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STRUCTURAL INVARIANTS FOR MODULAR ABELIAN GROUP ALGEBRAS

BY

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Abstract. Let G be an abelian group with a subgroup A and p a prime integer. We prove here that the vector spaces $((A \cap G^{p^\alpha})G^{p^{\alpha+1}})[p]/G^{p^{\alpha+1}}[p]$ and $(A \cap G^{p^\alpha})G^{p^{\alpha+1}}G_p^{p^\alpha}/G^{p^{\alpha+1}}G_p^{p^\alpha}$, which are closely related as being dual to the Ulm p-invariants and Warfield p-invariants, respectively, can be determined for each ordinal α from the pair of R-group algebras (RG, RA), whenever R is an unital commutative ring of prime characteritic p without nilpotent elements. This continues our recent results in (Math. Moravica, 2006).

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1. Introduction. Suppose, everywhere in the text, that RG is the group algebra of an arbitrary abelian group G, written via multiplicative record as is customary when discussing group algebras, over a commutative ring R with identity of prime characteristic, for instance, p. As usual, for a subgroup A of G, the letter $I(RG; A)$ is reserved for the augmentation ideal of RG with respect to A.

In [1]-[5], we have obtained some functional invariants for RG and the pair of group algebras (RG, RA) over R where $A \leq G$. The most important of them, closely related to the present paper, are as follows: In [1], we have proved that the vector space $(AG^{p^{\alpha+1}}) \cap G^{p^\alpha}[p]/(A \cap G^{p^\alpha}[p])G^{p^{\alpha+1}}[p] \cong (AG^{p^\alpha}[p]) \cap G^{p^{\alpha+1}}/(A \cap G^{p^{\alpha+1}})G^{p^{\alpha+1}}[p] \cong (G^{p^\alpha}[p]G^{p^{\alpha+1}}) \cap A/(A \cap G^{p^\alpha}[p])(A \cap G^{p^{\alpha+1}})$, called the α-th *p-overhang* of A in G, can be retrieved by (RG, RA)

when R is a field. In [2], we have also established that the relative Ulm (-Kaplansky) p-invariants

$$\begin{aligned} U_\alpha(G,A) &= G^{p^\alpha}[p]/(G^{p^{\alpha+1}}(A\cap G^{p^\alpha}))[p] = G^{p^\alpha}[p]/(G^{p^{\alpha+1}}A)\cap G^{p^\alpha}[p] \\ &\cong G^{p^\alpha}[p]G^{p^{\alpha+1}}A/G^{p^{\alpha+1}}A \end{aligned}$$

may be recaptured from (RG,RA) whenever R is a field. In particular, the ordinary Ulm p-invariants $U_\alpha(G)=G^{p^\alpha}[p]/G^{p^{\alpha+1}}[p]$ are deducible from RG.

In [4] and [5], we have shown that the relative Warfield p-invariants $W_\alpha(G,A)=G^{p^\alpha}/([(G^{p^{\alpha+1}}A)\cap G^{p^\alpha}]G_p^{p^\alpha}) = G^{p^\alpha}/G^{p^{\alpha+1}}G_p^{p^\alpha}(A\cap G^{p^\alpha})$ of G with respect to A can be recovered by (RG,RA). In particular, the ordinary Warfield p-invariants $W_\alpha(G)=G^{p^\alpha}/(G^{p^{\alpha+1}}G_p^{p^\alpha})$ are determined from RG.

The aim of this article to demonstrate that some new invariants of (FG,FA), which are respectively somewhat dual to the alluded to above $U_\alpha(G,A)$ and $W_\alpha(G,A)$, exist.

All notions and notation are the same as in [6].

2. Preliminary technicalities. Before proceed by proving the main result, we need the help of two technical assertions (see [1]-[5] too).

Lemma. *Assume that $B\le G$, $C\le G$ and that P is any commutative ring with identity. Then*

$$I(PG;BC)=I(PG;B)+I(PG;C).$$

Proof. The left hand-side obviously contains the right hand-side.

