefore, by dividing both sides by $\sin \pi i x$ we obtain

$$\sum_{i=1}^{\infty} u_i(t) + D_1 \pi^2 \sum_{i=1}^{\infty} i^2 + a \sum_{i=1}^{\infty} u_i - b \sum_{i=1}^{\infty} v_i(t) = \sum_{i=1}^{\infty} v_i(t) + D_2^2 \sum_{i=1}^{\infty} i^2$$
$$-c \sum_{i=1}^{\infty} u_i(t) + d \sum_{i=1}^{\infty} v_i(t)$$

since $\sin \pi i x \neq 0, \;\; 0 < x < 1$.

Rearranging and dropping the summation sign, we obtain

$$\dot{u}_1(t) - \dot{v}_1(t) = [D_2 \pi^2 1^2 + d + b] v_1(t) + [-D_1 \pi^2 1^2 - a - c] u_1(t)$$
$$\dot{u}_2(t) - \dot{v}_2(t) = [D_2 \pi^2 2^2 + d + b] v_2(t) + [-D_1 \pi^2 2^2 - a - c] u_2(t)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\dot{u}_n(t) - \dot{v}_n(t) = [D_2 \pi^2 n^2 + d + b] v_n(t) + [-D_1 \pi^2 n^2 - a - c] u_n(t).$$

We can put it in a compact form in the following manner: $\dot{u}_i(t) - \dot{v}_i(t) = Cv_i(t) + du_i(t)$, where $C = D_2 \pi^2 i^2 + d + b$, $i = 1, \ldots, n$, $D = -D_1 \pi^2 i^2 - a - c$, $i = 1, \ldots, n$.

Consequently, problem (1.17) reduces to

Minimize $\int_0^t \{v_1^2(t)+v_2^2(t)+\ldots+v_n^2(t)+u_1^2(t)+u_2^2(t)+\ldots u_n^2(t)\}dt$ subject to $\dot{u}_i(t)-\dot{v}_i(t)=Cv_i(t)+Du_i(t)$, where $C=D_2\pi^2 i^2+d+b$, $i=1,\ldots,n$, $D=-D_1\pi^2 i^2-a-c$, $i=1,\ldots,n$.

Hence we have our result.

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