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ULTIMATE BOUNDEDNESS AND PERIODICITY RESULTS FOR CERTAIN THIRD ORDER NONLINEAR MATRIX DIFFERENTIAL EQUATIONS

BY

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Abstract. This paper extends some known results on the boundedness of solutions and the existence of periodic solutions of certain vector equations to matrix equations.

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1. Introduction. Let $\mathcal{M}$ denote the space of all real $n\times n$ matrices, $\mathbb{R}^n$ the real n-dimensional Euclidean space and $\mathbb{R}$ the real line $-\infty < t < \infty$. We shall be concerned here with certain properties of solutions of differential equations of the form

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}) \tag{1.1}$$

where $X : \mathbb{R} \to \mathcal{M}$ is the unknown, $A, B \in \mathcal{M}$ are constants, $H : \mathcal{M} \to \mathcal{M}$ and $P : \mathbb{R}\times\mathcal{M}\times\mathcal{M}\times\mathcal{M} \to \mathcal{M}$. The specific properties we shall be interested in are the ultimate boundedness of all solutions and the existence of periodic solutions when P is periodic in t.

In [8], Tejumola establishes conditions under which all solutions of the matrix differential equation,

$$\ddot{X} + A\dot{X} + H(X) = P(t, X, \dot{X}), \tag{1.2}$$

are stable, bounded and periodic (depending on the choice of P). These results are extended to the equation (1.1).

For the special case in which (1.1) is an n-vector equation (so that$X : \mathbb{R} \to \mathbb{R}^n$, $H : \mathbb{R}^n \to \mathbb{R}^n$ and $P : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$) a number of boundedness, stability and existence of periodic solutions results have been established by EZEILO and TEJUMOLA [4], AFUWAPE [1], MENG [5] and others for a number of various vector third order differential equations. The conditions obtained in each of these previous investigations are generalizations of the well-known Routh-Hurwitz conditions

$$a > 0, \;\; c > 0, \;\; ab - c > 0 \tag{1.3}$$

for the stability of the trivial solution of the linear differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0 \tag{1.4}$$

with constant coefficients. Our present investigations are akin to those of TEJUMOLA [8], MENG [5], AFUWAPE [1] and we shall provide extensions of their results to matrix differential equations of the form (1.1).

2. Notations and definitions. Some standard matrix notation will be used. For any $X \in \mathcal{M}$, X^T and x_{ij} $i, j = 1, 2, \cdots, n$ denote the transpose and the elements of X respectively while $(x_{ij})(y_{ij})$ will sometimes denote the product matrix XY of the matrices $X, Y \in \mathcal{M}$. $X_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ and $X^j = (x_{1j}, x_{2j}, \cdots, x_{nj})$ stand for the ith row and jth column of X respectively and $\underline{X} = (X_1, X_2, \cdots, X_n)$ is the n^2 column vector consisting of the n rows of X.

We shall denote by $JH(X)$ the $n^2 \times n^2$ generalised Jacobian matrix associated with the function $H : \mathcal{M} \to \mathcal{M}$ and evaluated at X : that is, $JH(X)$ is the matrix associated with the Jacobian determinant $\frac{\partial(H_1, H_2, \cdots, H_n)}{\partial(X_1, X_2, \cdots, X_n)}$. Corresponding to the constant matrix $A \in \mathcal{M}$ we define an $n^2 \times n^2$ matrix $\tilde{A}$ consisting of n^2 diagonal $n \times n$ matrix$(a_{ij}I_n)$(I_n being the unit $n \times n$ matrix) and such that $(a_{ij}I_n)$ belongs to the $ith - n$ row and $jth - n$ column (that is, counting n at a time) of $\tilde{A}$. In the special case $n = 2, \tilde{A}$ is the 4×4 matrix

$$\begin{pmatrix} a_{11}I_2 & a_{12}I_2 \\ \\ a_{21}I_2 & a_{22}I_2 \end{pmatrix}.$$

Next we introduce an inner product $\langle ., . \rangle$ and a norm $\| \cdot \|$ on $\mathcal{M}$ as follows. For arbitrary $X, Y \in \mathcal{M}, \langle X, Y \rangle =$ trace XY^T. It is easy to check that $\langle X, Y\rangle = \langle Y, X\rangle$ and that $\|X - Y\|^2 = \langle X - Y, X - Y\rangle$ defines a norm of $\mathcal{M}$. Indeed, $\|X\| = |\underline{X}|_{n^2}$ where $|\cdot|_{n^2}$ denotes the usual Euclidean norm in $\mathbb{R}^{n^2}$ and $\underline{X} \in \mathbb{R}^{n^2}$ is as defined above.

