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NON-DEGENERATE REAL HYPERSURFACES OF A PARAQUATERNIONIC KÄHLER MANIFOLD

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Abstract. We obtain necessary and sufficient conditions for the integrability of the distributions on a non-degenerate real hypersurface of a paraquaternionic Kähler manifold. If these distributions are integrable, we show that the foliations determined by them are totally geodesic.

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Introduction. The paraquaternionic Kähler manifolds are semi-Riemannian manifolds endowed with two local almost product structures and one local almost complex structure satisfying some compatibility conditions (cf. Garcia-Rio et al [3]). The concept was defined having in mind the quaternionic Kähler manifolds which are equipped with three local almost complex structures (cf. Ishihara [4]).

In the present paper we study the geometry of non-degenerate real hypersurfaces of a paraquaterionic Kähler manifold. For the case of real hypersurfaces of a quaternionic Kähler manifold, such a study was done by Bejancu [1]. In the first section we define the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on a non-degenerate real hypersurface N of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$ and present their main properties. Then in Sect. 2, by using the second fundamental form of N we obtain necessary and sufficient conditions for the integrability of both distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$. Finally, in the last section, we show that in case these distributions are integrable, the foliations determined by them are totally geodesic. Also we show that the leaves of $\mathcal{D}$ are totally geodesic immersed in M and find a necessary and sufficient condition for leaves of $\mathcal{D}^{\perp}$ to be totally geodesic immersed in M.

1. Preliminaries. Throughout the paper all manifolds are smooth and paracompact. If M is a smooth manifold then we denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $F(M)$-module of smooth sections of the tangent bundle TM of M. We use these notations for any other manifold or vector bundle. If not stated otherwise, we use indices: $a, b, c, ... \in \{1, 2, 3\}$.

Let M be a manifold endowed with a *paraquaternionic structure* $\mathbf{V}$, that is, $\mathbf{V}$ is a rank-3 subbundle of $\mathrm{End}(TM)$ which has a local basis $\{J_1, J_2, J_3\}$ on a coordinate neighbourhood $\mathcal{U} \subset M$ satisfying (see Garcia-Rio et al [3])

$$\begin{aligned} &\text{(a)}\quad J_a^2 = \lambda_a I,\ a \in \{1,2,3\}, \\ &\text{(b)}\quad J_1 J_2 = -J_2 J_1 = J_3, \quad \text{(c)}\ \lambda_1 = \lambda_2 = -\lambda_3 = 1. \end{aligned} \tag{1.1}$$

A semi-Riemannian metric g on M is said to be *adapted* to the paraquaternionic structure $\mathbf{V}$ if it satisfies

$$g(X, Y) + \lambda_a g(J_a X, J_a Y) = 0,\ \forall a \in \{1, 2, 3\}, \tag{1.2}$$

for any $X, Y \in \Gamma(TM)$, and any local basis $\{J_1, J_2, J_3\}$ of $\mathbf{V}$. As a consequence of (1.1) and (1.2) we obtain

$$g(J_a X, Y) + g(X, J_a Y) = 0,\ \forall X, Y \in \Gamma(TM),\ a \in \{1, 2, 3\}. \tag{1.3}$$

If $\{\widetilde{J}_1, \widetilde{J}_2, \widetilde{J}_3\}$ is a local basis of $\mathbf{V}$ on $\widetilde{\mathcal{U}} \subset M$ and $\mathcal{U} \cap \widetilde{\mathcal{U}} \neq \emptyset$, then we have

$$\widetilde{J}_a = \sum_{b=1}^{3} A_{ab} J_b, \tag{1.4}$$

where the 3×3 matrix $[A_{ab}]$ is an element of the pseudo-orthogonal group $SO(2, 1)$. The above conditions impose special dimensions for M and special signatures for g. We take M of dimension $4(m+1)$ and g of neutral signature $(2(m + 1), 2(m + 1))$.

