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ABOUT ϵ-MONOTONICITY OF AN OPERATOR

BY

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Abstract. In [17], VESELY has introduced the $\epsilon-$monotone operators taking as starting point the approximative monotonicity property of $\epsilon-$subdifferential of a convex function.

In this paper we discuss some stability properties of $\epsilon-$monotonicity. Also, the special case of maximal $\epsilon-$monotone operators is considered.

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1. Introduction. Throughout this paper we will assume that X is a real Banach space and X^* is its dual. The norms in X and X^*will be denoted by $\|\cdot\|$. Also, by w and w^* we denote the weak and weak star topology in X, respectively in X^*. The symbol $(\cdot,\cdot)$ will be used for the usual pairing between X and X^*, while $\langle\cdot,\cdot\rangle$ will be used for the associated duality pairing, i.e. $\langle x, x^*\rangle = x^*(x)$, for all $x \in X, x^* \in X^*$.

Firstly, we recall some well known concepts of convex analysis (see [2], [11]).

Let $f : X \longrightarrow \mathbb{R} \cup\{\infty\}$ be given.We denote the *domain* of f by

$$Domf = \{x \in X/f(x) < \infty\}. \tag{1.1}$$

We recall that f is said to be *proper* if $Domf$ is nonempty.

The *subdifferential* of f at $x \in Domf$ is the set

$$\partial f(x) = \{x^* \in X^*/f(y) - f(x) \geq \langle y - x, x^*\rangle, \forall y \in Domf\}. \tag{1.2}$$

As it is well known, the subdifferential represents an outstanding example of *monotone operator*, namely:

$$\langle x - y, x^* - y^* \rangle \geq 0, \forall x^* \in \partial f(x), \forall y^* \in \partial f(y). \tag{1.3}$$

Moreover, *if f is a proper lower semicontinuous and convex function, then ∂f is a maximal monotone operator* (see [10], [11], 15], [17]).

Now, let us consider the *ϵ-subdifferential* of the function f given by

(1.4)
$$\partial_\epsilon f(x) = \{x^* \in X^*/f(y) - f(x) \geq \langle x^*, y - x\rangle - \epsilon, \forall y \in Dom f\}, \text{ where } \epsilon > 0.$$

It is well known that if f is a proper convex lower semicontinuous function, then $Dom(\partial_\epsilon f) = Dom f$ (see [3], [5], [9]).

When $\epsilon = 0$,the approximative subdifferential $\partial_\epsilon f$ becomes the subdifferential ∂f (the later can be empty at some points of $Dom f$, even in the case of proper convex lower semicontinuous functions).

Generally, $\partial_\epsilon f$ don't satisfy condition of monotonicity , but we have the following weaker property:

$$\langle x - y, x^* - y^* \rangle \geq -2\epsilon, \forall x^* \in \partial_\epsilon f(x), \forall y^* \in \partial_\epsilon f(y). \tag{1.5}$$

Thus, it is naturally to consider the operators acting between a linear normed space and its dual which fulfils the property (1.5) of approximative monotonicity. The operators of this type (see Definition 2.1) was considered by Vesely [18], to establish an *uniform local boundedness principle.* Moreover, $\epsilon-$monotonicity is closely related to the notion of *$\epsilon-$enlargement* introduced by Revalski and Théra [14] (see Definition 3.1).

In this paper we establish some stability properties with respect to usual operations. Also, special properties concerning the maximal $\epsilon-$ monotonicity are considered ([1], [13]).

2. Notations and preliminaries. Given a (multivalued) operator $A : X \rightrightarrows X^*$, we denote by:

$Gr(A) = \{(x, x^*) \in X \times X^*/x^* \in Ax\}$,

$D(A) = \{x \in X/Ax \neq \varnothing\}$,

$R(A) = \cup\{Ax/x \in D(A)\}$,

the *graph*, the *domain*, respectively the *range* of A. We recall that the *inverse operator* of A is defined on X^* by:

$$A^{-1}x^* = \{x \in X/x^* \in Ax\}, \ \ x^* \in X^*.$$

Obviously we have:

$D(A^{-1}) = R(A)$ and $R(A^{-1}) = D(A)$.

For the operator $A : X \rightrightarrows X^*$ and $\lambda > 0$ we associate the following operators:

$(coA)x = co(Ax), x \in X$

$\overline{A}x = \overline{Ax}^{w*}, x \in X$

$(\overline{co}A)x = \overline{co(Ax)}, x \in X$

$(\lambda A)x = \lambda(Ax)$, $x \in X$,

$\widetilde{A} : X \rightrightarrows X^*$ so that $Gr(\widetilde{A}) = \overline{Gr(A)}^{\|\cdot\| \times w*}$.

Here, in the right hand of these equalities, *co* means convex hull of a set and the overbar has the meaning of the closures in the norm topology or in the topology specified ([14]).