Lastly the symbol δ, with or without subscripts, denote finite positive constants whose magnitudes depend only on A, B, H and P. Any δ, with a subscript, retains a fixed identity throughout while the unnumbered ones are not necessarily the same each time they occur.

3. Statement of results. It will be assumed throughout the sequel that $H \in C'(\mathcal{M})$ and that $P \in C(\mathbb{R} \times \mathcal{M} \times \mathcal{M} \times \mathcal{M})$. Further, H and P satisfy conditions for the existence of solutions of (1.1) for any set of preassigned initial conditions.

Theorem 1. *Let $H(0) = 0$ and suppose that*

(i) *the Jacobian matrix $JH(X)$ of $H(X)$ is symmetric and furthermore that the eigenvalues $\lambda_i(JH(X))$ of $JH(X), (i = 1, 2, \ldots, n)$ satisfy for $X \in \mathcal{M}$,*

$$0 < \delta_h \leq \lambda_i(JH(X)) \leq \Delta_h \tag{3.1}$$

where δ_h, Δ_h are finite constants;

(ii) *the matrices $\tilde{A}, \tilde{B}, JH(X)$ are associative and commute pairwise. The eigenvalues $\lambda_i(\tilde{A})$ of $\tilde{A}$ and $\lambda_i(\tilde{B})$ of $\tilde{B}$ $(i = 1, 2, \ldots, n^2)$ satisfy*

$$0 < \delta_a \leq \lambda_i(\tilde{A}) \leq \Delta_a \tag{3.2}$$

$$0 < \delta_b < \lambda_i(\tilde{B}) \leq \Delta_b \tag{3.3}$$

where $\delta_a, \delta_b, \Delta_a, \Delta_b$ are finite constants. Furthermore,

$$\Delta_h \leq k\delta_a\delta_b, \tag{3.4}$$

where

$$k = \min \left\{ \frac{\alpha(1-\beta)\delta_b}{\delta_a(\alpha+\Delta_a)^2} \; ; \; \frac{\alpha(1-\beta)\delta_a}{2(\delta_a+2\alpha)^2} \right\} \tag{3.5}$$

$\alpha > 0, 0 < \beta < 1$ are some constants,

(iii) *P satisfies*

$$\|P(t,X,Y,Z)\| \leq \delta_0 + \delta_1(\|X\| + \|Y\| + \|Z\|) \tag{3.6}$$

for all arbitrary $X, Y, Z \in \mathcal{M}$, where $\delta_0 \geq 0, \delta_1 \geq 0$ are constants and δ_1 is sufficiently small.

Then every solution $X(t)$ of (1.1) satisfies

$$\|X(t)\| \leq \Delta_1, \;\; \|\dot{X}(t)\| \leq \Delta_1, \;\; \|\ddot{X}(t)\| \leq \Delta_1 \tag{3.7}$$

for all t sufficiently large, where Δ_1 is a constant the magnitude of which depends only on δ_0,δ_1,A,B,H and P.

This result provides an extension of a result of Afuwape [1] and Meng [5] for an n-vector.

Theorem 2. *Suppose, further to the conditions of Theorem 1, that P satisfies $P(t,X,Y,Z) = P(t+\omega,X,Y,Z)$ uniformly for all $X,Y,Z \in \mathcal{M}$. Then (1.1) admits of at least one periodic solution with period ω.*

4. Some preliminaries. The following results will be basic to the proofs of Theorems 1 and 2.

Lemma 1 ([8]). *Let $H(0) = 0$ and assume that the matrices $\tilde{A}$ and $JH(X)$ are symmetric and commute for all $X \in \mathcal{M}$. Then*

$$\langle H(X), AX \rangle = \int_0^1 \underline{X}^T \tilde{A} JH(\sigma X) \underline{X} d\sigma \tag{4.1}$$

Lemma 2 ([1]). *If D is a real symmetric $\ell \times \ell$ matrix, then for any $X \in \mathbb{R}^\ell$ we have*

$$\delta_d \|X\|^2 \leq \langle DX, X \rangle \leq \Delta_d \|X\|^2, \tag{4.2}$$

where δ_d, Δ_d are the least and greatest eigenvalues of D, respectively.

