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A SASAKIAN STRUCTURE ON THE INDICATRIX BUNDLE OF A CARTAN SPACE

BY

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Abstract. A Cartan space is a pair (M, K), where M is a smooth manifold and $K : T^*M \to \mathbb{R}$ is a regular Hamiltonian which is positively homogeneous of degree 1 in momenta. There exists a nonlinear connection N canonically derived from K. Using it and the metric tensor of the Cartan space, an almost complex structure on T^*M is defined and, based on this, we construct a framed $f(3,1)-$structure on $T^*M \setminus 0 := T_0^*M$. Restricting this $f(3,1)-$structure to the indicatrix bundle $IK = \{(x,p) \in T_0^*M | K(x,p) = 1\}$ we obtain an almost contact structure. We show that this is in fact a contact Riemannian structure and then we find that it is normal if and only if the curvature tensor of the nonlinear connection N has a simple form. A geometrical meaning of this form is provided in the case when the metric tensor of (M, K) does not depend on momenta. The results may be thought as dual to those from Finsler spaces case found in [1].

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1. Introduction. Let M be a smooth i.e. C^∞ manifold and $\tau^* : T^*M \to M$ its cotangent bundle. If $(x^i), i, j, k, ... = 1, 2, ...n = \dim M$ are local coordinates on M, the induced local coordinates on T^*M will be denoted by (x^i, p_i), where x^i is identified with $x^i \circ \tau^*$ and (p_i) are the components of a covector from $T_x^*M, x(x^i)$, in the cobasis $(dx^i)_x$. We put $T_0^*M = T^*M \setminus \{(x,0), x \in M\}$ and so we get an open submanifold of T^*M. All geometric objects to be considered will live on this submanifold.

The coordinates $p_1, ..., p_n$ will be called momenta. The kernel of the differential $D\tau^* : TT^*M \to TM$ is a subbundle, called *vertical* and denoted by VT^*M, of TT^*M. The vertical distribution $V : u \in T^*M \to V_uT^*M$ is locally spanned by $\left(\dot{\partial}^i := \frac{\partial}{\partial p_i}\right)$, hence it is integrable.

On T_0^*M we have the Liouville vector field $C^* = p_i\dot{\partial}^i$, the Liouville 1-form $\omega = p_i dx^i$ and a canonical symplectic structure $\theta = d\omega = dp_i \wedge dx^i$.

A Cartan space is a pair (M, K), where $K : T_0^*M \to R_+ \setminus \{0\}$ is smooth, positively 1-homogeneous, that is $K(x, \lambda p) = \lambda K(x, p)$ for any $\lambda > 0$ and the matrix with the entries $g^{ij}(x, p) = \frac{1}{2}\dot{\partial}^i\dot{\partial}^j K^2$ is positive defined. The functions $(g^{ij}(x, p))$ behave like the components of a tensor field on M, that is they define a $d-$ tensor field on T_0^*M. This is called the metric tensor of the Cartan space (M, K). See [3] for details.

If one sets $p^i = \frac{1}{2}\dot{\partial}^i K^2$, then $g^{ij} = \dot{\partial}^j p^i$. From the homogeneity condition it follows

$$p^i = g^{ij}p_j, p_i = g_{ij}p^j, K^2 = g^{ij}p_i p_j = p_i p^i, C^{ijk}p_k = 0, \tag{1.1}$$

where $C^{ijk} := -\frac{1}{2}\dot{\partial}^i g^{jk}$.

A *nonlinear connection* on T_0^*M is a distribution $u \longrightarrow H_u T^*M, u \in T^*M$, called *horizontal*, which is supplementary to the vertical distribution. This is usually given by a basis $\delta_i = \partial_i + N_{ik}\dot{\partial}^k$, for some functions (N_{ik}) having a special behavior by a change of coordinates on T^*M.

It was proved by Miron, Ch.6 in [3], that any regular Hamiltonian induces a nonlinear connection on T^*M. In particular, the regular Hamiltonian K^2 induces a nonlinear connection on T_0^*M whose local coefficients $N_{ij}(x, p) = N_{ji}(x, p)$ are positively homogeneous of degree 1 in momenta. For details we refer to Ch. 6 in [3].

