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CERTAIN CLASSES OF DOMAINS. APPLICATIONS IN EXTENSION THEOREMS FOR QUASICONFORMAL MAPPINGS

BY

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Abstract. The classes of domains which are present in this paper were introduced by Gehring, Martio, Sarvas, Jones, Koskela and we shall use for to give or to improve some boundary extension theorems for quasiconformal mappings.
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One of the basic problem of quasiconformal mappings theory is to determine conditions for that a quasiconformal mapping between two domains $f : D \to D'$ can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$. In this paper we deal with the boundary behavior for quasiconformal mappings involving domains that satisfy some properties.

Many results about boundary behavior of a quasiconformal mapping between two domains with certain properties due to Gehring and Martio [5], Väisälä [25], Herron and Koskela ([9], [14]). For the reader's convenience we include some of the definitions for various classes of domains and the remarks from [5], [8], [9], [10], [11], [12], [14], [18], [21] some of them perhaps are of independent interest. We, also establish a few relations between these classes of domains.

1. Certain classes of domains. Throughout this paper D and D' are domains in Euclidean n-dimensional space R^n or its the one-point compactification $\overline{R}^n = R^n \cup \{\infty\}, n \geq 2$.

At the beginning, we recall some definitions and properties which we shall use in this paper.

Gehring introduced the concept of linearly locally connected domain.

Definition 1.1. *A domain $D \subset \overline{R}^n$ is called a-linearly locally connected, $a \geq 1$, if for all $x \in R^n$ and all $r > 0$ we have:*

(1) *points in $D \cap \overline{B}(x,r)$ can be joined by a continuum in $D \cap \overline{B}(x,ar)$;*

(2) *points in $D \setminus B(x,r)$ can be joined by a continuum in $D \setminus B(x, \frac{r}{a})$.*

We abbreviate this by saying that D is a-LLC. We say that a domain is LLC if it is a-LLC for some $a \geq 1$.

Definition 1.2. *A domain $D \subset \overline{R}^n$ is called (a,ρ) -weakly linearly locally connected, or simply (a,ρ)-WLLC, if $a \geq 1, \rho > 0$ are constants such that the conditions (1) and (2) in Definition 1.1 valid for all $x \in R^n$ and all $0 < r < \rho$.*

We say that a domain is WLLC if it is (a,ρ)-WLLC for some $a \geq 1$ and $\rho > 0$.

Remark 1.1. LLC domains are locally connected on the boundary and WLLC are locally connected at every finite boundary point ([9], p. 190).

Remark 1.2. Obviously, LLC domains are WLLC.

Remark 1.3. WLLC domains are finitely connected at infinity ([9], Corollary 2.3) and hence these are finitely connected on the boundary.

Remark 1.4. A bounded WLLC domain is LLC ([9], Lemma 2.5).

The two generic examples of domains which are WLLC but not LLC are:

a) an infinite cylinder $\{x : x_1^2 + ... + x_{n-1}^2 < 1\}$ and

b) the complement of a semi -infinite band $R^n \setminus \{x : x_1 \geq 0, |x_n| \leq 1\}$.

Next, if E, F are subsets of $\overline{D}$, we denote by $\Delta(E,F;D)$ the family of all paths which join E and F in D and by $M(\Delta(E,F;D))$ the modulus of this family. If D is R^n or $\overline{R}^n$ we can denote $M(\Delta(E,F;D))$ by $M(\Delta(E,F))$ in view of the equality

$$M(\Delta(E,F;R^n)) = M(\Delta(E,F;\overline{R}^n))$$

Gehring and Martio [5] were introduced the class of quasiextremal distance domains $D \subset \overline{R}^n$. They showed that if a simply connected domain in R^2 is QED then it is uniform.

Definition 1.3. *A domain D in $\overline{R}^n$ is called k-quasiextremal distance domain, $k \geq 0$, if the following condition holds:*

$$M(\Delta(E,F)) \leq kM(\Delta(E,F;D))$$

for any disjoint continua E, F in D.

We abbreviate this by saying that D is k-QED. A domain is QED if it is k-QED for some constants $k \geq 0$.