Lemma 3 ([1]). *Let Q, D be any two real $\ell \times \ell$ commuting symmetric matrices. Then*

(i) *the eigenvalues* $\lambda_i(QD)$ $(i = 1, 2, \cdots, \ell)$ *of the product matrix* QD *are all real and satisfy*

$$\max_{i\leq j,k\leq \ell} \lambda_j(Q)\lambda_k(D) \geq \lambda_i(QD) \geq \min_{1\leq j,k\leq \ell} \lambda_j(Q)\lambda_k(D);$$

(ii) *the eigenvalues* $\lambda_i(Q + D)$ $(i = 1, 2, \cdots, \ell)$ *of the sum of matrices* Q *and* D *are real and satisfy*

$$\left\{\max_{i\leq j\leq \ell} \lambda_j(Q) + \max_{1\leq k\leq \ell} \lambda_k(D)\right\} \geq \lambda_i(Q+D) \geq \left\{\min_{1\leq j\leq \ell} \lambda_j(Q) + \min_{1\leq k\leq \ell} \lambda_k(D)\right\}$$

Proof of Theorem 1. Let us for convenience, replace Eq.(1.1) by the equivalent system form

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= Z \\ \dot{Z} &= -AZ - BY - H(X) + P(t, X, Y, Z). \end{aligned} \tag{4.3}$$

Our main tool in the proof is the scalar Lyapunov function

$$V : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathbb{R}$$

adapted from [5] and defined for any function $X, Y, Z \in \mathcal{M}$ by

$$\begin{aligned} 2V = {} & \big\{\langle \beta(1-\beta)BX, BX\rangle + \langle 2\alpha A^{-1}BY, Y\rangle + \langle \beta BY, Y\rangle \\ & + \langle \alpha A^{-1}Z, Z\rangle + \langle \alpha(Z + AY)), Y + A^{-1}Z\rangle \\ & \langle Z + AY + (1-\beta)BX, Z + AY + (1-\beta)BX\rangle\} \end{aligned} \tag{4.4}$$

where $\alpha > 0$, $0 < \beta < 1$ are some constants. For each term of this function it is clear that

$$\begin{aligned} \beta(1-\beta)\delta_b\|X\|^2 &\leq \langle \beta(1-\beta)BX, BX\rangle \\ &= \beta(1-\beta)\sum_{i=1}^{n} |BX^i|_n^2 \\ &\leq \beta(1-\beta)\Delta_b\|X\|^2, \end{aligned} \tag{4.4a}$$

$$\begin{aligned} 2\alpha\Delta_a^{-1}\delta_b\|Y\|^2 &\leq \langle 2\alpha A^{-1}BY, Y\rangle = 2\alpha\sum_{i=1}^{n} |A^{-1}BY^i|_n^2 \\ &\leq 2\alpha\delta_a^{-1}\Delta_b\|Y\|^2. \end{aligned} \tag{4.4b}$$

In a similar manner,

$$(4.4c)\qquad \beta\delta_b\|Y\| \le \langle \beta BY, Y\rangle = \beta\sum_{i=1}^{n}|BY^i|_n^2 \le \beta\Delta_b\|Y\|^2,$$

$$(4.4d)\qquad \alpha\Delta_a^{-1}\|Z\|^2 \le \langle \alpha A^{-1}Z, Z\rangle \le \alpha\delta_a^{-1}\|Z\|^2,$$

$$(4.4e)\qquad 0 \le \langle \alpha(Z+AY), Y + A^{-1}Z\rangle \le \nu(\|Y\|^2 + \|Z\|^2),$$

and

$$(4.4f)\qquad \begin{aligned} 0 &\le \langle Z + AY + (1-\beta)BX, Z + AY + (1-\beta)BX\rangle \\ &= \sum_{i=1}^{n}|Z^i + AY^i + (1-\beta)BX^i|_n^2 \\ &\le \mu(\|Z\|^2 + \|Y\|^2 + \|X\|^2), \end{aligned}$$

for some positive constants ν, μ. The estimates above are valid since

$$\sum_{i=1}^{n}|X^i|_n^2 = \sum_{i=1}^{n}|X_i|_n^2 = |\underline{X}|_{n^2}^2 \quad \text{for any} \quad X \in \mathcal{M}.$$