Thus K produces a decomposition

$$T_u T_0^*M = V_u T_0^*M \oplus H_u T_0^*M, \tag{1.2}$$

and we have a basis $(\delta_i, \dot{\partial}^i)$ adapted to it.

The dual basis is $(dx^i, \delta p_i = dp_i - N_{ij}(x, p)dx^j)$. The decomposition (1.2) provides an almost complex structure F on T_0^*M given in the adapted frame by

$$F(\delta_i) = -g_{ij}\dot{\partial}^j, \qquad F(\dot{\partial}^i) = g^{ik}\delta_k. \tag{1.3}$$

On using the matrix $(g^{ij}(x, p))$ one constructs a Riemannian structure of Sasaki type on T_0^*M:

$$G = g_{ij}dx^i \otimes dx^j + g^{ij}\delta p_i \otimes \delta p_j. \tag{1.4}$$

Here the matrix (g_{jk}) denotes the inverse of (g^{ij}). It defines also a $d-$tensor field on T_0^*M.

A direct calculation shows that

$$G(FX,FY)=G(X,Y),\qquad X,Y\in\chi(T_0^*M), \tag{1.5}$$

where $\chi(T_0^*M)$ denotes the $F(M)-$module of vector fields on T_0^*M.

A framed $f(3,1)-$structure of corank s on a manifold N is a triad $(f,(\xi_a),(\eta_a))$ for $a,b,c...=1,...,s$, where f is a tensor field of type (1,1), (ξ_a) are vector fields and (η_a) are 1-forms on N such that

$$\begin{aligned}&\eta^a(\xi_b)=\delta^a_b,\qquad f(\xi_a)=0,\eta^a\circ f=0,\\ &f^2=-I+\sum_a\eta^a\otimes\xi_a,\end{aligned} \tag{1.6}$$

where I denotes the Kronecker tensor field on N.

The term $f(3,1)-$structure was suggested by the identity $f^3+f=0$.

For an account of the structures of this type we refer to [2].

In this paper, Section 2, we show that if (M,K) is a Cartan space, then the manifold T_0^*M has a natural $f(3,1)-$structure of corank 2. It is in some sense compatible with G. In Section 3 we restrict this $f(3,1)-$structure to the indicatrix bundle $IK=\{(x,p)\in T_0^*M|K(x,p)=1\}$, thought of as a $(2n-1)-$dimensional submanifold of T_0^*M, and we obtain an almost contact structure on IK. We show that it is in fact a Riemannian contact structure and it is Sasakian if and only if the curvature tensor of the nonlinear connection N has a special form. A geometrical meaning of this special form is given for a particular form of K.

2. A framed $f(3,1)-$structure on T_0^*M when (M,K) is a Cartan space. Let us put $\xi_1=\frac{p^i}{K}\delta_i$, $\xi_2=\frac{p_i}{K}\dot\partial^i$. Thus we have two global vector fields on T_0^*M which are linearly independent. The first is horizontal and the second one is vertical.

From the definition of F it follows

Lemma 2.1. $F(\xi_1)=-\xi_2$, $F(\xi_2)=\xi_1$.

Let now be the 1-forms $\eta^1=\frac{1}{K}p_idx^i$ and $\eta^2=\frac{1}{K}p^i\delta p_i$. It easily follows that $\eta^a(\xi_b)=\delta^a_b$ as well as

Lemma 2.2. $\eta^1\circ F=\eta^2$, $\eta^2\circ F=-\eta^1$

Lemma 2.3. $\eta^1(X)=G(X,\xi_1)$, $\eta^2(X)=G(X,\xi_2)$, $X\in\chi(T_0^*M)$.