An equivalent description for QED domains can be given in terms of capacity thanks to Hesse ([12], 5.5) who established that $M(\Delta(E,F;D)) = cap(E,F;D)$ for any pair of disjoint compact sets $E, F \subset D$. If D is QED then equality remains valid for compacta in $\overline{D}$ ([8], 2.6]).

Definition 1.4. *The conformal capacity is defined by*

$$cap(E,F;D) = \inf_{u \in L} \int_D |\nabla u|^n$$

where $E, F \subset \overline{D}$ are disjoint non-empty compact sets and the infimum is taken over all functions in the class

$$L = L(E,F;D) = \{u \in L^1_n(D) \cap C(D \cup E \cup F); u\,|_E \leq 0, u\,|_F \geq 1\}.$$

We denote by $L^1_n(D)$ the Sobolev space of locally integrable functions $u : D \to R \cup \{-\infty, +\infty\}$ which satisfy $\int_D |\nabla u|^n < \infty$, where ∇u represents the distributional gradient of u.

Definition 1.5. *The Sobolev capacity of E, F relative to D is defined by:*

$$s - cap(E,F;D) = \inf_{u \in W} \int_D (|u|^n + |\nabla u|^n)$$

where

$$W = W(E,F;D) = \{u \in W^1_n(D) \cap C(D \cup E \cup F), u\,|_E \leq c, u\,|_F \geq c+1, c \in R\}.$$

Here, $W_n^1(D) = L_n^1(D) \cap L^n(D)$ where $L^n(D)$ is the Banach space of all measurable functions $u : D \to R \cup \{-\infty, \infty\}$ for which the norm $\|u\|_{L^n(D)} = (\int_D |u|^n \, dm)^{\frac{1}{n}}$ is finite. The space $W_n^1(D)$ is equipped with the norm $\|u\|_{W_n^1(D)} = \|\nabla u\|_{L^n(D)} + \|u\|_{L^n(D)}$.

Obviously, $W_n^1(D) \subset L_n^1(D)$ but $L_n^1(D) \nsubseteq W_n^1(D)$ even if D is bounded (see [18], 1.14).

If D is R^n or $\overline{R}^n$ and since $s - cap(E, F; R^n) = s - cap(E, F; \overline{R}^n)$ we denote $s - cap(E, F; D)$ by $s - cap(E, F)$.

Definition 1.6. *A domain D in R^n is called a Sobolev capacity domain with constant M ($M{-}SC$) if the following condition holds:*

$$s - cap(E, F) \leq M \cdot s - cap(E, F; D)$$

for any disjoint continua E, F in D. A domain is SC if it is M-SC for some constant M.

The Sobolev capacity domain was introduced by Koskela ([14], Definition 5.6).

It is important to remind the next remark:

Remark 1.5. (i) $cap(E, F; D) \leq s - cap(E, F; D)$;

(ii) $s - cap(E, F; D) = s - cap(F, E; D)$;

(iii) If $E, F \subset \overline{D}$ are disjoint compact sets, $D \subset D'$ and $E_0 \subset E, F_0 \subset F, E_0, F_0$ are compact sets, then $s - cap(E_0, F_0; D) \leq s - cap(E, F; D')$ ([14], Lemma 5.2).

We recall that a QED domain is invariant with respect to Möbius transformations but a SC domain is not invariant. In fact, a SC domain is not even affine invariant.Moreover, a bounded QED domain is a SC domain ([14], Theorem 5.14). An example of the SC domain which is not QED is $D = R^n \setminus \{x : 0 \leq x_n \leq 1, x_{n-1} \geq 0\}$.

Martio and Sarvas [21] were introduced the concept of a uniform domain in their studies on some problems in connection with univalent functions.

Definition 1.7. *A domain D in $\overline{R}^n$ is said to be c-uniform, $c > 0$, if each pair of points $x, y \in D$ can be joined by a rectifiable arc $\gamma \subset D$ satisfying:*

(1) $l(\gamma) \leq c\,|x - y|$, *and*

(2) $\min\{l(\gamma'), l(\gamma'')\} \leq c \cdot dist(z, \partial D)$ *for all* $z \in \gamma$. *Here,* $l(\gamma)$ *is the Euclidean arclengh of* γ *and* γ', γ'' *are the components of* $\gamma \setminus \{z\}$.

