ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIII, 2007, f.1

ON PROPERTY (P_λ) AND WEAK AMENABILITY OF $A(X)$

BY

O.T. MEWOMO and J.A. OGUNTUASE

Abstract. We investigate the weak amenability of the Banach algebra $A(X)$ of aproximable operators on a Banach space X and its relation with property (P_λ).

Mathematics Subject Classification 2000: Primary: 46H20; Secondary 46H10, 46H25.

Key words: weak amenability, Banach algebra, Banach space, bounded derivation, property (P_λ), approximable operators.

1. Introduction. Let A be a Banach algebra, and let X be an A-bimodule. Then X is a Banach A-bimodule, if X is a Banach space, and if there is a constant k such that:

$$\|a.x\| \leq k\|a\|\|x\| \text{ and } \|x.a\| \leq k\|a\|\|x\|, \ a \in A, x \in X.$$

Let A be a Banach algebra and let X be a Banach A-bimodule. Then X^*, the dual space of X, is a Banach A-bimodule with respect to the operations

$$(a.f)(x) = f(x.a), \ (f.a)(x) = f(a.x), \ \ a \in A, x \in X, f \in X^*,$$

X^* is the dual module of X, and in particular, A^* is the dual module of A.

Let A be a Banach algebra and let X be a Banach A-bimodule. A bounded linear map $D : A \to X$ is a derivation if

$$D(ab) = a.D(b) + D(a).b, \quad a, b \in A.$$

For any $x \in X$, the mapping $\delta_x : A \to X$ given by $\delta_x(a) = a.x - x.a, \ \ a \in A$ is a derivation, called an inner derivation.

Let $B^1(A,X)$ be the space of all continuous derivations from A into X and let $\mathcal{Z}^1(A,X)$ be the space of all inner derivations from A into X. Then the first cohomology group of A with coefficients in X is the quotient space $\mathcal{H}^1(A,X) = B^1(A,X)/\mathcal{Z}^1(A,X)$.

The Banach algebra A is amenable if $\mathcal{H}^1(A,X^*) = \{0\}$ for any Banach A-bimodule X, i.e. if bounded derivation from A into a dual Banach A-bimodule X^* is inner. A is weakly amenable if $\mathcal{H}^1(A,A^*) = \{0\}$. For instance, the group algebra $L^1(G)$ of a locally compact group G is always weakly amenable ([7]), and is amenable if and only if G is amenable in the classical sense ([6]). Also, a C^*-algebra is always weakly amenable ([5]) and is amenable if and only if it is nuclear ([5] and [3]).

Recently weak amenability has been investigated for algebras of type $A(X)$. In [4], it is shown that $A(X)$ is weakly amenable when $X = l_p(Y)$, with Y a reflexive Banach space having the approximation property, or when $X = E \oplus C_p$, where E is a Banach space with the bounded approximation property and C_p denotes any of the universal spaces introduced by Johnson in [8]. In [2], Blanco introduced a technical property of X, socalled trace unbounded triples, and by using it established weak amenability of $A(X)$ for a wide range of Banach spaces X. In [1] the author studied the weak amenability of $A(X)$, and its relation with the bounded approximation property. He showed that bounded approximation property is neither necessary nor sufficient for the weak amenability of $A(X)$.

In this paper, we shall show the relationship between the weak amenability of $A(X)$ and the property (P_λ), thus extending some of Blanco's result and we shall also provide a new proof for Proposition 3.3 of [2]. Here $A(X)$ is the closure in $B(X)$ of the ideal $F(X)$ of continuous finite-rank operators on X, where $B(X)$ denotes the algebra of all bounded linear operators on X.

We shall assume throughout this paper that all the Banach spaces are over the complex field.