Now we define a tensor field of type (1,1) f on T_0^*M by

$$f(X) = F(X) + \eta^1(X)\xi_2 - \eta^2(X)\xi_1, X \in \chi(T_0^*M). \tag{2.1}$$

We have

Theorem 2.1. *For the triple* $(f, (\xi_a), (\eta^a)), a, b = 1, 2$ *the followings hold*

$$(i) \qquad \eta^a(\xi_b) = \delta_b^a, f(\xi_a) = 0, \eta^a \circ f = 0,$$

$$(ii) \qquad f^2 = -I + \eta^1 \otimes \xi_1 + \eta^2 \otimes \xi_2,$$

$$(iii) \qquad f \text{ is of rank } 2n-2 \text{ and } f^3 + f = 0.$$

Proof. (i) follows from the definition (2.1) of f and the Lemmas 2.1 and 2.2. Using again (2.1) and (i) one verifies (ii). From (ii) quickly follows that $f^3 + f = 0$. It remains to prove that f is of rank $2n-2$. By (i), the space spanned by ξ_1 and ξ_2 is in $\ker f$. We show that $\ker f$ reduces to this 2-dimensional space. Let be $X \in \ker f$ written in the form $X = X^i\delta_i + Y_i\dot{\partial}^i$. The condition $f(X) = 0$ gives $X^i = \frac{X^j p_j}{K^2} p^i, Y_i = \frac{Y_j p^j}{K^2} p_i$, hence $X = \frac{X^j p_j}{K}\xi_1 + \frac{Y_j p^j}{K}\xi_2$, q.e.d. □

Theorem 2.2. *The Riemannian metric* G *verifies*

$$G(fX, fY) = G(X, Y) - \eta^1(X)\eta^1(Y) - \eta^2(X)\eta^2(Y), X, Y \in \chi(T_0^*M). \tag{2.2}$$

Proof. Inserting f from (2.1) in $G(fX, fY)$ one obtains a sum of seven term (two cancel) which reduces to the right side of (2.2) on using the Lemmas 2.1, 2.2 and 2.3.

A different proof of the Theorem 2.2 can be obtained if one uses the expression of f in the adapted frame $(\delta_i, \dot{\partial}^i)$.

From (2.1) it follows

$$\begin{aligned} f(\delta_i) &= \left(-g_{ij} + \frac{1}{K^2} p_i p_j\right)\dot{\partial}^j, \\ f(\dot{\partial}^i) &= \left(g^{ij} - \frac{1}{K^2} p^i p^j\right)\delta_j. \end{aligned} \tag{2.3}$$

By (2.3) one finds

$$
\begin{aligned}
&G(f\delta_i, f\delta_j) = g_{ij} - \frac{1}{K^2}p_i p_j,\\
&G(f\delta_i, f\dot\partial^j) = 0,\\
&G(f\dot\partial^i, f\dot\partial^j) = g^{ij} - \frac{1}{K^2}p^i p^j.
\end{aligned}
\tag{2.4}
$$

From (2.4) it easily follows (2.2).

Let us put

$$\phi(X,Y) = G(fX,Y), \qquad X,Y \in \chi(T_0^*M). \tag{2.5}$$

Using the Theorems 2.1 and 2.2 one verifies that

$$\phi(Y,X) = -\phi(X,Y), \qquad X,Y \in \chi(T_0^*M). \tag{2.6}$$

Thus ϕ is a 2-form on T_0^*M. It is degenerate with null space $span(\xi_1, \xi_2)$.

Using (2.3) one gets

$$\phi(\delta_i,\delta_j) = 0, \phi(\delta_i,\dot\partial^j) = -\delta_i^j + \frac{1}{K^2}p_i p^j, \phi(\dot\partial^i,\dot\partial^j) = 0. \tag{2.7}$$

On the other hand, we have

$$d\eta^1(\delta_i,\delta_j) = \frac{1}{K}(\delta_i p_j - \delta_j p_i) = \frac{1}{K}(N_{ij} - N_{ji}) = 0$$

since in a Cartan space $\delta_i K = 0$ and $N_{ij} = N_{ji}$, cf. Ch. 6 in [3].

$$d\eta^1(\delta_i,\dot\partial^j) = -\dot\partial^j\left(\frac{p_i}{K}\right) = \frac{1}{K}\left(-\delta_i^j + \frac{1}{K^2}p_i p^j\right),$$

where the equality $p^i = K\dot\partial^i K$ was used.

$$d\eta(\dot\partial^i,\dot\partial^j) = 0.$$

Comparing ϕ and $d\eta^1$, it comes out that

$$\phi = K d\eta^1. \tag{2.8}$$

From (2.8) it follows

$$\phi = \eta^2 \wedge d\eta^1 = \frac{1}{K}\eta^2 \wedge \phi,$$

because of $dK = \eta^2$.

