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BOCHNER FLAT TANGENT BUNDLES*

BY

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Abstract. We study the vanishing of the Bochner curvature tensor field on the tangent bundle TM of a Riemamnian manifold (M, g), when this tangent bundle is endowed with a Kähler structure (G, J) which is a natural lift of g, of diagonal type. It follows that the base manifold (M, g) must have constant sectional curvature and the Kähler manifold (TM, G, J) must have constant holomorphic sectional curvature.

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1. Introduction. The tangent bundle TM of a Riemannian manifold (M, g) can be endowed with an almost Hermitian structure (G, J), defined by using natural lifts of the metric g, of diagonal type. This lift depends on 8 initial parameters $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ which are smooth functions of the energy density $t = \frac{1}{2}\|y\|^2$, where $y \in TM$ is the current tangent vector from TM. From the condition for J to define an almost complex structure on TM one can obtain two parameters, e.g. a_2, b_2 as (rational) functions of a_1, b_1. From the condition for J to be integrable, it follows that the base manifold (M, g) must have constant sectional curvature c; then one can obtain the parameter b_1 as a (rational) function of c, a_1 and its first order derivative a_1'. From the condition for G to be Hermitian with respect to J, one can express the parameters c_1, c_2, d_1, d_2 with the help of the parameters a_1, a_2, b_1, b_2 and two proportionality factors λ, μ. From the condition for the almost Hermitian structure (G, J) to be almost Kählerian, one obtains

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that the new parameter μ must be the derivative of λ i.e. $\mu = \lambda'$. Next, combining the conditions for J to be integrable and for (G, J) to be almost Kählerian, one gets a Kählerian structure on M; this structure depends on two essential parameters: a_1 and λ. In this paper we study the vanishing of the Bochner curvature tensor field of (G, J), obtaining that the Kähler manifold (TM, G, J) must have constant holomorphic sectional curvature.

Several computations have been done by using the RICCI package under Mathematica for doing tensor calculations in differential geometry.

All geometric objects are assumed to be smooth. We use the computations in local coordinates in a fixed local chart though many results admit an invariant form via the vertical and horizontal lifts. The summation convention is used throughout over the indices h, i, j, k, l running $\{1, \dots, n\}$.

2. Natural almost complex structures of diagonal type on TM. Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \longrightarrow M$. To fix notation, the manifold structure of TM is obtained from the manifold structure of M whose local charts $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$ are induced from the local charts $(U, \varphi) = (U, x^1, \dots, x^n)$ on M, where the local coordinates $x^i, y^i,\ i = 1, \dots, n$, are defined as follows. The first n local coordinates of a tangent vector $y \in \tau^{-1}(U)$ are the local coordinates in the local chart (U, φ) of its base point, i.e. $x^i = x^i \circ \tau$, by an abuse of notation. The last n local coordinates $y^i,\ i = 1, \dots n$, of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis in the local chart (U, φ). A useful concept in the differential geometry of TM is that of M-tensor field (of type (p, q)) which is defined by sets of n^{p+q} components (functions of x and y) with p upper indices and q lower indices, assigned to induced local charts $(\tau^{-1}(U), \Phi)$ on TM, such that the local coordinate change rule is that of the local coordinate components of a (p, q)-tensor field on the base manifold M (see [3] for further details); e.g., the components $y^i,\ i = 1, \dots, n$, corresponding to the last n local coordinates of a tangent vector y, assigned to the induced local chart $(\tau^{-1}(U), \Phi)$ define an M-tensor field of type $(1, 0)$. Assume that $u : [0, \infty) \longrightarrow \mathbf{R}$ is a smooth function and let $\|y\|^2 = g_{\tau(y)}(y, y)$ be the square of the norm of the tangent vector y. If δ^i_j (the Kronecker symbols) are the local coordinate components of the identity $(1, 1)$-tensor field I on M, then the components $u(\|y\|^2)\delta^i_j$ define an M-tensor field of type $(1, 1)$ on TM. The components $u(\|y\|^2)g_{ij}$ define an M-tensor field of type $(0, 2)$ on TM, where g is the metric tensor field on M. The components

$g_{0i} = y^k g_{ki}$ define an M-tensor field of type $(0, 1)$ on TM.