We remark that

$$D(coA) = D(\overline{A}) = D(\lambda A) = D(A) \subseteq D(\widetilde{A}), \tag{2.1}$$

$$Gr(A) \subseteq Gr(coA) \subseteq Gr(\overline{coA}) \subseteq Gr(\widetilde{coA}). \tag{2.2}$$

Definition 2.1. *The operator $A : X \rightrightarrows X^*$ is called ϵ-monotone if the following property is fulfilled:*

$$\langle x - y, x^* - y^* \rangle \geq -2\epsilon, \quad \text{for all } (x, x^*), (y, y^*) \in Gr(A). \tag{2.3}$$

Definition 2.2. *An ϵ-monotone operator $A : X \rightrightarrows X^*$ is said to be maximal ϵ -monotone if its graph is not properly contained in the graph of any other ϵ-monotone operator from X to X^*.*

If $\epsilon = 0$ we obtain the usual property of monotonicity.

The ϵ-monotonicity above defined is different from the ϵ-monocity introduced in [8].

Remark 2.3. If X is a Hilbert space, using the Riesz identification between X and X^*, the ϵ-monotonicity of an operator $A : X \rightrightarrows X$, may be characterized by the following two equivalent properties:

(i) $\|x_1 - x_2 + t(y_1 - y_2)\|^2 \geq \|x_1 - x_2\|^2 - 4t\epsilon$, for all $t \in \mathbb{R}$, $(x_1, x_2), (y_1, y_2) \in Gr(A)$;

(ii) $\lim\limits_{t \to 0} \frac{\|x_1 - x_2 + t(y_1 - y_2)\|^2 - \|x_1 - x_2\|^2}{t} \geq -4\epsilon$, for any $(x_1, x_2), (y_1, y_2) \in Gr(A)$.

We can easily observe that for a given $\epsilon > 0$ every monotone operator is ϵ-monotone, but the converse is not true.

Definition 2.4. *The operator A is called locally bounded if for each point $x \in \overline{D(A)}$ there is a neighborhood U of x such that $A(U)$ is a bounded subset in X^*.*

Now we point out some simple properties of ϵ-monotone operators ([1], [13], [18]):

Proposition 2.5. *If $A : X \rightrightarrows X^*$ is an ϵ-monotone operator and $\lambda > 0$, then:*

a) *A is ϵ'-monotone, for any $\epsilon' \geq \epsilon$;*

b) *$A^{-1}, \overline{A}, coA, \overline{co}A$ are ϵ-monotone operators;*

c) *λA is $\lambda\epsilon$-monotone;*

d) *If $(A_i)_{i\in I}$ is a chain of $\epsilon-$monotone operators, then $\underset{i\in I}{\cup} A_i$ is also an $\epsilon-$monotone operator.*

Moreover, if $(A_i)_{i\in I}$ is a maximal chain, then $\underset{i\in I}{\cup} A_i$ is a maximal ϵ-monotone operator.

Proposition 2.6. *Let $A, B : X \rightrightarrows X^*$ and ϵ, ϵ' be two positive numbers. If A is $\epsilon-$monotone and B is ϵ'-monotone then:*

a) *$A + B$ is $\epsilon + \epsilon'$ monotone;*

b) *if A is locally bounded, then $\widetilde{A}, \overline{co}\widetilde{A}$ are ϵ-monotone operators.*

Proof. a) Let us consider $(x, x^*), (y, y^*) \in Gr(A+B)$. Therefore, there exist $x_1^* \in Ax, x_2^* \in Bx$, $y_1^* \in Ay, y_2^* \in By$ such that $x^* = x_1^* + x_2^*$, $y^* = y_1^* + y_2^*$ and so $\langle x - y, x^* - y^*\rangle = \langle x - y, x_1^* - y_1^*\rangle + \langle x - y, x_2^* - y_2^*\rangle \geq -2(\epsilon + \epsilon^{'})$ as claimed.

b) If $(x, x^*), (y, y^*) \in Gr(\tilde{A})$ then there exist two nets $(x_i, x_i^*)_{i\in I} \subset Gr(A), (y_j, y_j^*)_{j\in J} \subset Gr(A)$, such that

$$x_i \xrightarrow{\|\cdot\|} x\ ,\ x_i^* \xrightarrow{w^*} x^*,\ i \in I,$$
$$y_j \xrightarrow{\|\cdot\|} y\ ,\ y_j^* \xrightarrow{w^*} y^*,\ j \in J.$$

Since A is an $\epsilon-$ monotone locally bounded operator, according to the definition of $\widetilde{A}$,by passing to limit in the equality $\langle x-y, x^*-y^*\rangle = \langle x-x_i, x_i^*-y_j^*\rangle + \langle y_j-y, x_i^*-y_j^*\rangle + \langle x_i-y_j, x_i^*-y_j^*\rangle + +\langle x-y, x^*-x_i^*\rangle + \langle x-y, y_j^*-y^*\rangle$, we obtain that $\widetilde{A}$ is ϵ- monotone. Moreover, taking into account the Proposition II.5 (b) it follows that its $\overline{co}\widetilde{A}$ is also ϵ -monotone.