The triple $(M, \mathbf{V}, g)$ is called a *paraquaternionic Kähler manifold* if $\mathbf{V}$ is a parallel bundle with respect to Levi-Civita connection $\widetilde{\nabla}$ of g. Then taking into account that $\widetilde{\nabla}$ is a metric connection and by using (1.1) and (1.2) it is deduced that $(M, \mathbf{V}, g)$ is a paraquaternionic Kähler manifold if and only if for any local basis $\{J_1, J_2, J_3\}$ of $\mathbf{V}$ on $\mathcal{U} \subset M$ there exist the

1-forms $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ on $\mathcal{U}$ such that (cf. Garcia-Rio et al [3])

$$(1.5)\qquad \begin{aligned} &\text{(a)} \quad \left(\widetilde{\nabla}_X J_1\right) Y = \mathbf{q}(X) J_2 Y - \mathbf{r}(X) J_3 Y, \\ &\text{(b)} \quad \left(\widetilde{\nabla}_X J_2\right) Y = -\mathbf{q}(X) J_1 Y + \mathbf{p}(X) J_3 Y, \\ &\text{(c)} \quad \left(\nabla_X J_3\right) Y = -\mathbf{r}(X) J_1 Y + \mathbf{p}(X) J_2 Y, \ \forall X, Y \in \Gamma(T\mathcal{U}). \end{aligned}$$

Now, let N be a non-degenerate real hypersurface of of $(M, \mathbf{V}, g)$. This means that g induces a semi-Riemannian metric on N, which we denote by the same symbol g. Suppose that the unit normal vector field ξ to N on a coordinate neighhbourhood $\mathcal{V} = \mathcal{U} \cap N$ is space-like. Similar results are obtained when ξ is time-like.

Then we define on $\mathcal{V}$ the vector fields $\xi_a = J_a \xi$, $a \in \{1, 2, 3\}$.

From (1.2) we deduce that

$$(1.6)\qquad \text{(a) } g(\xi_1, \xi_1) = g(\xi_2, \xi_2) = -1, \ \text{(b) } g(\xi_3, \xi_3) = 1,$$

that is, ξ_1 and ξ_2 are unit time-like vector fields, while ξ_3 is a unit space-like vector field. In a similar way we define $\widetilde{\xi}_a = \widetilde{J}_a \xi$ on $\widetilde{\mathcal{V}} = \widetilde{\mathcal{U}} \cap N$. Suppose $\mathcal{V} \cap \widetilde{\mathcal{V}} \neq \emptyset$ and by (1.4) we obtain

$$(1.7)\qquad \widetilde{\xi}_a = \sum_{b=1}^{3} A_{ab} \xi_b, \quad \text{on } \mathcal{V} \cap \widetilde{\mathcal{V}}, \ a \in \{1, 2, 3\}.$$

Thus from (1.7) we deduce that there exists on N a globally defined distribution $\mathcal{D}^{\perp}$ which is locally represented on $\mathcal{V}$ by the orthonormal frame field $\{\xi_1, \xi_2, \xi_3\}$. Next, we denote by $\mathcal{D}$ the orthogonal complementary distribution to $\mathcal{D}^{\perp}$ in the tangent bundle TN. By using (1.1) and (1.2) we deduce that $\mathcal{D}$ is invariant by J_a, that is, we have

$$(1.8)\qquad J_a(\mathcal{D}_x) = \mathcal{D}_x, \quad \forall x \in N, \ a \in \{1, 2, 3\}.$$

We call $\mathcal{D}$ the *paraquaternionic distribution* on N. Note that $\mathcal{D}$ is of rank $4m$ and g has the signature $(2m, 2m)$ on $\mathcal{D}$. On the other hand, $\mathcal{D}^{\perp}$ is not invariant by J_a. More precisely, we have

$$(1.9)\qquad \begin{aligned} &\text{(a) } J_a \xi_a = \lambda_a \xi, && \text{(b) } J_1 \xi_2 = -J_2 \xi_1 = \xi_3, \\ &\text{(c) } J_1 \xi_3 = -J_3 \xi_1 = \xi_2, && \text{(d) } J_3 \xi_2 = -J_2 \xi_3 = \xi_1. \end{aligned}$$