There exist some equivalent definitions for the uniform domains (for example see [6], [7], [17], [26], [27]).

Definition 1.8. *A domain* D *in* $\overline{R}^n$ *is said to be* (c, r)*-locally uniform,* $c > 0, r > 0$, *if each pair of points* $x, y \in D$ *with* $|x - y| < r$ *can be joined by a rectifiable arc* $\gamma \subset D$ *satisfying the conditions* (1) *and* (2) *in Definition* 1.7. *A domain is locally uniform if it is* (c, r)*-locally uniform for some constants* $c > 0, r > 0$.

Remark 1.6. Obviously, the uniform domains are locally uniform.

Remark 1.7. A bounded locally uniform domain is uniform ([10], 2.12).

Examples of domains which are locally uniform but not uniform are an infinite cylinder and the complement of a semi-infinite band.

Remark 1.8. The uniform domains are QED which in turn are LLC ([5], 2.18, 2.11) and similarly locally uniform domains are SC which are WLLC ([14], 5.7, 5.8, 5.10).

By above remarks it is important to emphasize the following two remarks.

Remark 1.9. The QED domains are locally connected at every boundary point and the SC domains are locally connected at every finite boundary point.

Remark 1.10. The SC domains are finitely connected on the boundary.

Definition 1.9. $D \subset R^n$ *is locally quasiconvex if there exist* $0 < \delta \leq \infty$ *and* $b \geq 1$ *such that any* $x, y \in D$ *with* $|x - y| \leq \delta$ *can be joined in* D *by a rectifiable arc* $\gamma \subset D$ *satisfying* $l(\gamma) \leq b \cdot |x - y|$. *When* $b = \infty, D$ *is said to be b-quasiconvex or quasiconvex* (*see, for instance* [14], *Definition* 1.3 (ii)).

Definition 1.10. *A domain* D *is a* (b, δ)*-domain* [13], $0 < \delta \leq \infty, 1 \leq b$, *if for all* $x, y \in D$ *with* $|x - y| < \delta$, *there exists a path* $\gamma : [0, l(\gamma)] \to D$ *with* $\gamma(0) = x, \gamma(l(\gamma)) = y, l(\gamma) \leq b \cdot |x - y|$ *and* $B^n(\gamma(t), \frac{1}{b} \cdot \min\{t, l(\gamma) - t\}) \subset D$ *for* $t \in (0, l(\gamma))$. *A* (b, ∞)*-domain is called b-uniform* (*also, see* [14], [7], [13], [19] *and* [21]).

Remark 1.11. A bounded (b,δ)-domain is uniform. ([14], Remark 1.4 (iii)).

An example of the (b,δ)-domain is $D=(-1,1)^{n-1}\times(-\infty,\infty)$ but D is not QED (see [14], Example 6.7).

Definition 1.11. *A domain $D\subset R^n$ is a Loewner domain if there exists a decreasing function $\eta:(0,\infty)\to(0,\infty)$ such that $M(\Delta(E,F;D))\geq\eta(t)$ whenever E,F are disjoint non-degenerate continua in D, and $t\geq\frac{dist(E,F)}{\min\{d(E),d(F)\}}$.*

The Loewner condition on a metric space with a locally finite regular Borel measure introduced by Heinonen and Koskela [11]. They showed that, the Loewner domains are LLC. In Euclidean n-space R^n, $n\geq 2$ the Loewner's condition was first time using by Loewner [16] that observed that conform modulus has a Loewner uniform boundary which depends only the dimension n and the relative positions of those two continua.