2. Preliminaries. By $L_p(\mu) = L_p(X,\mathcal{M},\mu)$, $1 \le p \le \infty$, we denote the Banach space of equivalence classes of measurable functions on $(X,\mathcal{M},\mu)$ whose p^{th} power is integrable respectively, which are essentially bounded if $p = \infty$. The norm in $L_p(\mu)$ is denoted by

$$\|f\| = \left(\int \mid f(\omega) \mid^p d\mu(\omega) \right)^{\frac{1}{p}} \quad (\text{ess sup} \mid f(\omega) \mid \quad \text{if} \quad p = \infty).$$

Throughout this work, the term operator means a linear bounded operator. The space of all operators from X to Y is denoted by $B(X,Y)$.

Two Banach spaces X and Y are called isomorphic (denoted by $X \simeq Y$) if there exists an invertible operator from X onto Y. If there exists an invertible operator S from X onto Y with $\|S\| = \|S^{-1}\| = 1$, i.e $\|S(x)\| = \|x\|$, for every $x \in X$, then we say that X is isometric to Y, X and Y are called isometric and we write $X \cong Y$. We denote by $d(X,Y)$ the Banach-Mazur distance between X and Y, that is, the infimum of numbers $\|T\|\|T^{-1}\|$, where T is an isomorphism between X and Y.

A closed linear subspace Y of a Banach space X is said to be a complemented subspace of X if there is a projection from X onto Y, or if there exists a closed linear subspace Z of X such that X is the direct sum of Y and Z, i.e $X = Y \oplus Z$. We shall also use some direct sums of infinite sequences of Banach spaces. If $\{X_n\}_{n=1}^{\infty}$ is a sequence of Banach spaces we define the direct sum of these spaces in the sense of $l_p, 1 \leq p < \infty$, namely $(\sum_{n=1}^{\infty} \oplus X_n)_{l_p}$, as the space of all sequences $x = (x_1, x_2, ...)$ with $x_n \in X_n$ for all n for which

$$\|x\| = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}} < \infty.$$

Similarly, $(\sum_{n=1}^{\infty} \oplus X_n)_{c_o}$ denotes the direct sum of $\{X_n\}_{n=1}^{\infty}$ in the sense of c_o, i.e the space of all sequences $x = (x_1, x_2, ...)$ with $x_n \in X_n$ for all n for which $\lim_{n\to\infty} \|x_n\| = 0$. The norm in this direct sum is defined as

$$\|x\| = \max\{\|x_n\|\}.$$

If $X_n = X$ for all positive integer n we simply write $c_o(X)$ or $l_p(X), 1 \leq p \leq \infty$.

Given a Banach space X, we identify $F(X)$ and $X \otimes X^*$ in the usual way, that is we associate to the element $v = \sum_i x_i \otimes \lambda_i \in X \otimes X^*$ the operator $\tilde{v} \in F(X)$ defined by

$$\overline{v}(x) = \sum_i \lambda_i(x) x_i \quad (x \in X)$$

Denote by $X \hat{\otimes} Y$ the projective tensor product of the Banach spaces X and Y. The tensor algebra of X is $X \hat{\otimes} X^*$ with multiplication given by

$$(x \otimes x^*)(y \otimes y^*) = x^*(y)x \otimes y^*, \quad x, y \in X, x^*, y^* \in X^*.$$

The canonical trace tr and the operator trace Tr on $X \otimes X^*$ are, respectively, given by $tr : X \hat{\otimes} X^* \to C$ and $Tr : X \hat{\otimes} X^* \to N(X)$

$$tr(x \otimes x^*) = x^*(x), \quad Tr(x \otimes x^*)y = x^*(y)x, \quad x, y \in X, x^* \in X^*$$

where $N(X)$ is the ideal of nuclear operators on X.

3. Main results

Definition 3.1. *Let X be a Banach space, and let $\lambda \geq 1$.*

1. *X is said to have property (P_λ) if, for each finite-dimensional subspace $F \subset X$, there is a finite-dimensional subspace E of X containing F such that there is a projection of norm not greater than λ onto E (i.e there is a projection $P : X \to E$ with $\|P\| \leq \lambda$.)*

2. *X is said to have property (Q_λ) if, for each closed subspace F of X with finite codimension (i.e $\dim(X/F) < \infty$), there is a closed subspace E of F with finite codimension in X, such that the canonical surjection $\pi_E : X \to X/E$ has a bounded right inverse of norm not greater than λ (i.e there exists $\pi : X/E \to X$ such that π is bounded and $\pi_E \pi = I_{X/E}$ with $\|\pi_E\| \leq \lambda$).*

Example 3.1. Every Hilbert space has properties (P_1) and (Q_1).