2. Integrability of distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$. Let N be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$. Denote by ∇ the Levi-Civita connection determined by g on N. Then the equations of Gauss and Weingarten are expressed as follows:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi, \tag{2.1}$$

and

$$\widetilde{\nabla}_X \xi = -AX, \tag{2.2}$$

for any $X, Y \in \Gamma(TN)$, where h is the second fundamental form of N and A is the shape operator of N. Then we have

$$h(X,Y) = g(AX,Y), \quad \forall X, Y \in \Gamma(TN). \tag{2.3}$$

Now, denote by P the projection morphism of TN to $\mathcal{D}$ with respect to the decomposition

$$TN = \mathcal{D} \oplus \mathcal{D}^{\perp}. \tag{2.4}$$

Then any vector field Y on N can be expressed as follows

$$Y = PY + \sum_{a=1}^{3} \eta_a(Y)\xi_a, \tag{2.5}$$

where $\{\eta_a\}$ are 1-forms locally defined on N by

$$\eta_a(Y) = -\lambda_a g(Y, \xi_a), \quad a \in \{1,2,3\}. \tag{2.6}$$

We apply J_1, J_2, J_3 to (2.5) and by using (1.9) obtain

$$\begin{aligned} &\text{(a)} \quad J_1Y = J_1PY + \eta_1(Y)\xi + \eta_2(Y)\xi_3 + \eta_3(Y)\xi_2, \\ &\text{(b)} \quad J_2Y = J_2PY - \eta_1(Y)\xi_3 + \eta_2(Y)\xi - \eta_3(Y)\xi_1, \\ &\text{(c)} \quad J_3Y = J_3PY - \eta_1(Y)\xi_2 + \eta_2(Y)\xi_1 + \eta_3(Y)\xi. \end{aligned} \tag{2.7}$$

Lemma 2.1. *Let N be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$. Then we have:*

$$\begin{aligned} &\text{(a)} \ \nabla_X J_1Y - J_1P\nabla_X Y = \mathbf{q}(X)J_2Y - \mathbf{r}(X)J_3Y + h(X,Y)\xi_1 \\ &\qquad\qquad + \eta_3(\nabla_X Y)\xi_2 + \eta_2(\nabla_X Y)\xi_3, \\ &\text{(b)} \ \nabla_X J_2Y - J_2P\nabla_X Y = -\mathbf{q}(X)J_1Y + \mathbf{p}(X)J_3Y \\ &\qquad\qquad - \eta_3(\nabla_X Y)\xi_1 - \eta_1(\nabla_X Y)\xi_3 + h(X,Y)\xi_2, \\ &\text{(c)} \ \nabla_X J_3Y - J_3P\nabla_X Y = -\mathbf{r}(X)J_1Y + \mathbf{p}(X)J_2Y \\ &\qquad\qquad + \eta_2(\nabla_X Y)\xi_1 - \eta_1(\nabla_X Y)\xi_2 + h(X,Y)\xi_3, \end{aligned} \tag{2.8}$$

and

$$h(X, J_a Y) = \lambda_a \eta_a(\nabla_X Y), \quad \forall a \in \{1,2,3\}, \tag{2.9}$$

for any $X, Y \in \Gamma(\mathcal{D})$.

Proof. Take $X, Y \in \Gamma(\mathcal{D})$ in (1.5a) and by using (2.1) and (2.7a) we obtain

$$\begin{aligned} &\{\nabla_X J_1 Y - J_1 P \nabla_X Y - h(X,Y)\xi_1 - \eta_3(\nabla_X Y)\xi_2 - \eta_2(\nabla_X Y)\xi_3 \\ &-\mathbf{q}(X) J_2 Y + \mathbf{r}(X) J_3 Y\} + \{h(X, J_1 Y) - \eta_1(\nabla_X Y)\}\, \xi = 0. \end{aligned} \tag{2.10}$$

Then (2.8a) and (2.9) (for $a = 1$) are obtained by taking in (2.10) the tangent and normal parts, respectively. All other equalities in (2.8) and (2.9) are deduced using similar calculations. □

Now, we introduce two particular classes of hypersurfaces of $(M, \mathbf{V}, g)$. We say that N is $\mathcal{D}$-*geodesic* (resp. $\mathcal{D}^\perp$-*geodesic*) if any geodesic of N that is tangent to $\mathcal{D}$ (resp. $\mathcal{D}^\perp$) is a geodesic of M. Then, by using (2.1) we obtain the following.

