ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIII, 2007, f.1

FIXED POINT THEOREMS FOR SEQUENCE OF MAPPINGS IN FUZZY METRIC SPACES SATISFYING A NEW CONTRACTIVE CONDITION

BY

SHISHIR JAIN, MAMTA SINGH and SHOBHA JAIN

Abstract. The object of this paper is to establish a unique common fixed point theorem for a sequence of self mappings in a fuzzy metric space, which generalizes the result of Grabiec [3], Singh and Jain [5] and Vasuki [7]. All the results established in this paper are new.

Mathematics Subject Classification 2000: 54H25, 47H10.

Key words: Fuzzy metric space, t-norm, common fixed points, weak-compatible maps.

1. Introduction. Zadeh [8] initiated the concept of fuzzy sets in 1965. Many authors used this concept in Topology and Analysis and developed the theory of fuzzy sets and its applications. Kramosil and Michalek [4] introduced the concept of fuzzy metric space. George and Veeramani [2] modified this concept of fuzzy metric space and defined a Hausdroff topology on fuzzy metric space. Grabiec [3] obtained the fuzzy version of Banach contraction principle, which is a milestone in developing fixed point theory in fuzzy metric space.

2. Preliminaries

Definition 2.1. *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous t-norm if* $([0,1], *)$ *is an abelian topological monoid with unit* 1 *such that* $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *for all* a, b, c *and* $d \in [0,1]$.

Examples of t-norm are $a * b = ab$ and $a * b = min\{a, b\}$ etc.

Definition 2.2. ([4]) *The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:*
$(FM-1)$ $M(x, y, 0) = 0$;
$(FM-2)$ $M(x, y, t) = 1$, *for all* $t > 0$ *iff* $x = y$;
$(FM-3)$ $M(x, y, t) = M(y, x, t)$;
$(FM-4)$ $M(x, y, t) * M(y, z, t) \leq M(x, z, t + s)$;
$(FM-5)$ $M(x, y, .) : [0, \infty) \to [0, 1]$*is left continuous;*
for all $x, y, z \in X$ *and* $s, t > 0$.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$, for all $t > 0$. The following example shows that every metric space induces a fuzzy metric space.

Example 2.3. ([2]) Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and for all $x, y \in X$, $M(x, y, t) = t/(t + d(x, y))$, for all $t > 0$ and $M(x, y, 0) = 0$. Then $(X, M, *)$ is a fuzzy metric space. It is called the fuzzy metric space induced by the metric d.

Lemma 2.4. ([3]) *For all $x, y \in X$, $M(x, y, .)$ is a non-decreasing function.*

Definition 2.5. ([3]) *Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$. Further, the sequence $\{x_n\}$ is said to be a Cauchy sequence if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$, for all $t > 0$ and $p > 0$. The space is said to be complete if every Cauchy sequence in it converges to a point of it.*

Remark 2.6. Since * is continuous, it follows from $(FM-4)$ that the limit of a sequence in a fuzzy metric space is unique.

In this paper $(X, M, *)$ is considered to be the fuzzy metric space with condition
$(FM-6)$ $\lim_{t\to\infty} M(x, y, t) = 1$.

Lemma 2.7. ([1]) *Let $\{y_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$ with the condition* (FM-6). *If there exists a number $k \in (0, 1)$ such that $M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$, for all $t > 0$. Then $\{y_n\}$ is a Cauchy sequence in X.*

In [3], GRABIEC proved the following Banach contraction principle in fuzzy metric space

Theorem 2.8. ([3]) *Let $(X, M, *)$ be a complete fuzzy metric space where $*$ is continuous and T be a self mapping on X such that, $M(Tx, Ty, \alpha t) \geq M(x, y, t), \forall x, y \in X, t > 0$ and $\alpha \in (0, 1)$. Then T has a unique fixed point.*

In [5] SINGH and SHISHIR JAIN established the following result in fuzzy metric space:

Theorem 2.9. ([5]) *Let A be self mapping of a complete fuzzy metric space $(X, M, *)$ such that there exists some $k \in (0, 1)$ satisfying*

$$M(Ax, Ay, kt) \geq b_0 M(x, y, t) + c_0 M(Ax, x, t),$$

$\forall x, y \in X$ and $t > 0$, where b_0, c_0 are non-negative real numbers such that $b_0 + c_0 = 1$. Then A has a unique fixed point in X.

The following result is in the line of generalization of the above two results.