Definition 3.2. *We recall from* [10] *that a Banach space X is said to be an $\mathcal{L}_{p,\lambda}$- space, $1 \leq \lambda < \infty, 1 \leq p \leq \infty$ if for every finite-dimensional subspace F of X, there is a finite-dimensional subspace E of X containing F and such that*

$$d(F, l_p^n) \leq \lambda, \quad \text{where} \quad n = \dim F.$$

Then a Banach space X is an $\mathcal{L}_p$- space if it is an $\mathcal{L}_{p,\lambda}$- space for some λ.

Example 3.2. If X is an $\mathcal{L}_p$- space with $1 \leq p \leq \infty$, then X has property (P_λ) for some $\lambda \geq 1$ ([11, Theorem III(c)]).

Definition 3.3. (*Johnson Space*) *A Banach space J_p is said to a Johnson space if it has the form $(\sum \oplus_n G_n)_{l_p}$, $p = 0$ or $1 \leq p \leq \infty$, where (G_n) is a sequence of finite-dimensional Banach spaces such that, for each positive integer i, the set $\{n : G_n \cong G_i, \quad n \text{ a positive integer}\}$ is infinite.*

If the sequence (G_n) in the above definition (3.3) is dense (in the Banach -Mazur sense) in the class of all finite-dimensional Banach spaces, in this case for p fixed, all the Johnson spaces are pairwise isomorphic and the Banach-Mazur distance between any two of them is 1. Thus, in this situation, for each p there is essentially a unique Johnson space, which is denoted by C_p. (see [8] and [9]).

Example 3.3. l_p $(1 \leq p \leq \infty)$ and c_o are Johnson spaces ([2]).

Definition 3.4. *Let $J_p = (\sum \oplus_n G_n)_{l_p}$ be a Johnson space. A Banach space X is a J_p-space if there exists $\lambda \geq 1$ such that, for every finite-dimensional subspace E of X, there is a subspace F of X containing E and such that $d(F, G_i) \leq \lambda$ for some i.*

Example 3.4. If X is a J_p-space, then X has property (P_λ) for some $\lambda \geq 1$.

Definition 3.5. (*James Spaces*) *Let $1 < p < \infty$ be fixed. On the linear space C_{oo} of sequences of scalars with finite support we define a norm $\|.\|_{\mathcal{J}_p}$ by $\|(\alpha_n)\|_{\mathcal{J}_p} = \sup\{(\sum_{n=1}^{m-1} \mid \alpha_{in} - \alpha_{in+1} \mid^p)^{\frac{1}{p}} : m, i_1, i_2, ..., i_m$ are positive integers, $m \geq 2$ and $i_1 < i_2 < ... < i_m\}$ (for $(\alpha_n) \in C_{oo}$.)*

The p-th James space $\mathcal{J}_p$ is defined as the completion of C_{oo} in the above norm. Equivalently, $\mathcal{J}_p = \{(\alpha_n) : \alpha_n \in C$ (n is a positive integer), $\|(\alpha_n)\|_{\mathcal{J}_p} < \infty$ and $\lim_n \alpha_n = 0\}$).

Example 3.5. By Lemma 3.5 of [2], we have that for every $1 < p < \infty$, every James space $\mathcal{J}_p$ is a J_p-space and $\mathcal{J}_p \simeq \mathcal{J}_p \oplus J_p$, and so by Example 3.4, we have that every James space $\mathcal{J}_p$ has property (P_λ) for some $\lambda \geq 1$.

We first obtain a relation between properties (P_λ) and (Q_λ).