Thus ϕ is never closed. It defines an almost presimplectic structure on T_0^*M. □

3. An almost contact structure on the indicatrix bundle of a Cartan space (M, K)**.** The set $IK = \{(x, p) \in T_0^*M \mid K(x, p) = 1\}$ is a submanifold of dimension $2n-1$ of T_0^*M. It will be called the indicatrix bundle of the Cartan space (M, K).

Let be

$$\begin{cases} x^i = x^i(u^\alpha) \\ p_i = p_i(u^\alpha), \alpha = 1, 2, ..., 2n-1, \end{cases} \tag{3.1}$$

with $rank(\frac{\partial x^i}{\partial u^\alpha}, \frac{\partial p_i}{\partial u^\alpha}) = 2n-1$,a parametrization of this submanifold.

The local vector fields $\left(\frac{\partial}{\partial u^\alpha}\right)$ provides basis of the tangent space to IK. We have $\frac{\partial}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha}\partial_i + \frac{\partial p_i}{\partial u^\alpha}\dot{\partial}^i = \frac{\partial x^i}{\partial u^\alpha}\delta_i + \left(\frac{\partial p_j}{\partial u^\alpha} - N_{ij}\frac{\partial x^i}{\partial u^\alpha}\right)\dot{\partial}^j$.

Deriving the identity $K^2(x^i(u^\alpha), p_i(u^\alpha)) = 1$ with respect to (u^α) one gets $(\partial_i K^2)\frac{\partial x^i}{\partial u^\alpha} + (\dot{\partial}^i K^2)\frac{\partial p_i}{\partial u^\alpha} = 0$. But from $\delta_i K^2 = 0$, one obtains $\partial_i K^2 = -N_{ij}\dot{\partial}^j K^2$ and the previous equations reads $p^j\left(\frac{\partial p_j}{\partial u^\alpha} - N_{ji}\frac{\partial x^i}{\partial u^\alpha}\right) = 0$ on IK. On the other hand, $G\left(\frac{\partial}{\partial u^\alpha}, \xi_2\right) = G\left(\frac{\partial}{\partial u^\alpha}, \frac{1}{K}p_k\dot{\partial}^k\right) = \left(\frac{\partial p_j}{\partial u\alpha} - N_{ij}\frac{\partial x^i}{\partial u^\alpha}\right)g^{kj}\frac{1}{K}p_k$ $= \frac{1}{K}p^j\left(\frac{\partial p_j}{\partial u^\alpha} - N_{ji}\frac{\partial x^i}{\partial u^\alpha}\right)$.

Taking this equation on IK, it comes out that ξ_2 is normal to the submanifold IK. Recall that it is unitary with respect to G.

Now we restrict to IK all the geometrical objects introduced above on T_0^*M. We indicate this by lines over. Thus we have

- $\overline{\xi}_1 = \xi_1$ since ξ_1 is tangent to IK,
- $\overline{\eta}_2 = 0$ because of $\eta^2(X) = G(X, \xi_2) = 0$ for every vector field X that is tangent to IK,
- $\overline{G} = G|_{IK}$,
- $\overline{f} = \overline{F} + \overline{\eta}^1 \otimes \overline{\xi}_2$ is an endomorphism of the tangent bundle to IK because of $G(\overline{f}X, \xi_2) = G(\overline{F}X, \xi_2) + \eta^1(X)G(\xi_2, \xi_2) = \eta^2(\overline{F}X) + \eta^1(X) = 0$.