The Levi Civita connection $\dot{\nabla}$ of g on M gives the direct sum decomposition

$$TTM = VTM \oplus HTM \tag{1}$$

of the tangent bundle to TM into the vertical distribution $VTM = \text{Ker}\ \tau_*$ and the horizontal distribution HTM. The set of vector fields $(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n})$ on $\tau^{-1}(U)$ defines a local frame field for VTM and for HTM we have the local frame field $(\frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n})$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i}\frac{\partial}{\partial y^h}, \quad \Gamma^h_{0i} = y^k \Gamma^h_{ki}$$

and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of g.

The set $(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n})$ defines a local frame on TM, adapted to the direct sum decomposition (1). Remark that

$$\frac{\partial}{\partial y^i} = \left(\frac{\partial}{\partial x^i}\right)^V, \quad \frac{\delta}{\delta x^i} = \left(\frac{\partial}{\partial x^i}\right)^H,$$

where X^V and X^H denote the vertical and horizontal lift of the vector field X on M which help us to obtain invariant expressions later on. However, in local coordinates, the formulae are more direct, and more natural, in a certain sense.

We begin by considering the energy density of the tangent vector y

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y, y) = \frac{1}{2}g_{ik}(x)y^i y^k, \quad y \in \tau^{-1}(U). \tag{2}$$

Obviously, we have $t \in [0, \infty)$ for all $y \in TM$. By direct computation we obtain

Lemma. *If $n > 1$ and u, v are smooth functions on TM such that either $u g_{ij} + v g_{0i} g_{0j} = 0$, or $u\delta^i_j + v g_{0j} y^i = 0$, on the domain of any induced local chart on TM, then $u = v = 0$.*

Denote by $C = y^i \frac{\partial}{\partial y^i}$ the Liouville vector field on TM and by $\widetilde{C} = y^i \frac{\delta}{\delta x^i}$ the similar horizontal vector field on TM. Let $a_1, a_2, b_1, b_2 : [0, \infty) \to \mathbf{R}$ be some smooth functions. A natural 1-st order almost complex structure J of diagonal type on TM is given by (see [2])

$$J\frac{\delta}{\delta x^i} = a_1(t)\frac{\partial}{\partial y^i} + b_1(t) g_{0i} C, \;\; J\frac{\partial}{\partial y^i} = -a_2(t)\frac{\delta}{\delta x^i} - b_2(t) g_{0i}\widetilde{C}. \tag{3}$$

Proposition [5]. *The operator J defines an almost complex structure on TM if and only if*

$$a_1a_2 = 1, \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1. \tag{4}$$

Remark (i) As all coefficients $a_1, a_2, a_1 + 2tb_1, a_2 + 2tb_2$ from (4) are non-zero and of the same sign, we may assume them positive for any $t \geq 0$.

(ii) By (4), two of the coefficients $a_1,\ a_2,\ a_3,\ b_1,\ b_2$ are functions of the other two; e.g. we have:

$$a_2 = \frac{1}{a_1}, \quad b_2 = \frac{-a_2b_1}{a_1 + 2tb_1} = \frac{-b_1}{a_1(a_1 + 2tb_1)}. \tag{5}$$

For later use, we shall introduce the following M-tensor fields $J1^h_i,\ J2^h_i$, defined by

$$J1^h_i = a_1\delta^h_i + b_1g_{0i}y^h,\ J2^h_i = a_2\delta^h_i + b_2g_{0i}y^h.$$

Then the formula (3) can be written as

$$J\frac{\delta}{\delta x^i} = J1^h_i\frac{\partial}{\partial y^h},\ J\frac{\partial}{\partial y^i} = -J2^h_i\frac{\delta}{\delta x^h}.$$

The property of J to define an almost complex structure is equivalent to the property that $(J2^h_i)$ is the inverse of the matrix $(J1^h_i)$, i.e.

$$J1^h_iJ2^i_k = J2^h_iJ1^i_k = \delta^h_k.$$

To express the integrability condition of J we use the vanishing of its Nijenhuis tensor field N_J, defined by

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

for all vector fields X and Y on TM.