Definition 1.12. *A domain $D\subset R^n$ is a John domain* [21] *if there exist constants $a\geq b>0$ and a point $x_0\in D$ such that each $x\in D$ can be joined to x_0 by a curve $\gamma:[0,l(\gamma)]\to D$ with $\gamma(0)=x, l(\gamma)\leq a$ and $B^n(\gamma(t),\frac{bt}{l(\gamma)})\subset D$ for $0<t\leq l(\gamma)$; see* [20] *and* [23] *for various characterization of John domains.*

Definition 1.13. *A domain $D\subset R^n$ is an L^1_n-extension domain if there exists a bounded linear operator $E_n:L^1_n(D)\to L^1_n(R^n)$ with $E_nu\mid_D=u$ for all $u\in L^1_n(D)$. Boundedness of E_n means boundedness with respect to the seminorms $\|\nabla u\|_{L^n(D)}$ and $\|\nabla E_nu\|_{L^n(R^n)}$ ([14], Definition* 1.3 (v)*).*

Definition 1.14. *A domain D in R^n is a W^1_n -extension domain if there exists a bounded linear operator $E_n:W^1_n(D)\to W^1_n(R^n)$ with $E_nu\mid_D=u$ for all $u\in W^1_n(D)$ ([14], Definition* 1.3 (vi)*).*

In speciality literature, the L^1_n-extension domains and W^1_n-extension domains are called Sobolev extension domains.

Proposition 1.1. *Let D be a simply connected domain in the complex plane, which is distinct of the complex plane and has more than one boundary point. Then D is a L^1_2-extension domain if and only if it is uniform.*

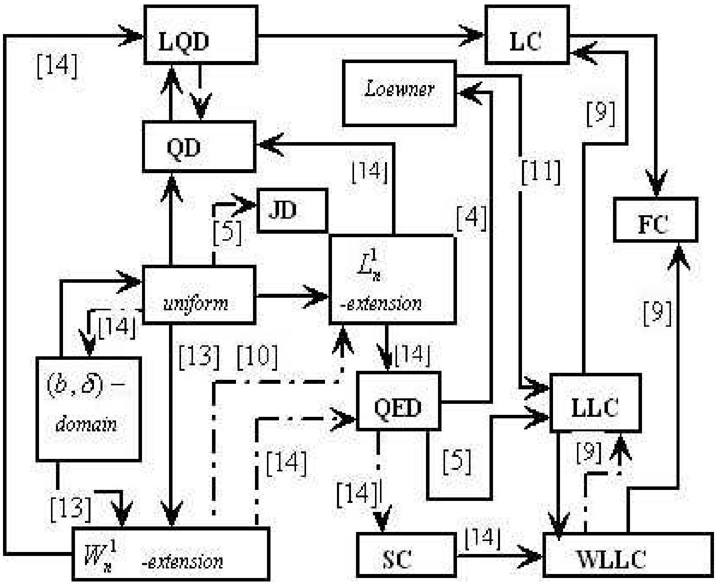
Proof. By Riemann mapping theorem ([15], Theorem 3.3, p. 143), D is conformally equivalent to the unit disk. Now, suppose that D is a

L_2^1-extension domain and by Theorem 2.2 [14] we conclude that D is QED. Since the unit disk is a uniform domain, the conditions in Corollary 3.6. [14] are satisfied and hence D is uniform.

Next assume that D is uniform and since the unit disk is uniform, by Theorem 6.3 (i) [14], it follows that D is a L_2^1-extension domain. $\square$

Remark 1.12. Since a simply connected domain D in R^2, $D \neq R^2$ is uniform if and only if D is a quasidisk (see [21]) and using Proposition 1.1 we establish that D is a L_2^1-extension domain if and only if D is a quasidisk.

We remind that a domain bounded by a quasicircle in the extended plane is called a quasidisk. The image of a circle under a quasiconformal mapping of the plane is called a quasicircle.



Remark 1.13. We present schematically in Figure 1, some relations between the classes of domains introduced above and that we shall use in this paper. The implications denote by $-- \blacktriangleright$ hold for bounded domains. Near implications we denoted the references where we find them. For simplicity, we abbreviate local connectedness on the boundary to LC and finite connectedness on the boundary to FC. We, also abbreviate a local quasiconvex domain by LQD and a quasiconvex domain by QD and a John domain by JD.

2. Extension theorems for quasiconformal mapping. First we shall make an analysis of the boundary behavior of quasiconformal mappings where one of the domains in question is QED or SC or Sobolev extension domain or uniform domain or a bounded $(b,\delta)-$ domain or a Loewner domain.