If A is an ϵ-monotone operator it is possible to be also ϵ'-monotone for $\epsilon' < \epsilon$. Thus, it is natural to consider the minimum value of $\epsilon' \geq 0$ for which A is ϵ'-monotone.

Consequently, for an arbitrary operator A, we define:

$$\epsilon_{\min}^A = \sup_{\substack{(x,x^*)\in Gr(A)\\(y,y^*)\in Gr(A)}} \left(\frac{\langle y-x, x^*-y^*\rangle}{2}\right). \tag{2.4}$$

Clearly, $\epsilon_{\min}^A \geq 0$.Also, $\epsilon_{\min}^A = \infty$ *if and only if* A *is not an* $\epsilon-$ *monotone operator for any* $\epsilon > 0$, and $\epsilon_{\min}^A = 0$ *if and only if* A *is a monotone operator.* Obviously, $\epsilon_{\min}^A \leq \epsilon$ if A is an ϵ-monotone operator. □

3. Some properties related to maximal $\epsilon-$ monotonicity. Among the most important examples of ϵ-monotone operators are the ϵ-subdifferentials of convex functions.

Generally, a reunion of ϵ-monotone operators is not an ϵ-monotone operator, but according to Proposition 2.5 (d), a*ny operator* ϵ *- monotone can be extended to a maximal* ϵ *-monotone operator which include it.*

Moreover, for a given ϵ-monotone operator can exist several maximal ϵ-monotone operators which include it.

While ∂f is a maximal monotone operator, $\partial_\epsilon f$ may be not maximal ϵ-monotone, as we will show by the following:

Example 3.1. Let us consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} 0 \quad , & x \leq 0, \\ +\infty, & x > 0. \end{cases}$.

This is a proper lower semicontinuous convex function.By an elementary computation, we find:

$$\partial f : \mathbb{R} \longrightarrow \mathbb{R} \text{ , } \partial f(x) = \begin{cases} 0, & \text{if } x < 0, \\ [0,\infty), & \text{if } x = 0, \\ \varnothing, & \text{if } x > 0, \end{cases}$$

$$\partial_\epsilon f : \mathbb{R} \longrightarrow \mathbb{R}, \partial_\epsilon f(x) = \begin{cases} [0, -\frac{\epsilon}{x}], & \text{if } \ x < 0, \\ [0, \infty), & \text{if } \ x = 0, \\ \varnothing, & \text{if } \ x > 0. \end{cases}$$

If we take the pair $(-2\epsilon, 1) \in \mathbb{R} \times \mathbb{R}$, then $(-2\epsilon, 1) \notin \partial_\epsilon f$ but for every $(x, \lambda) \in \partial_\epsilon f$ we obtain: $(x+2\epsilon)(\lambda-1) = x\lambda - x + 2\epsilon\lambda - 2\epsilon \geq -x - 2\epsilon \geq -2\epsilon$, for all $x < 0$. Also, for $x = 0$ we have $(0 + 2\epsilon)(\lambda - 1) = 2\epsilon(\lambda - 1) \geq -2\epsilon$.

Consequently, $\partial_\epsilon f \cup \{(-2\epsilon, 1)\}$ is an ϵ-monotone operator which strictly contains $\partial_\epsilon f$, i.e. $\partial_\epsilon f$ is not a maximal ϵ-monotone operator.

Therefore, it is possible to exist pairs $(x, x^*) \notin \partial_\epsilon f$ such that $\{(x, x^*)\} \cup Gr(\partial_\epsilon f)$ is also ϵ-monotone.On the other hand, there are some special types of convex functions for which the ϵ-subdifferential is even a maximal ϵ-monotone operator.

In this line we consider the case of *quadratic forms* on a Hilbert space.

Let H be a Hilbert space and let $f : H \longrightarrow \mathbb{R}$ be a quadratic form, i.e.:

$$f(x) = \frac{1}{2} \langle Ax, x \rangle + \langle b, x \rangle + c, x \in H,$$

where $A : X \longrightarrow X^*$ is one to one, linear, positive, and self adjoint operator, $b \in H$, $c \in \mathbb{R}$.

In this case (see [6]) f is a proper, convex and l.s.c. function and for every $x \in H$, we have: $\partial f(x) = Ax + b$ and $\partial_\epsilon f(x) = Ax + b + B_\epsilon$, where

$$B_\epsilon = \left\{x^* \in H / \left\langle x^*, A^{-1}x^* \right\rangle \leq 2\epsilon\right\} = \left\{x^* \in H / \left|A^{-\frac{1}{2}}x^*\right| \leq \sqrt{2\epsilon}\right\}. \tag{3.1}$$

The last equality is true because the operator A is self adjoint.