Theorem 3. [5] *Let (M, g) be an $n(> 2)$-dimensional connected Riemannian manifold. The almost complex structure J defined by* (3) *on TM is integrable if and only if (M, g) has constant sectional curvature c and the coefficient b_1 is given by:*

$$b_1 = \frac{a_1a_1' - c}{a_1 - 2ta_1'} \tag{6}$$

(*compare with the corresponding expressions from* [4] *and* [10]).

3. Natural diagonal Kählerian structures on TM. Consider a diagonal 1-st order, natural metric G on TM (see [2], see also, [1], [6]), given by

$$G\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)=c_1g_{ij}+d_1g_{0i}g_{0j},G\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)=c_2g_{ij}+d_2g_{0i}g_{0j}, \tag{7}$$

$$G\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)=G\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\right)=0,$$

where c_1,c_2,d_1,d_2 are smooth functions depending on the energy density $t\in[0,\infty)$. The conditions for G to be positive definite are assured if

$$c_1>0,\ c_2>0,\ c_1+2td_1>0,\ c_2+2td_2>0. \tag{8}$$

We establish here the conditions under which the metric G is almost Hermitian with respect to the almost complex structure J, considered in the previous section, i.e.

$$G(JX,JY)=G(X,Y),$$

for all vector fields X,Y on TM.

Considering the coefficients of g_{ij} in the conditions

$$G(J\frac{\delta}{\delta x^i},J\frac{\delta}{\delta x^j})=G(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}),\ \ G(J\frac{\partial}{\partial y^i},J\frac{\partial}{\partial y^j})=G(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}), \tag{9}$$

we obtain the following expressions

$$c_1=\lambda a_1,\quad c_2=\lambda a_2, \tag{10}$$

where $\lambda=\lambda(t)$ is a positive smooth function of $t\in[0,\infty)$. (Recall the assumptions $a_1,a_2>0$).

Next, considering the coefficients of $g_{0i}g_{0j}$ in the relations (9) and using (10), we obtain the following expressions

$$c_1+2td_1=(\lambda+2t\mu)(a_1+2tb_1),\ \ c_2+2td_2=(\lambda+2t\mu)(a_2+2tb_2), \tag{11}$$

where $\lambda+2t\mu=\lambda(t)+2t\mu(t)$ is a positive smooth function of $t\in[0,\infty)$. The conditions (8) are automatically fulfilled, due to the properties (4) of the coefficients a_1, a_2, b_1, b_2. From (14), d_1 and d_2 have the following explicit expressions

$$d_1 = \lambda b_1 + \mu(a_1 + 2tb_1), \quad d_2 = \lambda b_2 + \mu(a_2 + 2tb_2). \tag{12}$$

Remark If $\lambda = 1$ and $\mu = 0$, we obtain the almost Kählerian structure constructed in [5].

Consider now the two-form Ω defined by the almost Hermitian structure (G, J) on TM

$$\Omega(X, Y) = G(X, JY),$$

for all vector fields X, Y on TM.

The expression of the 2-form Ω in a local adapted frame $(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n})$ on TM, is given by

$$\Omega\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0, \ \Omega\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 0, \ \Omega\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = \lambda g_{ij} + \mu g_{0i} g_{0j}$$

or, equivalently

$$\Omega = (\lambda g_{ij} + \mu g_{0i} g_{0j}) \dot{\nabla} y^i \wedge dx^j, \tag{13}$$

where $\dot{\nabla} y^i = dy^i + \Gamma^i_{0h} dx^h$ is the absolute differential of y^i.