Lemma 2.2. *Let N be a non-degenerate real hypersurface of the para-quaternionic Kähler manifold $(M, \mathbf{V}, g)$. Then we have the assertions:*

(a) *N is $\mathcal{D}$-geodesic if and only if*

$$h(X, Y) = 0, \ \forall X, Y \in \Gamma(\mathcal{D}). \tag{2.11}$$

(b) *N is $\mathcal{D}^\perp$-geodesic if and only if*

$$h(\xi_a, \xi_b) = 0, \ \forall a, b \in \{1,2,3\}. \tag{2.12}$$

Next, we prove the following.

Lemma 2.3. *Let N be a non-degenerate real hypersurface of a para-quaternionic Kähler manifold $(M, \mathbf{V}, g)$. Then N is $\mathcal{D}$-geodesic if and only if its second fundamental form satisfies*

$$h(X, J_a Y) = h(Y, J_a X), \quad \forall a \in \{1,2,3\}, \ X, Y \in \Gamma(\mathcal{D}). \tag{2.13}$$

Proof. Suppose N is $\mathcal{D}$-geodesic. Then from (2.11) we obtain (2.13) since $\mathcal{D}$ is invariant by J_a for any $a \in \{1,2,3\}$. Conversely, suppose (2.13) is satisfied and by using (1.16) we deduce that

$$\begin{aligned} h(J_3X,Y) &= h(X,J_3Y) = h(X,J_1J_2Y) \\ &= h(J_1X,J_2Y) = h(J_2J_1X,Y) = -h(J_3X,Y), \end{aligned}$$

for any $X,Y \in \Gamma(\mathcal{D})$. Thus $h(J_3X,Y) = 0$ for any $X,Y \in \Gamma(\mathcal{D})$, which yields (2.11) since J_3 is an automorphism of the paraquaternionic distribution. □

Theorem 2.1. *Let N be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M,\mathbf{V},g)$. Then the paraquaternionic distribution $\mathcal{D}$ is integrable if and only if N is $\mathcal{D}$-geodesic.*

Proof. First, by using (2.6) into (2.9) we obtain

$$h(X,J_aY) = -g(\nabla_X Y,\xi_a), \quad \forall a \in \{1,2,3\}, X,Y \in \Gamma(\mathcal{D}).$$

Then, taking into account that ∇ is torsion-free we obtain

$$h(X,J_aY) - h(Y,J_aX) = g([Y,X],\xi_a), \tag{2.14}$$

for any $a \in \{1,2,3\}$ and $X,Y \in \Gamma(\mathcal{D})$. Thus the assertion of the theorem follows from (2.14), by using Lemma 2.3 and taking into account that $\{\xi_1,\xi_2,\xi_3\}$ is an orthonormal basis in $\Gamma(\mathcal{D}^\perp)$. □

Next, we say that N is $(\mathcal{D},\mathcal{D}^\perp)$-*geodesic* if its second fundamental form satisfies

$$h(X,\xi_a) = 0, \quad \forall X \in \Gamma(\mathcal{D}),\ a \in \{1,2,3\}. \tag{2.15}$$

Then we prove the following.

Theorem 2.2. *Let N be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M,\mathbf{V},g)$. Then the distribution $\mathcal{D}^\perp$ is integrable if and only if N is $(\mathcal{D},\mathcal{D}^\perp)$-geodesic.*

Proof. First, by using (1.9), (1.5c), (1.2) and (2.1) we obtain

$$\begin{aligned} g\left([\xi_1,\xi_2],J_3X\right) &= g\left(\widetilde{\nabla}_{\xi_1}\xi_2 - \widetilde{\nabla}_{\xi_2}\xi_1, J_3X\right) \\ &= -g\left(\widetilde{\nabla}_{\xi_1}J_3\xi_1 + \widetilde{\nabla}_{\xi_2}J_3\xi_2, J_3X\right) \\ &= g\left(\nabla_{\xi_1}\xi_1, X\right) + g\left(\nabla_{\xi_2}\xi_2, X\right), \end{aligned} \tag{2.16}$$

for any $X \in \Gamma(\mathcal{D})$. By similar calculations we deduce that

$$g\left([\xi_2, \xi_3], J_1 X\right) = g\left(\nabla_{\xi_3}\xi_3, X\right) - g\left(\nabla_{\xi_2}\xi_2, X\right), \tag{2.17}$$