We put $\xi = \overline{\xi}_1$ and $\eta = \overline{\eta}_1$. By the Theorem 2.1 we get

Theorem 3.1. *The triple* $(\overline{f}, \xi, \eta)$ *is an almost contact structure on* IK*, that is*

$$\eta(\xi) = 1, \eta \circ \overline{f} = 0, \overline{f}(\xi) = 0, \tag{i}$$

$$(ii) \qquad \overline{f}^2 = -I + \eta \otimes \xi,$$

$$(iii) \qquad \overline{f}^3 + \overline{f} = 0 \text{ and rank } \overline{f} = 2n - 2 = (2n-1) - 1.$$

Then by the Theorem 2.2 we obtain

Theorem 3.2. *The Riemannian metric $\overline{G}$ verifies*

$$\overline{G}(\overline{f}X, \overline{f}Y) = \overline{G}(X,Y) - \eta(X)\eta(Y), \tag{3.1}$$

for every vector fields X, Y that are tangent to IK.

Concluding, $(\overline{f}, \xi, \eta, \overline{G})$ is an almost contact Riemannian structure on IK.

We have that $(\delta_i, \overline{f}\delta_j)$ with $j = 1, 2...n-1$ is a local frame on IK. Indeed, $\overline{G}(\delta_i, \xi_2) = 0$, $\overline{G}(\overline{f}\delta_j, \xi_2) = \eta(\overline{f}(\delta_j)) = 0$ and the local vector fields δ_i and $\overline{f}\delta_j = (-g_{jh} + p_j p_h)\dot{\partial}^h$, $j = 1, ..., n-1$ are linearly independent since rank $(-g_{jh} + p_j p_h) = n - 1$.

We put $\overline{f}\delta_j = \dot{\delta}_j$, $j = 1, ..., n-1$ and we shall use the local frame $(\delta_i, \dot{\delta}_j)$ in the calculation below.

Let $\Omega(X,Y) = \overline{Gf}X, Y)$ be 2-form usually associated to an almost contact structure. This is nothing but the 2-form ϕ restricted to IK. Then (2.8) reduces to

$$\Omega = d\eta. \tag{3.2}$$

A direct calculation gives

$$\begin{aligned} &d\eta(\delta_i, \delta_k) = \Omega(\delta_i, \delta_k) = 0 \\ &d\eta(\delta_i, \dot{\delta}_j) = \Omega(\delta_i, \dot{\delta}_j) = g_{ij} - p_i p_j, j = 1, 2, ..., n-1 \\ &d\eta(\dot{\delta}_j, \dot{\delta}_h) = \Omega(\dot{\delta}_j, \dot{\delta}_h) = 0 \end{aligned} \tag{3.3}$$

This confirms again (3.2) in the frame $(\delta_i, \dot{\delta}_j)$, $j = 1, ..., n-1$. An almost contact Riemannian structure $(\overline{f}, \xi, \eta, G)$ with $\Omega = d\eta$ is called a contact Riemannian structure. Thus we have

Theorem 3.3. *Let $K^n = (M, K)$ be a Cartan space. Then the structure $(\overline{f}, \xi, \eta, G)$ is a contact Riemannian structure on IK.*

The almost contact structure $(\overline{f}, \xi, \eta)$ is called **normal** if the tensor field $N = N_{\overline{f}} + d\eta \otimes \xi = 0$, where $N_{\overline{f}}$ is the Nijenhuis tensor field of $\overline{f}$. And it

is called Sasakian if it is normal and $\Omega = d\eta$. We look for conditions that $(\overline{f}, \xi, \eta)$ be Sasakian.

Recall that $N_{\overline{f}}(X,Y) = [\overline{f}X, \overline{f}Y] + \overline{f}^2[X,Y] - \overline{f}[\overline{f}X, Y] - \overline{f}[X, \overline{f}Y]$ for $X, Y \in \chi(IK)$.