Theorem 2.1. *Suppose* $D \subset \overline{R}^n$ *is QED* (*or* $D \subset R^n$ *is SC*)*. Then* D *has property* P_1 *and* D *is quasiconformally accessible at each* (*finite*) *boundary point* ([9], *Lemma* 3.1).

We note that in speciality literature instead of D has property P_1 one says that D is quasiconformally flat.

Theorem 2.2. *Suppose* $D \subset \overline{R}^n$ *is QED (or* $D \subset R^n$ *is SC). Then* D *has the property* P_2^* *at each* (*finite*) *boundary point* ([2], *Theorem* 7).

For definition of property P_2^* see Definition 3 [1].

We mention that in general SC domains are not quasiconformally accessible at ∞ and haven't properties P_1, P_2^* at ∞. For example see ([9], Examples 3.2) and a straightforward calculation illustrates that D by Example 3.2 (b) [9] fails to has property P_2^* at ∞.

Next, we improve the assertion (d) of the Theorem 3.3 [9] saying that moreover D is locally connected at z and D is quasiconformally accessible at z. We add the assertion (h) and prove what we have added and improved.

Theorem 2.3. *Let* $f : D \to D'$ *be a quasiconformal mapping and* $D' \subset \overline{R}^n$ *is QED (or* $D' \subset R^n$ *is SC). Fix points* $z \in \partial D, w \in \partial D', (w \neq \infty)$.

(a) *If* D *is locally connected or it has property* P_1 *at* z *then* f *can be extended to a continuous mapping* $f^* : D \cup \{z\} \to D' \cup \{f^*(z)\}$ *where* $f^*(z) = C(f, z)$.

(b) *If* f *can be extended to a continuous mapping* $f^* : D \cup \{z\} \to D' \cup \{f^*(z)\}$ (*and* $f^*(z) \neq \infty$)*, then* D *has property* P_1 *at* z.

(c) *If* D *is finitely connected at each point of* $C(f^{-1}, w)$ *or if* D *is quasiconformally accessible at least one point of* $C(f^{-1}, w)$ *then* f^{-1} *can be extended to a continuous mapping* $(f^{-1})^* : D' \cup \{w\} \to D \cup C(f^{-1}, w)$.

(d) *If* f^{-1} *can be extended to a continuous mapping* $(f^{-1})^* : D' \cup \{w\} \to D \cup \{z\}$ *where* $\{z\} = C(f^{-1}, w)$, *then* D *is finitely connected and has property* P_1 *at* z*. Moreover* D *is quasiconformally accessible at* z *and* D *is locally connected at* z.

(e) *If* $w \in C(f,z), (\infty \notin C(f,z)), f$ *can be extended to a homeomorphism* $f^* : D \cup \{z\} \to D' \cup \{w\}$ *if and only if* D *is locally connected at each point of* $C(f^{-1},w)$.

(f) *If* $w \in C(f,z), (\infty \notin C(f,z)), f$ *can be extended to a homeomorphism* $f^* : D \cup \{z\} \to D' \cup \{w\}$ *if and only if* D *has property* P_1 *and is finitely connected at each point of* $C(f^{-1},w)$.

(g) *If* $w \in C(f,z), (\infty \notin C(f,z)), f$ *can be extended to a homeomorphism* $f^* : D \cup \{z\} \to D' \cup \{w\}$ *if and only if* D *has property* P_1 *at each point of* $C(f^{-1},w\}$ *and* D *is quasiconformally accessible at least one point of* $C(f^{-1},w)$.

(h) *If* $w \in C(f,z), (\infty \notin C(f,z))$ *then* f *can be extended to a homeomorphism* $f^* : D \cup \{z\} \to D' \cup \{w\}$ *if* D *has properties* P_1 *at* z *and* D *has property* P_2^* *at least one point of* $C(f^{-1},w)$.

Proof. (d) We prove that D is locally connected at z. Since D is finitely connected at z and D has property P_1 at z, by Theorem 1.18 [22], it follows that D is locally connected at z.

We prove that D is quasiconformally accessible at z. Since D is locally connected at z, and hence D is 1 -connected at z, in view of Theorem 2.3 [22] we obtain that $C(f,z)$ either have not points quasiconformally accessible from D', or consists of a single point. On the other hand, according with Theorem 2.1, D' is quasiconformally accessible on the boundary and hence $C(f,z)$ consists of a single point. But $\{z\} = C(f^{-1},w)$ and therefore $\{w\} = C(f,z)$.