and

$$g\left([\xi_3, \xi_1], J_2 X\right) = g\left(\nabla_{\xi_3}\xi_3, X\right) - g\left(\nabla_{\xi_1}\xi_1, X\right), \tag{2.18}$$

for any $X \in \Gamma(\mathcal{D})$. Thus, from (2.16), (2.17) and (2.18) we infer that $\mathcal{D}^{\perp}$ is integrable if and only if

$$\begin{aligned} &g\left(\nabla_{\xi_1}\xi_1, X\right) + g\left(\nabla_{\xi_2}\xi_2, X\right) = 0 \\ &g\left(\nabla_{\xi_3}\xi_3, X\right) - g\left(\nabla_{\xi_2}\xi_2, X\right) = 0, \ \forall X \in \Gamma(\mathcal{D}), \\ &g\left(\nabla_{\xi_3}\xi_3, X\right) - g\left(\nabla_{\xi_1}\xi_1, X\right) = 0 \end{aligned}$$

which are equivalent to

$$g\left(\nabla_{\xi_a}\xi_a, X\right) = 0, \ \forall a \in \{1, 2, 3\}, \ X \in \Gamma(\mathcal{D}). \tag{2.19}$$

On the other hand, by using (2.1), (1.2), (1.5), (1.9a), (2.2) and (2.3) we obtain

$$\begin{aligned} g\left(\nabla_{\xi_a}\xi_a, X\right) &= g\left(\widetilde{\nabla}_{\xi_a}\xi_a, X\right) = -\lambda_a g\left(J_a \widetilde{\nabla}_{\xi_a}\xi_a, J_a X\right) \\ &= -\lambda_a g\left(\lambda_a \widetilde{\nabla}_{\xi_a}\xi, J_a X\right) = g(A\xi_a, J_a X) = h(\xi_a, J_a X). \end{aligned} \tag{2.20}$$

Finally, taking into account that J_a are automorphisms of $\mathcal{D}$ for any $a \in \{1, 2, 3\}$ and using (2.19) and (2.20) we deduce that $\mathcal{D}^{\perp}$ is integrable if and only if (2.15) is satisfied. This completes the proof of the theorem. □

3. Foliations on a real hypersurface of $(M, \mathbf{V}, g)$. Let N be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$. In this section we study the geometry of leaves of the foliations on N provided $\mathcal{D}$ and/or $\mathcal{D}^{\perp}$ are integrable. First, we recall that a submanifold S of a Riemannian manifold Q is *totally geodesic* if its second fundamental form vanishes identically on S (cf. Chen [2]). Then we prove the following.

Theorem 3.1. *Let N be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$, whose paraquaternionic distribution $\mathcal{D}$ is integrable. Then any leaf of $\mathcal{D}$ is totally geodesic immersed in M.*

Proof. Let N^* be a leaf of $\mathcal{D}$ and h^* be the second fundamental form of the immersion of N^* in N. Then by Gauss equation we have

$$\nabla_X Y = \nabla^*_X Y + h^*(X, Y),\ \forall X, Y \in \Gamma(TN^*), \tag{3.1}$$

where ∇^* is the Levi-Civita connection on N^*. Then by using (2.1) and (3.1) we deduce that

$$\widetilde{\nabla}_X Y = \nabla_X Y = \nabla^*_X Y + h^*(X, Y),\ \forall X, Y \in \Gamma(TN^*), \tag{3.2}$$

since N is $\mathcal{D}$-geodesic (cf. Theorem 2.1). Thus by (3.2) h^* is also the second fundamental form of the immersion of N^* in M. Next, by using (1.5a), (2.7a) and (3.2) we obtain

$$\begin{aligned} \nabla^*_X J_1 Y + h^*(X, J_1 Y) &= J_1(\nabla^*_X Y) + \eta_1(\nabla_X Y)\xi \\ +\eta_2(\nabla_X Y)\xi_3 + \eta_3(\nabla_X Y)\xi_2 &+ \mathbf{q}(X) J_2 Y - \mathbf{r}(X) J_3 Y, \end{aligned} \tag{3.3}$$

for any $X, Y \in \Gamma(TN^*)$. Taking into account that N is $\mathcal{D}$-geodesic, from (2.9) we infer that

$$\eta_a(\nabla_X Y) = 0,\quad \forall a \in \{1, 2, 3\},\ X, Y \in \Gamma(TN^*).$$

Thus taking the vector fields from (3.3) that are normal to N^* in M we obtain

$$h^*(X, J_1 Y) = 0,\ \forall X, Y \in \Gamma(TN^*).$$

This completes the proof since J_1 is an automorphism of $\Gamma(TN^*)$. □

Now we recall that a foliation $\mathcal{F}$ on N is *totally geodesic* if any leaf of $\mathcal{F}$ is a totally geodesic submanifold of N (cf. Reinahrt [5]).