Proposition 3.1. *If f is a quadratic form, then for every $\epsilon > 0$, $\partial_\epsilon f$ is a maximal $\epsilon-$monotone operator and $\epsilon_{\min}^{\partial_\epsilon f} = \epsilon$.*

Proof. For the proof of first part of proposition, as we shell show in Proposition 3.8 (see also (3.7) below), it is enough to prove that $\partial_\epsilon f = (\partial_\epsilon f)^\epsilon$.

For an arbitrary element $x \in H$ we have

(3.2)

$$\begin{aligned} (\partial_\epsilon f)^\epsilon (x) &= \{x^* \in H / \langle x - y, x^* - y^* \rangle \geq -2\epsilon, \ \forall y \in H, \forall y^* \in \partial_\epsilon f(y)\} \\ &= \{x^* \in H / \langle u, x^* - y^* \rangle \leq 2\epsilon, \ \forall u \in H, \forall y^* \in \partial_\epsilon f(x + u)\} \\ &= Ax + b + \{x^* \in H / \langle u, x^* - y^* \rangle \\ &\leq \langle Au, u \rangle + 2\epsilon, \ \forall u \in H, \forall y^* \in B_\epsilon\} = Ax + b + \widehat{B}_\epsilon, \end{aligned}$$

where

$$\widehat{B}_\epsilon = \bigcap_{y^*\in B_\epsilon} \{x^* \in H/ \langle Au, u\rangle - \langle u, x^* - y^*\rangle + 2\epsilon \geq 0, \forall u \in X\}. \tag{3.3}$$

Now, we observe that is enough to show that $B_\epsilon = \widehat{B}_\epsilon$. Firstly, we prove that

$$\widehat{B}_\epsilon = \{x^* \in X^*/\|A^{-\frac{1}{2}}(x^* - y^*)\| \leq 2\sqrt{2\epsilon}, \forall y^* \in B_\epsilon\}. \tag{3.4}$$

Since A is one to one, linear, positive, and self adjoint ,it follows that: $\langle Au, u\rangle - \langle u, x^* - y^*\rangle + 2\epsilon \geq 0, \forall u \in X$ if and only if

$$\begin{aligned}\left\langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}u\right\rangle & -2\left\langle A^{\frac{1}{2}}u, \frac{1}{2}A^{-\frac{1}{2}}(x^* - y^*)\right\rangle \\ & +\left\langle \frac{1}{2}A^{-\frac{1}{2}}(x^* - y^*), \frac{1}{2}A^{-\frac{1}{2}}(x^* - y^*)\right\rangle \\ & +2\epsilon - \frac{1}{4}\left\langle x^* - y^*, A^{-1}(x^* - y^*)\right\rangle \geq 0, \forall u \in X,\end{aligned}$$

which can be write as follows

$$\begin{aligned}&\left\langle A^{\frac{1}{2}}u - \frac{1}{2}A^{-\frac{1}{2}}(x^* - y^*), A^{\frac{1}{2}}u - \frac{1}{2}A^{-\frac{1}{2}}(x^* - y^*)\right\rangle + 2\epsilon \\ &\geq \frac{1}{4}\left\langle x^* - y^*, A^{-1}(x^* - y^*)\right\rangle, \ \forall u \in X.\end{aligned}$$

Consequently, we obtain
$\left\langle x^* - y^*, A^{-1}(x^* - y^*)\right\rangle \leq 8\epsilon + 4\left\|A^{\frac{1}{2}}u - \frac{1}{2}A^{-\frac{1}{2}}(x^* - y^*)\right\|^2, \forall u \in X$,
i.e. $\left\langle x^* - y^*, A^{-1}(x^* - y^*)\right\rangle \leq 8\epsilon$.

According to (3.3), the equality (3.4) is proved.

In the next we establish the following equality

$$B_\epsilon = \widehat{B}_\epsilon. \tag{3.5}$$

Let x^* be an arbitrary element of B_ϵ.Taking into account the equality (3.1) , for every $y^* \in B_\epsilon$, we obtain
$\left\|A^{-\frac{1}{2}}(x^* - y^*)\right\| \leq \left\|A^{-\frac{1}{2}}x^*\right\| + \left\|A^{-\frac{1}{2}}y^*\right\| \leq 2\sqrt{2\epsilon}$.

Therefore, by (3.4) it follows that x^* belongs to $\widehat{B}_\epsilon$ i.e. $B_\epsilon \subset \widehat{B}_\epsilon$.

Conversely, if we suppose that exists x^* in $\widehat{B}_\epsilon$ such that x^* does not belong to B_ϵ then, $\left\|A^{-\frac{1}{2}}x^*\right\| > \sqrt{2\epsilon}$ and for every $y^* \in B_\epsilon$, we have that $\left\|A^{-\frac{1}{2}}(x^* - y^*)\right\| \leq 2\sqrt{2\epsilon}$.