From the following formula

$$d\Omega = \frac{1}{2}(\lambda' - \mu)(g_{ij} g_{0k} - g_{0i} g_{jk}) \dot{\nabla} y^k \wedge \dot{\nabla} y^i \wedge dx^j,$$

obtained by a straightforward computation and following the same idea as in [6], we obtain

Theorem 4. *The almost Hermitian structure (TM, G, J) is almost Kählerian if and only if*

$$\mu = \lambda'.$$

Thus the family of almost Kählerian structures of diagonal type on TM depends on three essential coefficients a_1, b_1, λ. Combining the results from Theorems 3 and 4, it follows that the coefficient b_1 can be expressed as a function of a_1 and its first derivative, so that a natural Kählerian structure (G, J) of diagonal type on TM is defined by two essential coefficients a_1, λ, which have to satisfy some additional conditions $a_1 > 0, a_1 + 2tb_1 > 0, \lambda >$

$0, \lambda + 2t\lambda' > 0$. Examples of such structures can be found in [10] (see also [4], [5], [7]).

For later use, we shall consider the following M tensor fields G^1_{ij}, G^2_{ij}, defined by G on TM

$$(14)\qquad \begin{aligned} G^1_{ij} &= G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) c_1 g_{ij} + d_1 g_{0i} g_{0j}, \\ G^2_{ij} &= G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) c_2 g_{ij} + d_2 g_{0i} g_{0j}, \end{aligned}$$

The conditions for G to be positive definite are assured iff G^1_{ij}, G^2_{ij} define positive matrices and this happens iff

$$(15)\qquad c_1 > 0,\ c_2 > 0,\ c_1 + 2td_1 > 0,\ c_2 + 2td_2 > 0.$$

We shall consider too the inverse matrices of (G^1_{ij}), (G^2_{ij}), denoted by $(H^{kl}_1), (H^{kl}_2)$, with the entries defined by the properties

$$H^{hi}_1 G^1_{ij} = \delta^h_j, \quad H^{hi}_2 G^2_{ij} = \delta^h_j.$$

It follows easily that the components H^{kl}_1, H^{kl}_2 define M-tensor fields of type $(2, 0)$ on TM and that they have the following expressio ns

$$H^{kl}_1 = p_1 g^{kl} + q_1 y^k y^l, \quad H^{kl}_2 = p_2 g^{kl} + q_2 y^k y^l,$$

where the components g^{kl} are the entries of the inverse of the matrix (g_{ij}) and coefficients p_1, p_2, q_1, q_2 are given by

$$p_1 = \frac{1}{c_1}, \quad p_2 = \frac{1}{c_2}, \quad q_1 = -\frac{-d_1}{c_1(c_1 + 2td_1)}, \quad q_2 = -\frac{-d_2}{c_2(c_2 + 2td_2)}.$$

2. The Levi Civita connection and its curvature tensor field on TM. Denote by $\dot\nabla$ the Levi Civita connection of the Riemannian metric g on M. Denote by $\dot R^h_{kij}$ the components of the curvature tensor field $\dot R$ of $\dot\nabla$, where$\dot R^h_{kij}\frac{\partial}{\partial x^h} = \dot R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k}$. Denote too by $\dot R^h_{0ij} = y^k \dot R^h_{kij}$. Consider a Riemannian metric G of diagonal type on TM and denote by ∇ its Levi Civita connection. Denote by $\delta_i = \frac{\delta}{\delta x^i}$, $\partial_i = \frac{\partial}{\partial y^i}$, $i = 1, \ldots, n$. The local expression of ∇ is given in an adapted local frame $(\partial_1, ..., \partial_n, \delta_1, ..., \delta_n)$ by

$$(16)\qquad \begin{cases} \nabla_{\partial_i}\partial_j = Q^h_{ij}\partial_h,\ \nabla_{\delta_i}\partial_j = \Gamma^h_{ij}\partial_h + P^h_{ji}\delta_h, \\ \nabla_{\partial_i}\delta_j = P^h_{ij}\delta_h,\ \nabla_{\delta_i}\delta_j = \Gamma^h_{ij}\delta_h + S^h_{ij}\partial_h, \end{cases}$$

where Γ^h_{ij} are the Christoffel symbols of the connection $\dot{\nabla}$ and the $M-$tensor fields P^h_{ij}, Q^h_{ij}, S^h_{ij} are given by