Lemma 3.1. *Let X be a reflexive Banach space, let $\lambda \geq 1$ and suppose that X^* has property (P_λ). Then X has property (Q_λ).*

Proof. Let F be a closed subspace of X with finite codimension. We show that there exists a closed subspace E of F with finite codimension in X, such that the canonical surjection $\pi_E : X \to X/E$ has a bounded right inverse of norm not greater than λ. F is a closed subspace of X with finite codimension, thus $F^\perp = \{f \in X^* : f(x) = 0$ for every $x \in F\}$ is a finite-dimensional subspace of X^*. Since X^* has property (P_λ), then by definition, there is a finite-dimensional subspace Y of X^* containing $F^\perp$ such that there

is a projection $P : X^* \to Y$ with $\|P\| \leq \lambda$. Let $E =^{\perp} Y\{x \in X : f(x) = 0$ for all $f \in X^*\}$. Since X is a reflexive Banach space, then E is a closed subspace of X with finite codimension such that

$$X/E \cong X^{**}/E^{**} \cong Y^*.$$

Let $\pi_E : X \to X/E$ be a canonical projection, with the appropriate identification, it follows that $P^* : X/E \to X^{**}$ is a right inverse of π_E^{**} with $\|P^*\| \leq \lambda$. Since $\pi_E^{**} : X^{**} \to (X/E)^{**}$ and X is reflexive, then the canonical projection $\pi_E^{**} : X \to X/E$ has a right inverse P^* with $\|P^*\| \leq \lambda$. Hence X has property (Q_λ). □

Our main result is Theorem 3.2. In order to establish our main result, we use Theorem 3.1 and Lemma 3.2.

Theorem 3.1 (1). *Let E be a Banach space and let A be a dense subalgebra of $(E' \otimes E, \|.\|_p)$ (and hence of $E'\hat{\otimes}E$). The algebra $A(E)$ is weakly amenable if and only if, whenever $T \in B(E')$ satisfies*

$$|\ tr(T(RS - SR)')\ | \leq K\|R\|\|S\|, \quad R, S \in A$$

for some constants K, the following holds:
(A) there exist $\lambda \in C$ and a constant $\tilde{K}$ such that

$$|\ tr((T - \lambda)W')\ | \leq \tilde{K}\|W\|, \ \ W \in A.$$

Moreover, (A) is a equivalent to the following condition:
(A^*) there exists a constant K_o, such that

$$|\ tr(TW')\ | \leq K_o\|W\|,$$

for all $W \in A$, with $trW = 0$.

Given a direct sum of Banach spaces $X = X_1 \oplus X_2$, we denote by $\gamma_k : X \to X_k$, $i_k :\to X_k \to X$, $k = 1, 2$, the canonical k-th coordinate projection and embedding, respectively. For each $V \in B(X), \tilde{V} \in B(X')$, we denote by V_{kj} (respectively $\tilde{V}_{kj}$) the operator $\gamma_k V i_j$ (respectively $i_k' \tilde{V} \gamma_j'$), $1 \leq k \leq 2$. Without loss of generality, we shall suppose that the norm on $X_1 \oplus X_2$ has been chosen such that $\|\gamma_k\| = 1$, $\|i_k\| = 1$ for $k = 1, 2$, i.e

$$V_{kj} = \gamma_k V i_j : X_j \to X_k, \tag{1}$$

since $i_j : X_j \to X$, $V : X \to X, \gamma_k : X \to X_k$, and also

$$\tilde{V}_{kj} = i'_k \tilde{V} \gamma'_j : X'_j \to X'_k, \tag{2}$$

where $\gamma'_j : X'_j \to X', \tilde{V}' : X' \to X', \ \ 1'_k : X' \to X'_k$.