Let U be a neighborhood of z. By $\lim\limits_{y\to w} f^{-1}(y) = z$, there exists a neighborhood U' of w such that$f^{-1}(U' \cap D') \subset U \cap D$.Since D' is quasiconformally accessible at w, there exist a continuum F' in D' and a constant $\delta > 0$ such that

$$(*) \qquad M(\Delta(E',F';D')) \geq \delta$$

whenever E' is a connected set in D' with $w \in \overline{E}'$ and $E' \cap \partial U' \neq \Phi$. Set $F = f^{-1}(F')$ which is a continuum in D. Take $\delta' = \frac{\delta}{K(f)} = \frac{\delta}{K(f^{-1})}$ and let E be a connected subset of D, with $z \in \overline{E}$ and $E \cap \partial U \neq \Phi$. The set $E_1' = f(E)$ is connected, $E_1' \cap \partial U' \neq \Phi$ and $w \in \overline{E}_1'$.

Using $(*)$ and the fact f is a quasiconformal mapping, it follows that $M(\Delta(E,F;D)) \geq \frac{1}{K(f^{-1})} M(\Delta(E_1',F';D')) \geq \frac{\delta}{K(f)} = \delta'$.

Therefore, D is quasiconformally accessible at z.

Now, we prove (h). By Remarks 1.9, 1.10, D' is finitely connected at each point of $C(f,z)$ and since D has property P_1 at z, according with Theorem 17.13 [24], we obtain that $C(f,z) = \{w\}$. By Remark 1.9, D' is locally connected at w and since D has property P_2^* at least one point of $C(f^{-1},w)$, and using Theorem 17.15 [24] with the remark that it is true if instead of property P_2 one considers property P_2^*, it follows that $C(f^{-1},w) = \{z\}$. Hence, the proof of (h) is complete. □

Remark 2.1. When D' is SC the conclusion of Theorem 2.3 (b,c) are not true without the restriction $f(z) \neq \infty, w \neq \infty$; e.g., to see that (b) can fail look at the identity map on an infinite cylinder, and for (c) consider the map from a ball onto an infinite cylinder. Note that we cannot conclude that D is locally connected at z in Theorem 2.3(b). ([9], Remarks 3.4).

Theorem 2.4. *Let $f : D \to D'$ be a quasiconformal mapping and $D' \subset \overline{R}^n$ is QED (or $D' \subset R^n$ is SC). Suppose that z is a boundary point of D and $w \in C(f,z), (\infty \notin C(f,z))$. If D has property P_1 at each point of $C(f^{-1},w)$ and property P_2^* at least one point of $C(f^{-1},w)$ then f can be extended to a homeomorphism $f^* : D \cup \{z\} \to D' \cup \{w\}$ (Theorem 8 [2]).*

Using above results, we give extension theorems on entire boundary for quasiconformal mapping.

Theorem 2.5. *Suppose that $f : D \to D'$ is a quasiconformal mapping and $D' \subset \overline{R}^n$ is QED (or $D' \subset R^n$ is SC, $\infty \notin \partial D'$). Then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D}'$ if and only if D has properties P_1 and P_2^* on the boundary (Corollary 1, [2]).*

Theorem 2.6. *Let $f : D \to D'$ be a quasiconformal mapping and $D' \subset \overline{R}^n$ is QED (or $D' \subset R^n$ is SC). If D is locally connected or D has property P_1 on the boundary then f can be extended to a continuous mapping $f^* : \overline{D} \to \overline{D'}$.*

Proof. The conclusion is a simple corollary of the Theorem 2.3 (a). □

Theorem 2.7. *Let $f : D \to D'$ a quasiconformal mapping and $D, D' \subset R^n$. Suppose that D' is a L_n^1 -extension domain or a bounded W_n^1-extension domain or a uniform domain or a bounded (b,δ) -domain. If D is locally connected or D has property P_1 on the boundary, then f can be extended to a continuous mapping $f^* : \overline{D} \to \overline{D'}$.*