$$\begin{aligned} P^h_{ij} &= \frac{1}{2}H_1^{hk}(\partial_i G^1_{jk} + G^2_{il}R^l_{0jk}) \\ &= \frac{c_1'}{2c_1}g_{0i}\delta^h_j + \frac{d_1}{2c_1}g_{0j}\delta^h_i + \frac{d_1}{2(c_1+2td_1)}g_{ij}y^h \\ &+\frac{c_1d_1' - c_1'd_1 - d_1^2}{2c_1(c_1+2td_1)}g_{0i}g_{0j}y^h - \frac{c_2}{2c_1}\dot{R}^h_{jik}y^k - \frac{c_2d_1}{2c_1(c_1+2td_1)}y^h\dot{R}_{ikjl}y^ky^l, \\ Q^h_{ij} &= \frac{1}{2}H_2^{hk}(\partial_i G^2_{jk} + \partial_j G^2_{ik} - \partial_k G^2_{ij}) \\ &= \frac{c_2'}{2c_2}(\delta^h_j g_{0i} + \delta^h_i g_{0j}) - \frac{c_2' - 2d_2}{2(c_2+2td_2)}g_{ij}y^h + \frac{c_2d_2' - 2c_2'd_2}{2c_2(c_2+2td_2)}g_{0i}g_{0j}y^h, \\ S^h_{ij} &= -\frac{1}{2}H_2^{hk}(\partial_k G^1_{ij} + G^2_{kl}\dot{R}^l_{0ij}) \\ &= -\frac{d_1}{2c_2}\delta^h_j g_{0i} - \frac{d_1}{2c_2}\delta^h_i g_{0j} - \frac{c_1'}{2(c_2+2td_2)}g_{ij}y^h \\ &-\frac{c_2d_1' - 2d_1d_2}{2c_2(c_2+2td_2)}g_{0i}g_{0j}y^h - \frac{1}{2}\dot{R}^h_{kij}y^k. \end{aligned}$$

The curvature tensor field of ∇ is denoted by R. Its expression in the local adapted frame $(\partial_1, ..., \partial_n, \delta_1, ..., \delta_n)$ is given by

$$\begin{aligned} &R(\delta_i,\delta_j)\delta_k = XXX^h_{ijk}\delta_h,\ R(\delta_i,\delta_j)\partial_k = XXY^h_{ijk}\partial_h, \\ (17)\qquad &R(\partial_i,\partial_j)\partial_k = YYY^h_{ijk}\partial_h,\ R(\partial_i,\partial_j)\delta_k = YYX^h_{ijk}\delta_h, \\ &R(\partial_i,\delta_j)\delta_k = YXX^h_{ijk}\partial_h,\ R(\partial_i,\delta_j)\partial_k = YXY^h_{ijk}\delta_h, \end{aligned}$$

where the components $XXX^h_{ijk}, \ldots$ are given by

$$\begin{aligned} XXX^h_{kij} &= \dot{R}^h_{kij} + P^h_{lk}\dot{R}^l_{0ij} + P^h_{li}S^l_{jk} - P^h_{lj}S^l_{ik}, \\ XXY^h_{kij} &= \dot{R}^h_{kij} + S^h_{il}P^l_{kj} - S^h_{jl}P^l_{ki} + Q^h_{lk}\dot{R}^l_{0ij}, \\ YYX^h_{kij} &= \frac{\partial}{\partial y^i}P^h_{jk} - \frac{\partial}{\partial y^j}P^h_{ik} + P^h_{il}P^l_{jk} - P^h_{jl}P^l_{ik}, \\ YYY^h_{kij} &= \frac{\partial}{\partial y^i}Q^h_{jk} - \frac{\partial}{\partial y^j}Q^h_{ik} + Q^h_{il}Q^l_{jk} - Q^h_{jl}Q^l_{ik}, \end{aligned}$$

$$YXX^h_{kij} = \frac{\partial}{\partial y^i} S^h_{jk} + Q^h_{il} S^l_{jk} - S^h_{jl} P^l_{ik},$$
$$YXY^h_{kij} = \frac{\partial}{\partial y^i} P^h_{kj} + P^h_{il} P^l_{kj} - P^h_{lj} Q^l_{ik}.$$

The expanded expressions of the above components (depending on a_1 and λ) can be obtained by using RICCI.