Since B_ϵ is a closed and absorbent set, then there exists $\lambda_0 > 0$ such that $\lambda_0 x^* \in B_\epsilon$ and $\left\|A^{-\frac{1}{2}}(\lambda_0 x^*)\right\| = \sqrt{2\epsilon}$. Hence $\left\|A^{-\frac{1}{2}}x^*)\right\| = \frac{\sqrt{2\epsilon}}{\lambda_0}$ and consequently, because x^* does not belong to B_ϵ,we obtain

$$\lambda_0 = \frac{\sqrt{2\epsilon}}{\left\|A^{-\frac{1}{2}}x^*)\right\|} < 1. \tag{3.5}$$

On the other hand, since B_ϵ is a circled set and $x^* \in \widehat{B}_\epsilon$, it follows that $\left\|A^{-\frac{1}{2}}\left(x^* - (-\lambda_0 x^*)\right)\right\| \le 2\sqrt{2\epsilon}$, and so $(1+\lambda_0)\left\|A^{-\frac{1}{2}}x^*)\right\| \le 2\sqrt{2\epsilon}$. Thus, we obtain that $(1+\lambda_0)\frac{\sqrt{2\epsilon}}{\lambda_0} \le 2\sqrt{2\epsilon}$, i.e. $\lambda_0 \ge 1$, which contradicts the inequality (3.5).

For the second part, we have to evaluate $\epsilon_{\min}^{\partial_\epsilon f}$. Let us consider $(x,x^*),(y,y^*) \in Gr(\partial_\epsilon f)$. According to (3.1) and (3.3) there exist $u^*, v^* \in B_\epsilon$ such that $x^* = Ax + b + u^*$ and $y^* = Ay + b + v^*$.

Now, we can write

$$\begin{aligned}
\epsilon_{\min}^{\partial_\epsilon f} &= \sup_{\substack{x,y\in X\\ u^*,v^*\in B_\epsilon}} \left(\frac{\langle y-x, Ax+b+u^*-Ax-b-v^*\rangle}{2}\right)\\
&= \sup_{\substack{x,y\in X\\ u^*,v^*\in B_\epsilon}} \left(\frac{\langle y-x, A(x-y)+u^*-v^*\rangle}{2}\right)\\
&= \sup_{\substack{u\in X\\ u^*,v^*\in B_\epsilon}} \left(-\frac{\langle u, Au\rangle}{2} - \frac{\langle u, u^*-v^*\rangle}{2}\right) \ge \sup_{\substack{u\in X\\ u^*\in B_\epsilon}} \left(-\frac{\langle u, Au\rangle \ + \ 2\langle u, u^*\rangle}{2}\right)\\
&= \sup_{\substack{u\in X\\ u^*\in B_\epsilon}} \left(-\frac{\left\langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}u\right\rangle \ + \ 2\left\langle A^{\frac{1}{2}}u, A^{-\frac{1}{2}}u^*\right\rangle}{2}\right)\\
&= \sup_{\substack{u\in X\\ u^*\in B_\epsilon}} \left(-\frac{\left\langle A^{\frac{1}{2}}u + A^{-\frac{1}{2}}u^*, A^{\frac{1}{2}}u + A^{-\frac{1}{2}}u^*\right\rangle \ - \left\langle A^{-\frac{1}{2}}u^*, A^{-\frac{1}{2}}u^*\right\rangle}{2}\right)\\
&= \sup_{\substack{u\in X\\ u^*\in B_\epsilon}} \left(\frac{1}{2}\left\|A^{-\frac{1}{2}}u^*\right\|^2 - \frac{1}{2}\left\|A^{\frac{1}{2}}u + A^{-\frac{1}{2}}u^*\right\|^2\right) = \sup_{u^*\in B_\epsilon}\left(\frac{1}{2}\left\|A^{-\frac{1}{2}}u^*\right\|^2\right) = \epsilon.
\end{aligned}$$

Therefore $\epsilon_{\min}^{\partial_\epsilon f} \ge \epsilon$ and so $\epsilon_{\min}^{\partial_\epsilon f} = \epsilon$ (because the inequality $\epsilon_{\min}^{\partial_\epsilon f} \le \epsilon$ is always true) as claimed. □

Definition 3.2. *We say that the pair* $(x, x^*) \in X \times X^*$ *is* ϵ*-monotone related to* $Gr(A)$ *if*

$$\langle x - y, x^* - y^* \rangle \geq -2\epsilon, \forall (y, y^*) \in Gr(A). \tag{3.6}$$

In [14] Revalski and Théra consider the set of all pairs $\epsilon-$monotone related to $Gr(A)$, if A is a monotone operator. Thus they introduced the $\epsilon-$enlargement of a monotone operator denoted by A^ϵ. But, this notion can be also considered for an arbitrary operator.