About the other inclusion, given $x\in I(PG;BC)$. Hence,

$$\begin{aligned} x &= \sum_{g\in G} u_g(1-b_gc_g) \\ &= \sum_{g\in G} u_gc_g(1-b_g)+\sum_{g\in G} u_g(1-c_g)\in I(PG;B)+I(PG;C), \end{aligned}$$

where $u_g\in PG$, $b_g\in B$ and $c_g\in C$. Thus $I(PG;BC)$ is contained in the last sum, as wanted. The proof is ended. □

Proposition. *Assume that $B\le C\le G$, $A\le G$ and $K\le P$ contains the same identity as the commutative ring P. Then*

$$I(PC;B)\cap KA=I(K(A\cap C);A\cap B).$$

Proof. Take x to belong in the intersection from the left hand-side. Consequently, $x = \sum_{a_c \in A\cap C} f_{a_c} a_c$, where $f_{a_c} \in K$ and $\sum_{a_c \in a'_c B} f_{a_c} = 0$ for every $a'_c \in A \cap C$. We therefore observe that $\sum_{a_c \in (a'_c B)\cap A \cap C} f_{a_c} = \sum_{a_c \in a'_c (A\cap B)} f_{a_c} = 0$ since $(a'_c B) \cap A \cap C = a'_c(A \cap B)$ whenever $a'_c \in A \cap C$. Finally, it is a straightforward argument that $x \in I(K(A \cap C); A \cap B)$ and thereby we are done.

The converse inclusion is trivial. The proof is completed. □

The following key affirmation due to Karpilovsky (see, for more details, [1] and [3]).

Theorem (Karpilovsky, 1982). *Let $B \leq C \leq G$ and $C/B \cong \coprod_\lambda \langle p \rangle$, where λ is a cardinal number and $\langle p \rangle$ is a cyclic group of order p. Then*

$$\lambda = dim_R(I(RG; C)/(I(RG; G)I(RG; C) + I(RG; B))).$$

For facilitating of the exposition, we define the following two vector spaces. Firstly, following [7] or [8], we state

$$I_\alpha(G, A) = ((A \cap G^{p^\alpha})G^{p^{\alpha+1}})[p]/G^{p^{\alpha+1}}[p] = (G^{p^\alpha}[p] \cap (AG^{p^{\alpha+1}}))/G^{p^{\alpha+1}}[p].$$

This is a relative invariant of the pair (G, A) i.e., in other words, of G with respect to A. Clearly, $I_\alpha(G, A)$ is a subspace of $U_\alpha(G)$, that is $I_\alpha(G, A) \leq U_\alpha(G)$. An interesting relationship is the following

$$U_\alpha(G)/I_\alpha(G, A) \cong U_\alpha(G, A).$$

Secondly, we put by analogy

$$J_\alpha(G, A) = (A \cap G^{p^\alpha})G^{p^{\alpha+1}}G_p^{p^\alpha}/G^{p^{\alpha+1}}G_p^{p^\alpha} \cong A \cap G^{p^\alpha}/A \cap (G^{p^{\alpha+1}}G_p^{p^\alpha}).$$

This is a relative invariant of (G, A). Evidently, $J_\alpha(G, A) \leq W_\alpha(G)$. A relation between them is the following one

$$W_\alpha(G)/J_\alpha(G, A) \cong W_\alpha(G, A).$$

3. Main result. Now, we are ready to show the validity of the next statement.

Theorem (Invariants). *Let G be an abelian group and R a commutative unital ring of prime characteristic p. Then the vector spaces $I_\alpha(G, A)$ and $J_\alpha(G, A)$ may be deduced from (RG, RA).*

Proof. In order to verify this, it suffices to illustrate that their dimension is an invariant for (RG, RA). Toward this end, appealing to the Karpilovsky's theorem, it is enough to check that the ideals $I(RG; G^{p^\alpha}[p] \cap (AG^{p^{\alpha+1}}))$ and $I(RG; G^{p^{\alpha+1}}[p])$ as well as $I(RG; A \cap G^{p^\alpha})$ and $I(RG; A \cap (G^{p^{\alpha+1}} G_p^{p^\alpha}))$ can be gotten by (FG, FA).