Proof. The required conclusion follows by Remark 1.13 and Theorem 2.6. $\square$

Theorem 2.8. *Let $f : D \to D'$ be a quasiconform mapping and $D \subset \overline{R}^n$ is QED (or $D \subset R^n$ is SC, $\infty \notin \partial D$). Then f can be extended to a continuous mapping $f^* : \overline{D} \to \overline{D'}$ if D' is finitely connected on the boundary.*

Proof. In according with Theorem 2.1 it follows that D has property P_1 on the boundary and since D' is finitely connected on the boundary, by means of Corollary 17.14 [24], we obtain that f can be extended to a continuous mapping $f^* : \overline{D} \to \overline{D'}$. $\square$

Theorem 2.9. *Let $f : D \to D'$ be a quasiconformal mapping and $D, D' \subset R^n$. Suppose that D is a L_n^1 -extension domain or a bounded W_n^1-extension domain or a uniform domain or a bounded (b, δ) -domain. Then f can be extended to a continuous map $f^* : \overline{D} \to \overline{D'}$ if D' is finitely connected on the boundary.*

Proof. In view of Remark 1.13 and Theorem 2.8 we get the desired conclusion. $\square$

Remark 2.2. Theorem 2.9 is valid if D' is one of the domains from the schema of Figure 1.

Theorem 2.10. *Let $f : D \to D'$ a quasiconformal mapping. If $D, D' \subset \overline{R}^n$ are QED (or $D, D' \subset R^n$ are SC, $\infty \notin \partial D, \infty \notin \partial D'$) then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$.*

Proof. Remarks 1.9, 1.10 involve that, D, D' are finitely connected on the boundary and according with Theorem 2.1, D, D' have property P_1 on the boundary. Thus, the conclusion is a simple corollary of Theorem 17.13 [24]. $\square$

Theorem 2.11. *Let $f : D \to D'$ a quasiconformal mapping and $D, D' \subset R^n$, $\infty \notin \partial D, \infty \notin \partial D'$. If D is QED and D' is SC (or D is SC and D' is QED) then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$.*

Proof. The proof is likewise with the proof of Theorem 2.10. $\square$

Theorem 2.12. *Let $f : D \to D'$ a quasiconformal mapping and $D, D' \subset R^n$ If D and D' are L_n^1 -extension domains or bounded W_n^1 -extension domains or bounded (b, δ) -domains or uniform domains (it is not indispensable that D and D' to be same kind), then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$.*

Proof. Using Remark 1.13, D and D' are QED and by Theorems 2.10, 2.11 follows the desired conclusion. □

Theorem 2.13. *Let $f : D \to D'$ a quasiconformal mapping.*

If $D \subset \overline{R}^n$ is QED (or $D \subset R^n$ is SC, $\infty \notin \partial D$) then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$ if and only if D' is locally connected on the boundary (Theorem 9 [2]).

Theorem 2.14. *Let $f : D \to D'$ be a quasiconformal mapping and $D, D' \subset R^n$. If D is a L_n^1 -extension domain or a bounded W_n^1 -extension domain or uniform domain or a bounded (b, δ) -domain then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$ if and only if D' is locally connected on the boundary.*

Proof. Remark 1.13 and Theorem 2.13 involve the desired conclusion.□

Theorem 2.15. *Let $f : D \to D'$ be a quasiconformal mapping and $D, D' \subset R^n$ and D is QED. If D' is a Loewner domain or a locally quasiconvex domain or a LLC domain then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$.*

Proof. The proof rests on Remark 1.13 and Theorem 2.13. □

Theorem 2.16. *Let $f : D \to D'$ a quasiconformal mapping. If $D \subset \overline{R}^n$ is QED (or $D \subset R^n$ is SC, $\infty \notin \partial D$), then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$ if and only if D' has property P_1 on the boundary and is finitely connected on the boundary.*

Proof. By Remarks 1.9, 1.10 it follows that D is finitely connected on the boundary and from Theorem 2.1, D has property P_1 on the boundary.

For the necessity, suppose that f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D}'$. Since D is finitely connected and has property P_1 on the boundary, according with Theorem 3.1 [22], D' is finitely connected and has property P_1 on the boundary.