In the sequel, for a given operator $A : X \rightrightarrows X^*$ we attach its $\epsilon-$*enlargement,* denoted by A^ϵ, as follows:

(3.7)
$$Gr(A^\epsilon) = \{(x, x^*) \in X \times X^* / \langle x - y, x^* - y^* \rangle \geq -2\epsilon, \forall (y, y^*) \in Gr(A)\}.$$

Proposition 3.3. *Let* $A : X \rightrightarrows X^*$ *be an arbitrary operator. Then the following properties are true:*

$$A^\epsilon(x) \textit{ is } w^* \textit{ closed}, \forall x \in X. \tag{3.8}$$

$$A^\epsilon(x) \textit{ is convex}, \forall x \in X. \tag{3.9}$$

$$Gr(A^{\epsilon_1}) \subseteq Gr(A^{\epsilon_2}) \textit{ if } 0 \leq \epsilon_1 \leq \epsilon_2. \tag{3.10}$$

$$\textit{If } Gr(A) \subseteq Gr(B) \textit{ then } Gr(B^\epsilon) \subseteq Gr(A^\epsilon), \textit{ for every } \epsilon \geq 0. \tag{3.11}$$

$$Gr(A) \subseteq Gr(A^\epsilon) \textit{ if and only if } A \textit{ is } \epsilon - \textit{monotone}. \tag{3.12}$$

Taking into account the steps presented in [14] in case of monotone extensions, we can prove the following proposition:

Proposition 3.4. *Let* A *be a multivalued operator between* X *and* X^**. Then for every* $\epsilon > 0$ *we have:*

(a) $A^\epsilon(x) = \bigcap_{\epsilon' > \epsilon} A^{\epsilon'}(x)$, $\forall x \in X$;

(b) $A^\epsilon = \overline{A}^\epsilon = (coA)^\epsilon = (\overline{co}A)^\epsilon$;

(c) *If A is locally bounded, then:* $A^\epsilon = \tilde{A}^\epsilon = (co\widetilde{A})^\epsilon = [\overline{co}(\tilde{A})]^\epsilon$.

Remark 3.5.

• $A^\epsilon x$ can be the empty set, even the set Ax is nonempty.

• If A is ϵ-monotone, then for every $\epsilon' \geq \epsilon$, $A^{\epsilon'}$is an enlargement of A, but the ϵ-monotonicity property can not be maintained.

In the sequel we denote by $M_\epsilon(A)$ the family of all maximal $\epsilon-$monotone operators which contain A.

Now, we can associate to A the following operators $A^\cup_\epsilon$, $A^\cap_\epsilon : X \rightrightarrows X^*$

$$A^\cup_\epsilon(x) = \bigcup_{B\in M_\epsilon(A)} B(x),\ A^\cap_\epsilon(x) = \bigcap_{B\in M_\epsilon(A)} B(x), \forall x \in X. \tag{3.13}$$

We can easily show that if A is ϵ-monotone then $A \subseteq A^\cap_\epsilon \subseteq A\epsilon^\cup$. The operator $A^\cap_\epsilon$ is always ϵ-monotone but the operator $A^\cup_\epsilon$ can not be ϵ-monotone. In fact, $A^\cup_\epsilon$ is ϵ-monotone if and only if there exists an unique maximal ϵ-monotone operator including A, equivalently $A^\cap_\epsilon = A^\cup_\epsilon$.

Example 3.2. Let $A : X \rightrightarrows X^*$, defined by

$$A(x) = \begin{cases} 0\ , & x < 0, \\ \varnothing, & x \in (0,1), \\ 1, & x > 1. \end{cases}$$

By standard calculus we find

$$A^\cup_\epsilon x = \begin{cases} [1 - \frac{2\epsilon}{x-1}, 1], & x > 1 + 2\epsilon, \\ [0,1], & x \in [-2\epsilon, 1+2\epsilon], \\ [0, -\frac{2\epsilon}{x}], & x < -2\epsilon. \end{cases}$$

Taking $(0,1),(1,0) \in Gr(A^\cup_\epsilon)$ we have $(0-1)(1-0) = -1 < -2\epsilon$, for $\epsilon\in [0, \frac{1}{2})$. Since A is ϵ-monotone for every $\epsilon \in [0, \infty)$ it follows that $A^\cup_\epsilon$ is not ϵ-monotone for $\epsilon\in [0, \frac{1}{2})$.