In fact, for the ideal $I(RG; A \cap G^{p^\alpha})$, we observe in virtue of the Proposition that

$$\begin{aligned} I(RG; A \cap G^{p^\alpha}) &= RG.I(RA; A \cap G^{p^\alpha}) \\ &= RG.(I(RG; G^{p^\alpha}) \cap RA) \\ &= RG.((RG.I(R^{p^\alpha} G^{p^\alpha}; G^{p^\alpha})) \cap RA) \\ &= RG.((RG.I^{p^\alpha}(RG; G)) \cap RA) \end{aligned}$$

which depends only on RG and RA.

As for the ideal $I(RG; A \cap (G^{p^{\alpha+1}} G_p^{p^\alpha}))$, we apply the Lemma and the Proposition to infer that

$$\begin{aligned} I(RG; A \cap (G^{p^{\alpha+1}} G_p^{p^\alpha})) &= RG.I(RA; A \cap (G^{p^{\alpha+1}} G_p^{p^\alpha})) \\ &= RG.(I(RG; G^{p^{\alpha+1}} G_p^{p^\alpha}) \cap RA) \\ &= RG.((I(RG; G^{p^{\alpha+1}}) + I(RG; G_p^{p^\alpha})) \cap RA). \end{aligned}$$

That the ideal

$$I(RG; G^{p^{\alpha+1}}) = RG.I(R^{p^{\alpha+1}} G^{p^{\alpha+1}}; G^{p^{\alpha+1}}) = RG.I^{p^{\alpha+1}}(RG; G)$$

depends only on RG is now immediate.

About the obtaining the ideals $I(RG; G_p^{p^\alpha})$ and $I(RG; G^{p^{\alpha+1}}[p])$, we refer to [1] or [2].

What remains to show is the determining of the ideal $I(RG; G^{p^\alpha}[p] \cap (AG^{p^{\alpha+1}}))$ by (RG, RA), which follows like this. Employing the Lemma and the Proposition, we write

$$\begin{aligned} I(RG; G^{p^\alpha}[p] \cap (AG^{p^{\alpha+1}})) &= RG.I(R(AG^{p^{\alpha+1}}); (AG^{p^{\alpha+1}}) \cap G^{p^\alpha}[p]) \\ &= RG.(I(RG; G^{p^\alpha}[p]) \cap R(AG^{p^{\alpha+1}})). \end{aligned}$$

Owing as above to [1] or [2], we retrieve the ideal $I(RG; G^{p^\alpha}[p])$. Finally, the R-algebra $R(AG^{p^{\alpha+1}}) = RA.R^{p^{\alpha+1}} G^{p^{\alpha+1}} = RA.(RG)^{p^{\alpha+1}}$ being depending on (RG, RA) is completely obtained.

The proof is finished in all generality. □

REFERENCES

1. Danchev, P.V. – *Numerical invariants for commutative group algebras*, Ric. Mat. (2) 50(2001), 323–336.
2. Danchev, P.V. – *Invariants for group algebras of abelian groups*, Rend. Circ. Mat. Palermo (3) 51(2002), 391–402.
3. Danchev, P.V. – *Numerical invariants for modular group algebras of abelian groups*, Ric. Mat. (2) 52(2003), 197–202.
4. Danchev, P.V. – *Numerical invariants for group algebras of abelian groups*, Compt. rend. Acad. bulg. Sci. (11) 57(2004), 5–6.
5. Danchev, P.V. – *Cardinal invariants for commutative group algebras*, Math. Moravica 10(2006), 1–7.
6. Fuchs, L. – *Infinite Abelian Groups*, I and II, Mir, Moskva, 1974 and 1977 (in Russian).
7. Warfield Jr., R.B. – *A classification theorem for abelian p-groups,* Trans. Amer. Math. Soc. (1) 210(1975), 149–168.
8. Warfield Jr., R.B. – *Classification theory of abelian groups. I: Balanced projectives,* Trans. Amer. Math. Soc. (1) 222(1976), 33–63.

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