For each $T \in B(X')$, we set

$$l_T : W \to tr(TW') \ \ i.e \ \ l_T(W) = tr(TW'), \ \ W \in F(X) \tag{3}$$

and

$$\begin{aligned} &b_T : (R,S) \to tr(T(RS - SR)'), \ i.e \ b_T(R,S) \\ &= tr(T(RS - SR)'), \ \ (R,S) \in F(X) \times F(X). \end{aligned} \tag{4}$$

We define

$$\Delta_X = \{T \in B(X') : b_T/_{F(X)\times F(X)} \text{is bounded}\}. \tag{5}$$

Given a Banach space X and a continuous finite-rank operator W on a Banach space Y, we set

$$| W |_X = \inf\{\|R\|\|S\| : R \in F(X,Y), S \in F(Y,X) \ \ and \ \ RS = W\}. \tag{6}$$

Lemma 3.2. *Let X_1 and X_2 be infinite-dimensional Banach space, and let $X = X_1 \oplus X_2$. Suppose X_1 has property (P_λ).*
Then, for each $T \in \Delta_X$, there exists a constant K_T such that

$$| l_T(i_1 W_{12} \gamma_2 + i_2 W_{21} \gamma_1 | \leq K_T \|W\|, \ \ W \in F(X). \tag{7}$$

Proof. Let $T \in \Delta_X$, and let $W \in F(X)$. Since X_1 has property (P_λ), there exists $P \in F(X_1)$ and $Q \in F(X_1)$ with $\|P\| \leq \lambda$ and $\|Q\| \leq \lambda$, with $PW_{12} = W_{12}$ and $W_{21}Q = W_{21}$. Then, since $\gamma_1 i_1 = I_X, \gamma_2 i_1 = 0$ and $\gamma_1 i_2 = 0$, we have $P\gamma_1 i_1 = P i_1 W_{12} \gamma_2 i_1 P \gamma_1 = 0$. Thus

$$\begin{aligned} | \, l_T(i_1 W_{12} \gamma_2) \, | \ &= | \, l_T(i_1 P \gamma_1 i_1 W_1 2 \gamma_2 - i_1 W_{12} \gamma_2 i_1 P \gamma_1) \, | \\ &= | \, b_T(i_1 P \gamma_1, i_1 W_{12} \gamma_2) \, | \quad \text{using (3) and (4)} \\ &\leq \|b_T\| \|i_1 P \gamma_1\| \|i_1 W_{12} \gamma_2\| \quad (\text{since } T \in \Delta_X) \\ &\leq \lambda \|b_T\| \|W\|, \end{aligned} \tag{8a}$$

since $\|P\| \leq \lambda, \|\gamma_k\| = 1, \|i_k\| = 1, \ k = 1, 2$ and $W = i_1 W_{12} \gamma_2 : X \to X$.

Similarly, $\gamma_1 i_1 = I_X, \gamma_1 i_1 Q = Q, W_{21}\gamma_1 i_1 Q = W_{21}Q = W_{21}, \gamma_1 i_2 = 0$, implies $i_1 Q \gamma_1 i_2 W_{21}\gamma_1 = 0$, and so we get

$$
\begin{aligned}
(8b) \qquad | \, l_T(i_2 W_{21}\gamma_1) \, | \; &= | \, l_T(i_2 W_{21}\gamma_1 i_1 Q\gamma_1 - i_1 Q\gamma_1 i_2 W_{21}\gamma_1) \, | \\
&= | \, b_T(i_1 Q\gamma_1, i_2 W_{21}\gamma_1) \, | \quad \text{(using(3) and (4))} \\
&\leq \|b_T\| \|i_1 Q\gamma_1\| \|i_2 W_{21}\gamma_1\| \quad \text{(since } \; T \in \Delta_X) \\
&\leq \lambda \|b_T\| \|W\|.
\end{aligned}
$$