3. Main results

Theorem 3.1. *Let $\{A_n\}$ be a sequence of self mappings and S be a self mapping of a complete fuzzy metric space $(X, M, *)$ satisfying:*

(3.1) $A_n(X) \subseteq S(X), \forall n$;

(3.2) $A_n S = S A_n, \forall n$;

(3.3) *S is continuous;*

(3.4) *There exist $k \in (0, 1)$ such that $\forall x, y \in X, t > 0$ and for all i and j*

$$M(A_i x, A_j y, kt) \geq aM(A_i x, Sx, t) + bM(Sx, Sy, t) + cM(A_j y, Sy, t),$$

for some $a, b, c \in [0, 1]$ with $a + b + c = 1$. Then for $x_0 \in X$ the sequence $\{y_n\}$ in X defined by $A_n x_{n-1} = Sx_n = y_n$ for all n, converges and its limit is the unique common fixed point of the sequence $\{A_n\}$ and self mapping S.

Proof. First we shall show $\{y_n\}$ is a Cauchy sequence in X. Putting $x = x_{n-1}$ and $y = x_n$ for the maps A_n and A_{n+1} in (3.4) we obtain that

$$\begin{aligned} M(A_n x_{n-1}, A_{n+1} x_n, kt) \quad &\geq aM(A_n x_{n-1}, Sx_{n-1}, t) + bM(Sx_{n-1}, Sx_n, t) \\ &\quad + cM(A_{n+1} x_n, Sx_n, t), \end{aligned}$$

i.e.

$$\begin{aligned} M(y_n, y_{n+1}, kt) &\geq aM(y_n, y_{n-1}, t) + bM(y_n, y_{n-1}, t) + cM(y_n, y_{n+1}, t) \\ &= (a+b)M(y_n, y_{n-1}, t) + cM(y_n, y_{n+1}, t) \\ &\geq Min\{M(y_n, y_{n-1}, t), M(y_n, y_{n+1}, t)\}. \end{aligned}$$

Thus, $M(y_n, y_{n+1}, t) \geq Min\ M(y_n, y_{n-1}, t/k), M(y_n, y_{n+1}, t/k)$. By putting the value of $M(y_n, y_{n+1}, t)$ in above we obtain that

$$\begin{aligned} M(y_n, y_{n+1}, kt) &\geq Min\{M(y_n, y_{n-1}, t), M(y_n, y_{n-1}, t/k), M(y_n, y_{n+1}, t/k)\} \\ &= Min\{M(y_n, y_{n-1}, t), M(y_n, y_{n+1}, t/k)\} \\ &\geq Min\{M(y_n, y_{n-1}, t), M(y_n, y_{n+1}, t/k^2)\} \\ &.... \\ &\geq Min\{(y_n, y_{n-1}, t), (y_n, y_{n+1}, t/k^m)\}. \end{aligned}$$

Taking limit as $m \to \infty$, we have $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$. Thus, for any n and t we have, $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$. Hence by Lemma 2.7, $\{y_n\}$ is a Cauchy sequence in X, which is complete. Therefore $\{y_n\}$ converges to some $z \in X$ and thus the sequences $\{A_n x_{n-1}\} = \{Sx_n\} \to z$. Also as S is continuous, we have $\{SSx_{n-1}\} \to Sz$ and as $A_n S = SA_n$, we obtain that $\lim_{n\to\infty} A_n S x_{n-1} = \lim_{n\to\infty} SA_n x_{n-1} = \lim_{n\to\infty} Sy_n = Sz$.

Step I. For the maps A_n and $A_p, p \in N$ putting $x = Sx_{n-1}, y = z$ in (3.4), we obtain

$$\begin{aligned} M(A_n Sx_{n-1}, A_p z, kt) &\geq aM(A_n Sx_{n-1}, SSx_{n-1}, t) \\ &\quad +bM(SSx_{n-1}, Sz, t) + cM(A_p z, Sz, t). \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned} M(Sz, A_p z, kt) &\geq aM(Sz, Sz, t) + bM(Sz, Sz, t) + cM(A_p z, Sz, t) \\ &= a + b + cM(A_p z, Sz, t) \\ &\geq (a+b+c)M(A_p z, Sz, t) \\ &= M(A_p z, Sz, t), \forall t > 0. \end{aligned}$$

It gives $A_p z = Sz$ for all p. Thus z is a coincidence point of the sequence $\{A_n\}$ and S.