Corollary 3.1. *Let N be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$, whose paraquaternionic distribution $\mathcal{D}$ is integrable. Then the foliation determined by $\mathcal{D}$ on N is totally geodesic.*

Proof. Let N^* be a leaf of $\mathcal{D}$. Then by Theorem 3.1 we have $h^* = 0$ on N. Thus by (3.1) we deduce that N^* is totally geodesic immersed in N. Hence the foliation determined by $\mathcal{D}$ is totally geodesic. □

Next, we study the geometry of leaves of $\mathcal{D}^\perp$. First, we prove the following.

Theorem 3.2. *Let N be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$, whose distribution $\mathcal{D}^\perp$ is integrable. Then the foliation determined by $\mathcal{D}^\perp$ on N is totally geodesic.*

Proof. Let N' be a leaf of $\mathcal{D}^\perp$ and $h^\perp$ be the second fundamental form of the immersion of N' in N. Then by using the Gauss equation for the submanifold N' of N, (1.5) and (2.1)-(2.3) we obtain

$$\begin{aligned} g\left(h^\perp(X,\xi_a),Y\right) &= g\left(\nabla_X\xi_a,Y\right) = g\left(J_a\widetilde{\nabla}_X\xi,Y\right) \\ &= -g\left(\widetilde{\nabla}_X\xi,J_aY\right) = g(AX,J_aY) = h(X,J_aY), \end{aligned} \tag{3.4}$$

for any $X \in \Gamma(TN')$, $Y \in \Gamma(\mathcal{D})$ and $a \in \{1,2,3\}$. By Theorem 2.2 N is $(\mathcal{D},\mathcal{D}^\perp)$-geodesic and thus (3.4) implies

$$h^\perp(X,Z) = 0, \quad \forall X, Z \in \Gamma(TN'), \tag{3.5}$$

which completes the proof. □

From Corollary 3.1 and Theorem 3.2 we deduce the following corollary.

Corollary 3.2. *Let N be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$. If both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are integrable then N is locally a semi-Riemannian product $N^* \times N'$ where N^* and N' are leaves of $\mathcal{D}$ and $\mathcal{D}^\perp$ respectively.*

Finally, we state a necessary and sufficient condition for leaves of $\mathcal{D}^\perp$ to be totally geodesic immersed in M.

Theorem 3.3. *Let N be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$, whose distribution $\mathcal{D}^\perp$ is integrable. Then any leaf of $\mathcal{D}^\perp$ is totally geodesic immersed in M if and only if N is $\mathcal{D}^\perp$-geodesic.*

Proof. By using Gauss equation for both the immersion of N in M and the immersion of a let N' of $\mathcal{D}^\perp$ in N and (3.5) we obtain

$$g\left(\widetilde{\nabla}_X Z, Y\right) = 0, \quad \forall X, Z \in \Gamma(TN'),\ Y \in \Gamma(\mathcal{D}). \tag{3.6}$$

On the other hand, by using (2.2) and (2.3) we get

$$g\left(\widetilde{\nabla}_X Z, \xi\right) = h(X,Z), \quad \forall X, Z \in \Gamma(TN'). \tag{3.7}$$

Now, we denote by $\widetilde{h}$ the second fundamental form of the immersion of N' in M. Then by using Gauss equation in (3.6) and (3.7) we deduce that

$$g(\widetilde{h}(X,Z),Y)=0,\quad \forall X,Z\in\Gamma(TN'),\ Y\in\Gamma(\mathcal{D}), \tag{3.8}$$

and

$$g(\widetilde{h}(X,Z),\xi)=h(X,Z),\quad \forall X,Z\in\Gamma(TN'), \tag{3.9}$$

respectively. Thus the assertion of the theorem follows by using (3.8), (3.9) and (2.12). $\square$

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