By combining (7), (8a) and (8b), it results

$$
\begin{aligned}
| \, l_T(i_1 W_{12}\gamma_2 + i_2 W_{21}\gamma_1) \, | \; &= | \, l_T(i_1 W_{12}\gamma_2) + l_T(i_2 W_{21}\gamma_1) \, | \\
&\leq | \, l_T(i_1 W_{12}\gamma_2) \, | + | \, l_T(i_2 W_{21}\gamma_1) \, | \\
&\leq \lambda \|b_T\| \|W\| + \lambda \|b_T\| \|W\| \\
&= K_T \|W\|,
\end{aligned}
$$

where $K_T = 2\lambda\|b_T\|$, and so the proof is complete. □

Theorem 3.2. *Let X_1 and X_2 be infinite-dimensional Banach spaces, and let $X = X_1 \oplus X_2$. Suppose X_1 has property (P_λ) and $A(X_1)$ is weakly amenable. If there exists $\rho \geq 0$ such that $|W|_{X_1} \leq \rho\|W\|$ $(W \in F(X_2))$, then $A(X)$ is weakly amenable.*

Proof. We use (A^*) of Theorem 3.1 to prove the result.
Let $T \in B(X')$ and $W \in F(X)$ be such that $tr(W) = 0$. We show that

$$| \, tr(TW') \, | = | \, l_T(W) \, | \leq K\|W\|.$$

Let $T \in \Delta_X$, (this implies $T \in B(X')$), also let $W \in F(X)$ be such that $tr(W) = 0$. From (2), $T_{11} = i_1' T \gamma_1' : X_1' \to X_1'$ and so $T_{11} \in B(X_1')$, and since $T \in \Delta_X$, then b_T is bounded, and this implies that b_{T11} is also bounded, and so $T_{11} \in \Delta_{X11}$.

Since $A(X_1)$ is weakly amenable, then by Theorem 3.1, there exists $\lambda \in C$ and a constant $K_{1T} > 0$ such that

$$| \, ((T_{11} - \lambda)\tilde{W}') \, | \leq K_T \|\tilde{W}\|, \qquad \tilde{W} \in F(X_1). \tag{9}$$

Taking into account the fact that $trW_{11} + trW_{22} = trW = 0$, we see that

$$
\begin{aligned}
(10) \qquad | \, l_T(i_1 W_{11}\gamma_1 + i_2 W_{22}\gamma_2) \, | \; &= | \, l_{T11}(W_{11}) + l_{T22}(W_{22}) \, | \\
&\leq | \, l_{T11}(W_{11}) \, | + | \, l_{T22}(W_{22}) \, | \\
&\leq K_{1T}\|W_{11}\| + | \, tr((T_{22} - \lambda)W_{22}') \, |
\end{aligned}
$$

(since $A(X_1)$ is weakly amenable, then by (A^*) of Theorem 3.1, we obtain

$$| l_{T11}(W_{11}) | = | tr(T_{11}W'_{11}) | \leq K_{1T}\|W_{11}\|, \quad W_{11} \in F(X_1).$$

We show next that there exists a constant $K_{2T} > 0$, independent of W, such that

$$tr((T_{22} - \lambda)W'_{22}) | \leq K_{2T}\|W_{22}\|.$$

If $R \in F(X_1, X_2)$ and $S \in F(X_2, X_1)$, we have

$$\begin{aligned} & | tr(T_{11}(SR)') - tr(T_{22}(RS)' | \\ & = | tr(T(i_1 S\gamma_2 i_2 R\gamma_1 - i_2 R\gamma_1 i_1 S\gamma_2)') | \\ & = | b_T(i_i S\gamma_2, i_2 R\gamma_1) | \quad \text{(using (4))} \\ & \leq \|b_T\|\|i_1 S\gamma_2\|\|i_2 R\gamma_1\| \quad \text{(since } T \in \Delta_X) \\ & \leq \|b_T\|\|R\|\|S\|. \end{aligned} \tag{11}$$

Using (9) for $\tilde{W} = SR \in F(X_1)$ and (11), we obtain

$$| tr((T_{22}-\lambda)(RS)') | \leq | tr((T_{11}-\lambda)(SR)') | + | tr(T_{11}(SR)')-tr(T_{22}(RS)') |$$

$$\leq K_{1T}\|R\|\|S\| + \|b_T\|\|R\|\|S\|.$$

Then by using the definition in equation (6), we have

$$| tr((T_{22} - \lambda)W') | \leq K_{1T} \mid W_{22}|_{X_1} + \|b_T\| \mid W_{22}|_{X_11}, \tag{12}$$

with $W_{22} = RS \in F(X_2)$.