Step II. For the maps A_n and $A_p, p \in N$ putting $x = x_{n-1}, y = z$ in (3.4) we obtain

$$\begin{aligned} M(A_n x_{n-1}, A_p z, kt) &\geq aM(A_n x_{n-1}, Sx_{n-1}, t) + bM(Sx_{n-1}, Sz, t) \\ &\quad +cM(A_p z, Sz, t). \end{aligned}$$

Taking limit as $n \to \infty$ and using $A_p z = Sz$, we have

$$\begin{aligned} M(z, Sz, kt) &\geq aM(z, z, t) + bM(z, Sz, t) + cM(z, Sz, t) \\ &= a + (b + c)M(z, Sz, t) \\ &\geq (a + b + c)M(z, Sz, t) \\ &= M(z, Sz, t), \forall t > 0. \end{aligned}$$

It gives $Sz = z$. Thus for all p, we have $A_p z = Sz = z$. Hence z is a common fixed point of the sequence of self mappings $\{A_n\}$ and S.

Uniqueness. Let u be another common fixed point of $\{A_n\}$ and S. Then $A_p u = Su = u, \forall p$. For the maps A_1 and A_2 using (3.4) with $x = z$ and $y = u$, we have

$$M(A_1 z, A_2 u, kt) \geq aM(A_1 z, Sz, t) + bM(Sz, Su, t) + cM(A_2 u, Su, t),$$

i.e.

$$\begin{aligned} M(z, u, k_1 t) &= a + bM(z, u, t) + c \geq (a + b + c)M(z, u, t) \\ &= M(z, u, t), \forall t > 0. \end{aligned}$$

It gives $z = u$ and we get that z is the unique common fixed point of sequence of mappings $\{A_n\}$ and S. □

Taking $S = I$ in Theorem 3.1, we obtain the following result for sequence of mappings in a fuzzy metric space:

Corollary 3.2. *Let $\{A_n\}$ be a sequence of self mappings and of a complete fuzzy metric space $(X, M, *)$ satisfying:*
(3.5) *There exists $k \in (0, 1)$ such that $\forall x, y \in X, t > 0$ and for all i and j, $M(A_i x, A_j y, kt) \geq aM(A_i x, x, t) + bM(x, y, t) + cM(A_j y, y, t)$, for some $a, b, c \in [0, 1]$ with $a+b+c = 1$. Then for each x_0 in X, the sequence defined by $A_n x_{n-1} = x_n$ converges in X and its limit is the unique common fixed point of the self mappings of the sequence $\{A_n\}$.*

Taking $A_n = A, \forall n$ we obtain the following important result:

Corollary 3.3. *Let A be a self mapping of a complete fuzzy metric space $(X, M, *)$ satisfying:*
(3.6) *There exists $k \in (0, 1)$ such that $\forall x, y \in X, t > 0$*

$$M(Ax, Ay, kt) \geq aM(Ax, x, t) + bM(x, y, t) + cM(Ay, y, t),$$

for some $a, b, c \in [0,1]$ with $a+b+c = 1$. Then for each $x_0 \in X$ the sequence $\{x_n\}$ defined by $Ax_{n-1} = x_n = A^n x_0$, converges in X and its limit is the unique fixed point of the mapping A.

Remark 3.4. The above corollary is a generalization of the above said Banach contraction principle established in GRABIEC [3] and also of the result of [6].

A more general form of Theorem 3.1 is the following:

Theorem 3.5. *Let $\{A_n\}$ be a sequence of self mappings and S be a self mapping of a complete fuzzy metric space $(X, M, *)$ satisfying* (3.1), (3.2), (3.3) *and there exists $k \in (0,1)$ such that for some integers m_n, $\forall x, y \in X, t > 0$ and for all i and j,*

(3.7)
$$M(A_i^{m_i}x, A_j^{m_j}y, kt) \geq aM(A_i^{m_i}x, Sx, t) + bM(Sx, Sy, t) + cM(A_j^{m_j}, Sy, t),$$

for some $a, b, c \in [0,1]$ with $a+b+c = 1$. Then for $x_0 \in X$ the sequence $\{y_n\}$ in X defined by $A_n^{m_n}x_{n-1} = Sx_n = y_n$, for all n, converges and its limit is the unique common fixed point of the sequence $\{A_n\}$ and self mapping S.