Now we obtain the Ricci tensor field of ∇ by using the well known contraction

$$Ric(X,Y) = trace(Z \to R(Z,X)Y).$$

The components of Ric in the adapted local frame $(\delta_1, \dots \delta_n, \partial_1, \dots, \partial_n)$ are obtained as

$$\begin{aligned} &RicXX_{jk} = Ric(\delta_j, \delta_k) = XXX^h_{khj} + YXX^h_{khj} \\ (18)\qquad &RicYY_{jk} = Ric(\partial_j, \partial_k) = YYY^h_{khj} - YXY^h_{kjh} \\ &Ric(\delta_j, \partial_k) = Ric(\partial_k, \delta_j) = 0. \end{aligned}$$

Their explicit expressions can be obtained by using RICCI.

In the following we shall use too the following $(1,1) - M$-tensor fields $RicX^h_k, RicY^h_k$ on TM, obtained from the components $RicXX_{jk}$, $RicYY_{jk}$ by raising an index with the help of the metric G

$$RicX^h_k = H^{hj}_1 RicXX_{jk},\ RicY^h_k = H^{hj}_2 RicYY_{jk}.$$

Finally, the scalar curvature r of G is obtained as the trace of the tensor field Ric

$$r = RicX^h_h + RicY^h_h = H^{hj}_1 RicXX_{jh} + H^{hj}_2 RicYY_{jh}.$$

Its explicit expression can be obtained by using, again, the package RICCI.

3. The Bochner curvature tensor field on TM and its vanishing. The Bochner curvature tensor field B of the $2n$-dimensional Kählerian

manifold (TM, G, J) is defined by the well known formula (see e.g. [12])
(19)

$$B(X,Y)Z = R(X,Y)Z - \frac{1}{2n+4}[Ric(Y,Z)X - Ric(X,Z)Y$$

$$+G(Y,Z)QX - G(X,Z)QY + G(JY,Z)QJX - G(JX,Z)QJY$$

$$+Ric(JY,Z)JX - Ric(JX,Z)JY - 2Ric(JX,Y)JZ - 2g(JX,Y)QJZ]$$

$$+\frac{r}{(2n+2)(2n+4)}[G(Y,Z)X - G(X,Z)Y$$

$$+G(JY,Z)JX - G(JX,Z)JY - 2G(JX,Y)Z],$$

where the tensor field Q of type $(1,1)$ is obtained from the tensor field Ric as the corresponding linear operator with respect to the metric G, i.e.

$$Ric(X,Y) = G(X,QY),$$

and $r = trace\ Q$ is the scalar curvature. If we use the adapted local frame $(\delta_1, \dots \delta_n, \partial_1, \dots, \partial_n)$, we get the following components of B

$$BXXX^h_{kij}\delta_h = B(\delta_i,\delta_j)\delta_k = [XXX^h_{kij}$$
$$-\frac{1}{2n+4}(RicXX_{jk}\delta^h_i - RicXX_{ik}\delta^h_j + G^1_{ik}RicX^h_j - G^1_{jk}RicX^h_i)$$
$$+\frac{r}{(2n+2)(2n+4)}(G^1_{jk}\delta^h_i j - G^1_{ik}\delta^h_j)]\delta_h,$$
$$BXXY^h_{kij}\partial_h = B(\delta_i,\delta_j)\partial_k = [XXY^h_{kij}$$
$$-\frac{1}{2n+4}(J1^l_jG^2_{lk}J1^r_iRicY^h_r - J1^l_iG^2_{lk}J1^r_jRicY^h_r$$
$$+J1^l_jRicYY_{lk}J1^h_i - J1^l_iRicYY_{lk}J1^h_j)$$
$$+\frac{r}{(2n+2)(2n+4)}(J1^l_jG^2_{lk}J1^h_i - J1^l_iG^2_{lk}J1^h_j)]\partial_h,$$
$$BYYY^h_{kij}\partial_h = B(\partial_i,\partial_j)\partial_k = [YYY^h_{kij}$$
$$-\frac{1}{2n+4}(RicYY_{jk}\delta^h_i - RicYY_{ik}\delta^h_j + G^2_{jk}RicY^h_i j - G^2_{ik}RicY^h_j)$$
$$+\frac{r}{(2n+2)(2n+4)}(G^2_{jk}\delta^h_i - G^2_{ik}\delta^h_j)]\partial_h,$$