Thus by our hypothesis that $| W|_{X_1} \leq \rho\|W\|$, $W \in F(X_2)$, it follows from (12) that

$$\begin{aligned} | tr((T_{22} - \lambda)W'_{22}) | & \leq (K_{1T} + \|b_T\|) \mid W_{22}|_{X_1} \\ & \leq (K_{1T} + \|b_T\|).\rho\|W_{22}\| \\ & = K_{2T}\|W_{22}\|, \end{aligned} \tag{13}$$

where $K_{2T} = \rho(K_{1T} + \|b_T\|)$.

Finally, using (7) in Lemma 3.2, (10) and (13), we have

$$\begin{aligned} | l_T(W) | & = | l_T(i_1 W_{11}\gamma_1 + i_2 W_{22}\gamma_2 + i_1 W_{12}\gamma_2 + i_2 W_{21}\gamma_1) | \\ & \leq | l_T(i_1 W_{11}\gamma_1 + i_2 W_{22}\gamma_2 | + | l_T(i_1 W_{12}\gamma_2 + i_2 W_{21}\gamma_1) | \end{aligned}$$

$$\begin{aligned}
&\leq K_{1T}\|W_{11}\| + \mid tr((T_{22}-\lambda)W'_{22}) \mid K_T\|W\| \\
&\leq K_{1T}\|W_{11}\| + K_{2T}\|W_{22}\| + K_T\|W\| \\
&\leq K_{1T}\|W\| + K_{2T}\|W\| + K_T\|W\| \\
&\qquad\qquad (\text{since } W_{11} = \gamma_1 W i_1 \text{ and } W_{22} = \gamma_2 W i_2) \\
&= \tilde{K_T}\|W\|,
\end{aligned}$$

where $\tilde{K_T} = (K_{1T} + K_{2T} + K_T) > 0$
Thus,

$$\mid l_T(W) \mid = \mid tr(TW') \mid \leq \tilde{K_T}\|W\|,$$

with $W \in F(X)$ such that $tr(W) = 0$ and so the condition (A^*) of Theorem 3.1 is satisfied, hence $A(X)$ is weakly amenable. □

Corollary 3.1. *For every Banach space X, the algebra $A(\mathcal{J}_p \oplus X)$ is weakly amenable for $1 < p < \infty$.*

Proof. Since by Example 3.5, $\mathcal{J}_p$ has property (P_λ) for some $\lambda \geq 1$ and by Theorem 3.6 of [2], $A(\mathcal{J}_p)$ is weakly amenable. So the result follows from Theorem 3.2 and the definition of the Banach space $\mathcal{J}_p$. □

Corollary 3.2. *For $1 < p < \infty$, $A(\mathcal{J}_p \oplus J_p)$ is weakly amenable.*

Proof. This follows from Corollary 3.1 with $X = J_p$. □

Remark 3.1. Corollary 3.2 is the same as Proposition 3.3 of [2] in which the author gave a more technical proof. Our proof is simpler and new.

Corollary 3.3. *For $1 < p < \infty$, $A(l_p^2(\mathcal{J}_p))$ is weakly amenable.*

Proof. With $X = \mathcal{J}_p$ in Corollary 3.1, we have $A(\mathcal{J}_p \oplus \mathcal{J}_p) = A(l_p^2(\mathcal{J}_p))$ is weakly amenable. □

Corollary 3.4. *For every Banach space X, the algebra $A(\mathcal{L}_p \oplus X)$ is weakly amenable, for $1 \leq p \leq \infty$.*