Proof. As $A_nS = SA_n, \forall n$, we get $A_n^{m_n}S = SA_n^{m_n}$ and by Theorem 3.1, the sequence $\{A_n^{m_n}\}$ and self mapping S have a unique common fixed point say z in X. Thus $A_n^{m_n}z = Sz = z, \forall n$. Now, $A_1^{m_1}(A_1z) = A_1(A_1^{m_1}z) = A_1z$ and $S(A_1z) = A_1(Sz) = A_1z$. Now using (3.7) for the maps $A_1^{m_1}$ and $A_2^{m_2}$ with $x = A_1z$ and $y = z$ we obtain that

$$\begin{aligned} M(A_1^{m_1}A_1z, A_2^{m_2}z, kt) &\geq aM(A_1^{m_1}A_1z, SA_1z, t) + bM(SA_1z, Sz, t) \\ &\quad + cM(A_2^{m_2}A_1z, Sz, t), \end{aligned}$$

i.e.

$$\begin{aligned} M(A_1z, z, kt) &\geq aM(A_1z, A_1z, t) + bM(A_1z, z, t) + cM(A_1z, z, t) \\ &= a + bM(A_1z, z, t) + cM(A_1z, z, t) \\ &\geq (a+b+c)M(A_1z, z, t) \\ &= M(A_1z, z, t). \end{aligned}$$

It gives $A_1z = z$. Similarly, we can prove $A_nz = z$ for all n. Thus z is a common fixed point of the sequence $\{A_n\}$ and self mapping S. The uniqueness follows from the fact that if u is another common fixed point of the sequence $\{A_n\}$ and self mapping S then it becomes the common fixed

point of the sequence $A_n^{m_n}$ and S, which is unique. Thus it follows that z is the unique common fixed point of the sequence $\{A_n\}$ and self mapping S.

In [7], VASUKI established the following theorem for sequence of mappings.

Theorem 3.6. ([7]) *Let $\{T_n\}_n$ be a sequence of mappings of a complete fuzzy metric space $(X, M, *)$ into itself where * is continuous t-norm such that for any two mappings T_i and T_j, we have*

$$M(T_i^m x, T_j^m y, \alpha_{i,j} t) \geq M(x, y, t)$$

for some m and $0 < \alpha_{i,j} < 1, i, j = 1, 2, ...,$ and $x, y \in X$. Then $\{T_n\}_n$ has a unique common fixed point in X.

In [6], SONG made the correction in the above result of VASUKI [7] and pointed out the following fact.

Theorem 3.7. ([6]) *Suppose all the conditions of Theorem* 3.6 *are satisfied and t-norm * satisfies for any $a, b \in [0, 1], a * b = \min\{a, b\}$ or $a * a = a$. Then the conclusion of above Theorem* 3.6 *remains true.*

Remark 3.8. In view of above our Theorem 3.5 is a generalization of the above theorem of VASUKI [7]. The said result of [7] follows by taking $S = I$ and $a = 1, b = c = 0$ in Theorem 3.5.

REFERENCES

1. CHO, Y.J. – *Fixed point in fuzzy metric space,* Journal of Fuzzy Mathematics, 4(1997) 949-962.
2. GEORGE, A.; VEERAMANI, P. – *On some results in fuzzy metric spaces,* Fuzzy Sets and Systems, 64 (1994), 395-399.
3. GRABIEC, M. – *Fixed points in fuzzy metric spaces,* Fuzzy Sets and Systems, 7(1988), 385-389.
4. KRAMOSIL, I.; MICHALEK, J. – *Fuzzy metric and statistical metric spaces,* Kybernetica 11 (1975) 326-334.
5. SINGH, B.; JAIN, SHISHIR – *Semi-compatibility and fixed point theorems in fuzzy metric space using implicit relation,* International Journal of Mathematics and Mathematical Sciences, USA, 16 (2005), 2617-2630.

6. Song, G. – *Comments on a common fixed point theorem in a fuzzy metric space,* Fuzzy Sets and Systems, 135 (2003), 409-413.

7. Vasuki, R. – *Common fixed point theorem in a fuzzy metric space,* Fuzzy Sets and Systems, 97 (1998), 395-397.

8. Zadeh, L. A. – *Fuzzy sets,* Inform and Control, 89 (1965), 338-353.

Received: 8.VI.2006

Shri Vaishnav Institute of Technology and Science,
Indore (M.P.),
INDIA
jainshishir11@rediffmail.com

Department of Mathematics and Computer Applications,
Bundelkhand University,
Jhansi (U.P.),
INDIA

Shri Vaishnav Institute of Management,
Indore (M.P.),
INDIA
shobajain1@yahoo.com