Proposition 3.6. *If A is an ϵ-monotone operator, then the following properties hold:*

a) *$Gr(A) \subseteq Gr(A^\cap_\epsilon)\subseteq Gr(A^\cup_\epsilon)$;*

b) *$A^\cap_\epsilon$ is ϵ-monotone;*

c) *The ranges of operators $A^\cap_\epsilon$and $A^\cup_\epsilon$ are convex and w^*-closed;*

d) *$Gr(\ \overline{co}A) \subseteq Gr(A_\epsilon^\cap)$;*

e) *If A is locally bounded, then $Gr(\overline{co}\widetilde{A}) \subseteq Gr(A_\epsilon^\cap)$;*

f) $A_\epsilon^\cup x = \bigcap_{\epsilon' > \epsilon} A^{\epsilon'} x = A^\epsilon x$;

g) $(A_\epsilon^\cap)^{\epsilon'} = A^{\epsilon'}$, $\forall\ \epsilon' \geq \epsilon$;

h) $(A_\epsilon^\cap)_{\epsilon'}^\cap = A_{\epsilon'}^\cap$, $\forall\ \epsilon' \geq \epsilon$.

Proof. The properties a), b), c) follows from definition given by (3.13).

d) Let B be an arbitrary maximal ϵ-monotone operator which included A.If (x, x^*) belongs to $Gr(\overline{A})$ then there exists a net $(x_j^*)_j \subset Ax$ such that $x_j^* \overset{w*}{\to} x^*$. For every couple $(x_1, x_1^*) \in Gr(B)$, by passing to limit in the equality $\left\langle x - x_1, x^* - x_1^* \right\rangle = \left\langle x - x_1, x^* - x_j^* \right\rangle + \left\langle x - x_1, x_j^* - x_1^* \right\rangle$ we obtain that (x, x^*) is monotone related to $Gr(B)$. Since B is maximal ϵ-monotone we obtain that $Gr(\overline{A}) \subseteq Gr(B)$, i.e. $Gr(\overline{A}) \subseteq Gr(A_\epsilon^\cap)$. Since the operator $A_\epsilon^\cap$ is always with convex and w^*-closed images we obtain that $Gr(\overline{co}A) \subseteq Gr(A_\epsilon^\cap)$.

e) Let (x, x^*) be an arbitrary couple of $Gr(\widetilde{A})$. Then we find a net $(x_i, x_i^*)_i \subset Gr(A)$,so that $(x_i, x_i^*) \overset{\|\cdot\| \times w*}{\longrightarrow} (x, x^*)$.

If we take $B \in M_\epsilon(A)$, for every $(x_1, x_1^*) \in Gr(B)$, we have
$\langle x - x_1, x^* - x_1^* \rangle = \langle x - x_i, x^* - x_1^* \rangle + \langle x_i - x_1, x_i^* - x_1^* \rangle + \langle x_i - x_1, x^* - x_i^* \rangle$

By passing to limit, because A is locally bounded, we obtained that (x, x^*) is ϵ- monotone related to $Gr(B)$. Since B is maximal ϵ-monotone it follows that $Gr(\widetilde{A}) \subseteq Gr(B)$ i.e.$\tilde{A}(x) \subseteq \bigcap_{B \in M_\epsilon(A)} B(x) = A_\epsilon^\cap(x), \forall x \in X$ and so $Gr(\tilde{A}) \subseteq Gr(A_\epsilon^\cap)$.

Using once again that the operator $A_\epsilon^\cap$is always with convex and w^*-closed images we obtain that $Gr(\overline{co}\widetilde{A}) \subseteq Gr(A_\epsilon^\cap)$.

f) If $(x, x^*) \in Gr(A_\epsilon^\cup)$, then $\langle x - y, x^* - y^* \rangle \geq -2\epsilon$ for every $(y, y^*) \in Gr(A)$. Now, if we consider an $\epsilon' \geq \epsilon$, then for every $(y, y^*) \in Gr(A)$ we have that $\langle x - y, x^* - y^* \rangle \geq -2\epsilon'$,and so $(x, x^*) \in Gr(A^{\epsilon'})$, i.e.

$$A_\epsilon^\cup(x) \subseteq \bigcap_{\epsilon' > \epsilon} A^{\epsilon'}(x).$$

Conversely, if $x^* \in \bigcap_{\epsilon' > \epsilon} A^{\epsilon'}(x)$, then $\langle x - y, x^* - y^* \rangle \geq -2\epsilon'$, for all $\epsilon' > \epsilon$, and $(y, y^*) \in Gr(A)$. It follows that (x, x^*) is $\epsilon-$monotone related to $Gr(A)$ and so, there exist an maximal ϵ-monotone operator B such that $Gr(A) \cup \{(x,x^*)\} \subset Gr(B)$, i.e. $(x,x^*) \in Gr(A_\epsilon^\cup)$.

g) Since $Gr(A)$ is included in $Gr(A_\epsilon^\cap)$, using (3.11) we get $Gr(A_{\epsilon'}^\cup) \subset Gr((A_\epsilon^\cap)_{\epsilon'}^\cup)$.