Now we shall compute N in the frame $(\delta_i, \dot{\delta}_j)$, $j = 1, 2, ..., n-1$. First, we notice that

$$\begin{array}{l} \dot{\delta}_j = \overline{f}\delta_j = a_{jk}\dot{\partial}^k \text{ for } a_{jk} := -g_{jk} + p_j p_k, \\ \overline{f}(\dot{\partial}^k) = g^{kh}\delta_h, \\ \overline{f}\dot{\delta}_j = \overline{f}^2\delta_j = b_j^k\delta_k \text{ for } b_j^k = -\delta_j^k + p_j p^k \end{array} \tag{3.4}$$

and recall the equations (4.14), Ch. 4 in [3]:

$$[\delta_i, \delta_j] = R_{kij}\dot{\partial}^k, [\delta_i, \dot{\partial}^h] = -\dot{\partial}^h N_{jr}\dot{\partial}^r,$$

where $(N_{ij} = N_{ji})$ are the local coefficients of the canonical nonlinear connection of the Cartan space K^n and $R_{kij} = \delta_j N_{ik} - \delta_j N_{jk}$.

A direct calculation gives

$$\begin{array}{l} N(\delta_i, \delta_j) = A_{hij}g^{hk}\delta_k + (B_{kij} - R_{kij})\dot{\partial}^k, \\ N(\delta_i, \dot{\delta}_j) = (a_{ih}\dot{\partial}^h b_j^s - b_j^k R_{hik}g^{hs} - B_{hij}g^{hs} - a_{ij}p^s)\delta_s \\ \qquad\qquad -[A_{kij} + p_j p^r(\delta_r a_{ik} - a_{ih}\dot{\partial}^h N_{rk}) + a_{kr}\delta_i b_j^r]\dot{\partial}^k \\ N(\dot{\delta}_i, \dot{\delta}_j) = (D_{ij}^s - D_{ji}^s)\delta_s + (b_i^k b_j^h R_{rkh} - B_{rij} - a_{rk}E_{ij}^k)\dot{\partial}^r, \end{array} \tag{3.5}$$

with the following notations

$$\begin{array}{l} A_{kij} = \delta_i a_{jk} - \delta_j a_{ik} + a_{ih}\dot{\partial}^h N_{jk} - a_{jh}\dot{\partial}^h N_{ik}, \\ B_{kij} = a_{ih}\dot{\partial}^h a_{jk} - a_{jh}\dot{\partial}^h a_{ik}, \\ D_{ij}^s = b_i^k\delta_k b_j^s + (b_j^k\delta_k a_{ih} + b_i^k a_{jh}\dot{\partial}^h N_{kr})g^{hs} \\ E_{ij}^k = a_{ih}\dot{\partial}^h b_j^k - a_{jh}\dot{\partial}^h b_i^k. \end{array} \tag{3.6}$$

In order to simplify the above expressions we need more formulae from the geometry of the Cartan space K^n.

We collect the needed formulae as follows:

$$C^{ijk}p_k = C^{ikj}p_k = C^{kij}p_k = 0 \text{ for } C^{ijk} = -\frac{1}{2}\dot{\partial}^k g^{ij}. \tag{3.7}$$

$$p_k\dot{\partial}^k N_{ij} = N_{ij} \text{ (the homogeneity of } N_{ij} \text{ in momenta).} \tag{3.8}$$

We assume that the Cartan space $K^n = (M, K)$ is endowed with the Cartan connection ($H^i_{jk} = H^i_{kj}, C^{jk}_i = -\frac{1}{2} g_{is} \dot\partial^s g^{jk} = g_{is} C^{sjk}$).

Then we have

$$\begin{aligned} &K^2_{|j} = 0, \qquad K^2|^j = 2p^j, \qquad p_{i|j} = 0, \qquad p_i|^j = \delta^j_i \\ &p^i_{|j} = 0, \qquad p^i|^j = g^{ij}. \end{aligned} \tag{3.9}$$

Here short and long vertical bars denote the horizontal and vertical covariant derivatives, respectively.

$$P^i_{jk} := H^i_{jk} - \dot\partial^i N_{jk}, \qquad P^i_{jk} p^j = 0. \tag{3.10}$$

$$\delta_i g_{ik} = H^s_{ji} g_{sk} + H^s_{ki} g_{js}. \tag{3.11}$$

Using (3.7) one easily obtains

$$B_{kij} = p_i g_{ik} - p_j g_{ik}. \tag{3.12}$$

In the vertical part of $N(\dot\delta_i, \dot\delta_j)$ the first term vanishes since $R_{kij} = -R_{kji}$ and using $\dot\partial^h b^k_j = \delta^h_j p^k + p_j g^{kh}$ one obtains $E^k_{ij} = (p_i g_{jh} - p_j g_{ih}) g^{hk}$ and after some cancelations the whole vertical part of $N(\dot\delta_i, \dot\delta_j)$ vanishes.