$$BYYX^h_{kij}\delta_h = B(\partial_i,\partial_j)\delta_k = [YYX^h_{kij} - \frac{1}{2n+4}(J2^l_j G^1_{lk} J2^r_i RicX^h_r$$
$$-J2^l_i G^1_{lk} J2^r_j RicX^h_r + J2^l_j RicXX_{lk} J2^h_i - J2^l_i RicXX_{lk} J2^h_j)$$
$$+\frac{r}{(2n+2)(2n+4)}(J2^l_j G^1_{lk} J2^h_i - J2^l_i G^1_{lk} J2^h_j)]\delta_h,$$
$$BYXX^h_{kij}\partial_h = C(\partial_i,\delta_j)\delta_k = [YXX^h_{kij} - \frac{1}{2n+4}(RicXX_{jk}\delta^h_i$$
$$+G^1_{jk} RicY^h_i + J2^l_i G^1_{lk} J1^r_j RicY^h_r + J2^l_i RicXX_{lk} J1^h_j$$
$$+2J2^l_i RicXX_{lj} J1^h_k + 2J2^l_i G^1_{lj} J1^r_k RicY^h_r)$$
$$+\frac{r}{(2n+2)(2n+4)}(G^1_{jk}\delta^h_i + J2^l_i G^1_{lk} J1^h_j + 2J2^l_i G^1_{lj} J1^h_k)]\partial_h,$$
$$BYXY^h_{kij}\delta_h = B(\partial_i,\delta_j)\partial_k = [YXY^h_{kij} - \frac{1}{2n+4}(-RicYY_{ik}\delta^h_j$$
$$-G^2_{ik} RicX^h_j - J1^l_j G^2_{lk} J2^r_i RicX^h_r - J1^l_j RicYY_{lk} J2^h_i$$
$$-2J2^l_i RicXX_{lj} J2^h_k - 2J2^l_i G^1_{lj} J2^r_k RicX^h_r)$$
$$+\frac{r}{(2n+2)(2n+4)}(-G^2_{ik}\delta^h_j - J1^l_j G^2_{lk} J2^h_i - 2J2^l_i G^1_{lj} J2^h_k)]\delta_h.$$

We are interested in the conditions under which $B = 0$. From the condition $BXXY^h_{kij} = 0$ we get, by taking the terms containing n and n^2 in the coefficient of $g_{jk}\delta^h_i$, the following relation

$$n(n+4)\lambda(a_1 - 2a'_1 t)^3(\lambda + 2\lambda' t)^2(a_1^2 a'_1\lambda + 2a_1 c\lambda + a_1^3\lambda' - 2a'_1 c\lambda t + 2a_1 c\lambda' t) = 0$$

Considering the essential relation only, we get after an easy integration

$$\lambda = A\frac{a_1}{a_1^2 + 2ct}, \qquad (20)$$

where A is a positive constant. In this case we get easily the expression of the scalar curvature

$$r = \frac{4c}{A}n(n+1).$$

Next, we obtain by a straightforward but quite long computation that all components of B are zero.

We should remark that the relation (21) implies that the Kählerian manifold (TM, G, J) has constant holomorphic curvature. Hence we may state

Theorem. *The Kählerian manifold (TM, G, J) is Bochner flat if and only if it has constant holomorphic sectional curvature.*

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