Conversely, if $(x,x^*) \in Gr(A_{\epsilon'}^\cup)$, then there exists a maximal ϵ'-monotone operator B, so that $Gr(A) \cup \{(x,x^*)\} \subset Gr(B)$. Therefore $Gr(A) \subseteq Gr(A_\epsilon^\cap) \subseteq Gr(B)$ and $Gr(B_{\epsilon'}^\cup) \subseteq Gr((A_\epsilon^\cap)_{\epsilon'}^\cup)$. But B being maximal ϵ'-monotone we have that $Gr(B) = Gr(B_{\epsilon'}^\cup)$ and so, $Gr(B)$ is contained in $Gr((A_\epsilon^\cap)_{\epsilon'}^\cup)$. Hence $(x,x^*) \in Gr((A_\epsilon^\cap)_{\epsilon'}^\cup)$ i.e. $Gr(A_{\epsilon'}^\cup) \subseteq Gr((A_\epsilon^\cap)_{\epsilon'}^\cup)$.

h) Since $Gr(A) \subset Gr(A_\epsilon^\cap)$, taking into account (3.13), for every $\epsilon' \geq \epsilon$ we obtain that $Gr((A_\epsilon^\cap)_{\epsilon'}^\cap) \subseteq Gr(A_{\epsilon'}^\cap)$.

If B is an arbitrary maximal ϵ'-monotone operator which contain $Gr(A_\epsilon^\cap)$ then $Gr(B) = Gr(B_{\epsilon'}^\cup)$ and $Gr(A) \subseteq Gr(B)$. Thus, we get $Gr(A_{\epsilon'}^\cap) \subseteq Gr(B)$ i.e. $Gr(A_{\epsilon'}^\cap) \subseteq Gr((A_\epsilon^\cap)_{\epsilon'}^\cap)$, for every $\epsilon' \geq \epsilon$.

The proof is completed. □

Proposition 3.7. *Let A be a maximal ϵ-monotone operator. Then*

$$A = A_\epsilon^\cap = A_\epsilon^\cup. \tag{3.14}$$

Proof. It is obvious that $Gr(A) \subseteq Gr(A_\epsilon^\cap) \subseteq Gr(A_\epsilon^\cup)$. Conversely, let (x,x^*) be an arbitrary element of $Gr(A_\epsilon^\cup)$. We have $\langle x-y, x^*-y^*\rangle \geq -2\epsilon$, for every $(y,y^*) \in Gr(A)$ and so, we obtain that the operator with the graph $Gr(A) \cup \{(x,x^*)\}$ is ϵ-monotone and contains $Gr(A)$. Hence $(x,x^*) \in Gr(A)$ because A is maximal ϵ-monotone. □

Proposition 3.8. *If A is an ϵ-monotone operator, then A^ϵ is ϵ-monotone if and only if there is an unique maximal ϵ-monotone operator which contains A. In this case, A^ϵ is the unique maximal ϵ-monotone operator which contains A.*

Proof. Since A^ϵ is ϵ-monotone, there exists $B \in M_\epsilon(A)$ such that $Gr(A^\epsilon) \subseteq Gr(B) \subseteq Gr(A_\epsilon^\cup)$. On the other hand, $A^\epsilon = A_\epsilon^\cup$ and so $B = A^\epsilon$ i.e. $A_\epsilon^\cap = A_\epsilon^\cup = A^\epsilon$.

Conversely, if there exists an unique element in $M_\epsilon(A)$, we have that this operator is just $A_\epsilon^\cap = A_\epsilon^\cup$. Thus, according to Proposition 3.6 (f) the proof is finished. □

Proposition 3.9. *If A is a maximal ϵ-monotone operator, then A is also maximal $\epsilon_{\min}^A$-monotone operator.*

Proof. If we suppose that A is not maximal $\epsilon_{\min}^{A}$-monotone, then there exists a pair $(x, x^*) \in X \times X^*$ which is $\epsilon_{\min}^{A}$-monotone related to $Gr(A)$ and $(x, x^*) \notin Gr(A)$.Since $\epsilon_{\min}^{A} \leq \epsilon$, we obtained that the pair (x, x^*) is ϵ-monotone related to $Gr(A)$. Because A is maximal ϵ-monotone operator, we get $(x, x^*) \in Gr(A)$ which is a contradiction. □

There are ϵ-monotone operators which are maximal $\epsilon_{\min}^{A}$-monotone, but are not maximal ϵ-monotone, as we shown in the following:

Example 3.3. Let $A : \mathbb{R} \longrightarrow \mathbb{R}$, be defined by

$$A(x) = \begin{cases} 0, & x < 0, \\ [0, & \infty), x = 0, \\ \varnothing, & x > 0. \end{cases}$$

It is easily to prove that A is maximal monotone. Hence $\epsilon_{\min}^{A} = 0$, but A is not maximal ϵ-monotone for every $\epsilon > 0$, since A^{ϵ} strictly contains A.

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