From $p^k a_{kr} = 0$ it results $(\delta_i p^k) a_{kr} = -\delta_i (a_{kr}) p^k$. We have $\delta_i p_k = N_{ik}$. Based on these formulae one finds that the vertical part of $N(\delta_i, \dot\delta_j)$ has the form $-b^k_j A_{rik} \dot\partial^r$ and its horizontal part is $b^k_j (p_i g_{kh} - p_k g_{ih} - R_{hik}) g^{hs} \delta_s$.

A tedious computation leads to

$$D^s_{ij} - D^s_{ji} = A_{hij} g^{hs} + p^k P^r_{hk} (p_i g_{jr} - p_j g_{ir}) = A_{hij} g^{hs}$$

by (3.10).

The tensor field A_{kij} takes the form

$$A_{kij} = \delta_j g_{ik} - \delta_i g_{jk} + g_{jh} \dot\partial^h N_{ik} - g_{ih} \dot\partial^h N_{jk}$$

and by (iii) of Prop. 2, Ch. 7 in [3], it vanishes for Cartan spaces.

Summarizing up we have proved

Theorem 3.4. *The tensor field N is given in the frame $(\delta_i, \dot\delta_j)$, $j = 1, ..., n-1$, as follows:*

$$\begin{aligned} &N(\delta_i, \delta_j) = (p_i g_{jk} - p_j g_{ik} - R_{kij}) \dot\partial^k, \\ &N(\delta_i, \dot\delta_j) = b^k_j (p_i g_{kh} - p_k g_{ih} - R_{hik}) g^{hs} \delta_s, \\ &N(\dot\delta_i, \dot\delta_j) = 0. \end{aligned}$$

Corollary 3.1. *The almost contact structure $(\overline{f}, \xi, \eta)$ is Sasakian if and only if*

$$R_{kij} = p_i g_{jk} - p_j g_{ik}. \tag{3.13}$$

A geometrical meaning of (3.13) can be easily obtained when the functions g^{ij} do not depend on p, that is we have $K^2 = g^{ij}(x)p_i p_j$. In this case the local coefficients N_{ij} of the canonical nonlinear connection take the form $N_{ij} = \gamma^k_{ij} p_k$, where (γ^k_{ij}) are the Christoffel symbols constructed with $(g_{ij}(x))$.

We have $R_{kij} = \delta_j(\gamma^h_{ik}p_h) - \delta_i(\gamma^h_{jk}p_h) = (\partial_j\gamma^h_{ik} - \partial\gamma^h_{jk} + \gamma^s_{ik}\gamma^h_{sj} - \gamma^s_{jk}\gamma^h_{si})p_h = R^h_{kij}p_h$, where R^h_{kij} is the curvature tensor derived from $g_{ij}(x)$.

The equation (3.13) becomes $R_k{}^h{}_{ij}p_h = (\delta^h_i g_{jk} - \delta^h_j g_{ik})p_h$ and it is equivalent to

$$R_{kh\ ij} = g_{kj}g_{hi} - g_{ik}g_{jh}. \tag{3.14}$$

But (3.14) says that the manifold $(M, g_{ij}(x))$ is of constant curvature -1. Thus we have proved

Theorem 3.5. *Let be the Cartan space (M, K) with $K^2 = g^{ij}(x)p_i p_j$. The almost contact structure $(\overline{f}, \xi, \eta)$ on the indicatrix bundle IK is Sasakian if and only if the Riemannian manifold $(M, g_{ij}(x))$ is of constant curvature -1.*

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