Conversely, suppose that D' has property P_1 on the boundary and D' is finitely connected on the boundary. Since D has property P_1 and is finitely connected on the boundary, in view of Corollary 17.14 [24], we obtain that f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$. $\square$

Theorem 2.17. *Let $f : D \to D'$ be a quasiconformal mapping and $D, D' \subset R^n$. If D is a L_n^1 -extension domain or a bounded W_n^1 -extension domain or a uniform domain or a bounded (b, δ) -domain, then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$ if and only if D' has property P_1 on the boundary and D' is finitely connected on the boundary.*

Proof. From Remark 1.13, D is QED and by Theorem 2.16 it follows the required conclusion. $\square$

Theorem 2.18. *Let $f : D \to D'$ be a quasiconformal mapping, $D, D' \subset R^n$ and D is QED. If D' is a L_n^1 -extension domain or a bounded W_n^1 -extension domain or a uniform domain or a bounded (b, δ) -domain, then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$.*

Proof. Using Remark 1.13, D' is QED and by Theorem 2.10 we get the conclusion. $\square$

Theorem 2.19. *Let $f : D \to D'$ be a quasiconformal mapping and $D, D' \subset R^n$, D, D' are bounded Loewner domains. Then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$.*

Proof. Since R^n is complete $n-$regular metric spaces, we can apply Theorem 2.12 [3] and we obtain the conclusion of theorem. $\square$

Theorem 2.20. *Let $f : D \to D'$ be a quasiconformal mapping, $D \subset \overline{R}^n$ D is a uniform domain. If there exists a constant positive c such that*

$$|f(x) - f(y)| \leq c \cdot |f(x) - f(z)|\,, (\forall)\, |x - y| \leq |x - z|\,, x, y, z \in D$$

then$f$ can be extended to a homeomorphim $f^* : \overline{D} \to \overline{D'}$.

Proof. Using the result of 2.12 [9], it follows that D' is a uniform domain and hence, D and D' are QED. Apply Theorem 2.10, we have the desired conclusion. $\square$

Gehring and Martio have observed that every quasiconformal mapping $f : D \to D'$, where D is LLC and D' is QED, always can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$, ([5], 3.1). This result it follows by Theorem 2.1, Remark 2 [1], Theorem 2.2, Corollary 17.14 [24] and Remarks 1.1, 1.9.

Analogous results hold if replace or not the domains LLC with WLLC and respectively, QED with SC. Thus, we obtain the following theorem.

Theorem 2.21. *Let $f : D \to D'$ a quasiconformal mapping and $D, D' \subset R^n, \infty \notin \partial D, \infty \notin \partial D'$. If D is WLLC and D' is SC then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$.*

Proof. Let $b \in \partial D$. By Remark 1.1 it follows that D is locally connected at b. Since D' is SC, in view of Theorem 2.2, we have that D' has property P_2^* on the boundary. Making use of Remark 2 [1] we get that f has a limit at b. One the other hand, by Remark 1.3 , D is finitely connected on the boundary and by Theorem 2.1, D' has property P_1 on the boundary.We apply Corollary 17.14 [24], we obtain that f^{-1} has a limit at each boundary point of D'. □

Corollary 2.1. *Let $f : D \to D'$ a quasiconformal mapping and $D, D' \subset R^n, \infty \notin \partial D'$. If D is LLC and D' is SC then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$.*

Theorem 2.22. *Let $f : D \to D'$ a quasiconformal mapping and $D, D' \subset R^n, \infty \notin \partial D$. If D is WLLC and D' is QED, then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$.*

Proof. Since D is WLLC and $\infty \notin \partial D$, by Remark 1.1, we have that D is locally connected on the boundary. Since D' is QED, by Theorem 2.2, it follows that D' has the property P_2^* at each boundary point. Using Remark 2 [1], we obtain that f has a limit at each boundary point of D. Since D' is QED, by Theorem 2.1, we get D' has property P_1 on the boundary.By the first part of proof, D is locally connected on the boundary and hence D is finitely connected on the boundary. We apply Corollary 17.14 [24], we get that f^{-1} has a limit at each boundary point of D'. □

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