Proof. From Theorem 3.6 and Corollary 3.8 of [1], it follows that $A(\mathcal{L}_p)$ is weakly amenable and from the definition of $\mathcal{L}_P$-space and Example 3.2, by using Theorem 3.2, we conclude that $A(\mathcal{L}_p \oplus X)$ is weakly amenable. □

Corollary 3.5. *For $1 \leq p \leq \infty$, the algebra $A(l_p^2(\mathcal{L}_p))$ is weakly amenable.*

Proof. For $X = \mathcal{L}_p$ in Corollary 3.4, we obtain that $A(\mathcal{L}_p \oplus \mathcal{L}_p) = A(l_p^2(\mathcal{L}_p))$ is weakly amenable. □

In [2], Corollary 3.3, it was shown that for every Banach space E, the algebra $A(E \oplus C_p)$ is weakly amenable. We use this fact in our next result in order to prove that there are Banach spaces X without Property (P_λ) for which $A(X)$ is weakly amenable.

Proposition 3.1. *There are Banach spaces X without Property (P_λ) for $\lambda \geq 1$ for which $A(X)$ is weakly amenable.*

Proof. The Banach space $X = \mathcal{L}_1$ has property (P_λ) for $\lambda = 1$ and the Banach space $X = \mathcal{L}_\infty$ has Property (P_λ) for $\lambda = \infty$. By Corollary 1 and Corollary 2 of [12], every complementary subspace of an $\mathcal{L}_1$-space is an $\mathcal{L}_1$ space and every complementary subspace of an $\mathcal{L}_\infty$-space is an $\mathcal{L}_\infty$-space. Thus if E is not an $\mathcal{L}_1$-space ($\mathcal{L}_\infty$-space), the Banach space $X = E \oplus C_p$ is not $\mathcal{L}_1$-space ($\mathcal{L}_\infty$-space) and hence does not necessarily has property (P_λ) for $\lambda = 1, (\lambda = \infty)$ or even $\lambda \geq 1$. □

Yet $A(X) = A(E \oplus C_P)$ is weakly amenable by Corollary 3.3 of [2].

Remark 3.2. The above Proposition shows that X having Property (P_λ) for $\lambda \geq 1$ is not a necessary condition for the weak amenability of $A(X)$.

REFERENCES

1. Blanco, A. – *On the weak amenability of $A(X)$ and its relation with the approximation property*, J. Funct. Analysis, 203 (2003), 1-26.
2. Blanco, A. – *Weak amenability of $A(E)$ and the geometry of E*, J. London Math. Soc. (2) 66(2002), 721-740.
3. Connes, A. – *On the cohomology of operator algebras*, J. Funct. Anal. 28 (1978), 248-253.
4. Dales, H.G.; Ghahramani, F.; Gronbaek, N. – *Derivations into iterated duals of Banach algebras*, Studia Math. 128(1998), 19-54.
5. Haagerup, U. – *All nuclear C^*-algebras are amenable*, Invent. Math. 74 (1983), 305-319.
6. Johnson, B.E. – *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. 127, 1972.

7. Johnson, B.E. – *Weak amenability of group algebras*, Bull. London Math. Soc. 23, 1991.

8. Johnson, W.B. – *Factoring Compact operators*, Israel J. Math., 9 (1972), 337-345.

9. Johnson, W.B. – *A complementary universal conjugate Banach spaces and its relation to approximation problem*, Isreal J. Math., 18 (1972), 301-310.

10. Lindenstrauss, J.; Pelczynski, A. – *Absolutely summing operators,* Studia Math., 29 (1968), 275-326.

11. Lindenstrauss, J.; Rosenthal, H.P. – *The $\mathcal{L}_p$-spaces*, Isreal J. Math., 7 (1969), 325-349.

12. Rutherford, J.R. – *Operator characterizations of $\mathcal{L}_p$-spaces*, Israel J. Math. Vol. 13, (1972), 336-347.

Received: 10.XI.2006

Department of Mathematics,
University of Agriculture,
Abeokuta,
NIGERIA
tosinmewomo@yahoo.com
oguntuase@yahoo.com