Combining these estimates $(4.4a - 4.4f)$ in (4.4) we obtain that

$$(4.5)\qquad \delta_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \le 2V \le \delta_3(\|X\|^2 + \|Y\|^2 + \|Z\|^2),$$

$$\delta_2 = \min\{\beta(1-\beta)\delta_b; 2\alpha\Delta_a^{-1}\delta_b + \beta\delta_b; \alpha\Delta_a^{-1}\}$$

and

$$\delta_3 = \max\{\beta(1-\beta)\Delta_b + \mu; 2\alpha\delta_a^{-1}\Delta_b + \beta\Delta_b + \nu + \mu; \alpha\delta_a^{-1} + \nu + \mu\}.$$

From (4.5), we have that $V(X, Y, Z) \to \infty$ as $\|X\|^2 + \|Y\|^2 + \|Z\|^2 \to \infty$. To prove our result, it suffices to prove that there exists a constant $\Delta_1 > 0$ such that

$$(4.6)\qquad \|X\|^2 + \|Y\|^2 + \|Z\|^2 \le \Delta_1, \quad \text{for} \quad t \ge T(X_0, Y_0, Z_0),$$

for any solution (X, Y, Z) for (4.3), $(X_0 = X(0), Y_0 = Y(0), Z_0 = Z(0))$.

Let (X, Y, Z) be any solution of (4.3). Then the total derivative of V with respect to t along this solution path is

$$\dot{V} = -U_1 - U_2 - U_3 + U_4 \tag{4.7}$$

where

$$\begin{aligned}
U_1 &= \left\langle \frac{1-\beta}{2} BX, H(X) \right\rangle + \langle \beta ABY, Y \rangle + \left\langle \frac{\alpha}{2} Z, Z \right\rangle \\
U_2 &= \left\langle \frac{1-\beta}{2} BY, H(X) \right\rangle + \langle \alpha BY, Y \rangle + \langle (A + \alpha I)Y, H(X) \rangle \\
U_3 &= \left\langle \frac{1-\beta}{4} BX, H(X) \right\rangle + \left\langle \frac{\alpha}{2} Z, Z \right\rangle + \langle (I + 2\alpha A^{-1})Z, H(X) \rangle \\
U_4 &= \langle (1-\beta)BX + (A + \alpha I)Y + (I + 2\alpha I)Y \\
&\quad + (I + 2\alpha A^{-1})Z, P(t, X, Y, Z) \rangle .
\end{aligned}$$

To arrive at (4.6), we first prove the following:

Lemma 4. *Subject to a conveniently chosen value for k in* (3.5)*, we have for all X, Y, Z*

$$U_j \geq 0, \quad (j = 2, 3).$$

Proof of Lemma 4. For strictly positive constants k_1, k_2 conveniently chosen later, we have

$$\begin{aligned}
&\langle (\alpha I + A)Y, H(X) \rangle \\
&= \left\| k_1 (\alpha I + A)^{1/2} Y + 2^{-1} k_1^{-1} (\alpha I + A)^{1/2} H(X) \right\|^2 \\
&- \langle k_1^2 (\alpha I + A)Y, Y \rangle \\
&- 4^{-1} k_1^{-2} \langle (\alpha I + A) H(X), H(X) \rangle
\end{aligned} \tag{4.8a}$$

and

$$\begin{aligned}
&\langle (I + 2\alpha A^{-1})Z, H(X) \rangle \\
&= \left\| k_2 \left(I + 2\alpha A^{-1}\right)^{1/2} Z + 2^{-1} k_2^{-1} \left(I + 2\alpha A^{-1}\right)^{1/2} H(X) \right\|^2 \\
&- \langle k_2^2 (I + 2\alpha A^{-1})Z, Z \rangle \\
&- \left\langle 4^{-1} k_2^{-2} (I + 2\alpha A^{-1}) H(X), H(X) \right\rangle ,
\end{aligned} \tag{4.8b}$$

thus,

$$\begin{aligned} U_2 &= \|k_1(\alpha I + A)^{1/2}Y + 2^{-1}k_1^{-1}(\alpha I + A)^{1/2}H(X)\|^2 \\ &\quad + \langle 4^{-1}(1-\beta)BX - 4^{-1}k_1^{-1}(\alpha I + A)H(X), H(X)\rangle \\ &\quad + \langle (\alpha B - k_1^2(\alpha I + A)Y, Y\rangle \end{aligned}$$

and

$$\begin{aligned} U_3 &= \|k_2(I + 2\alpha A^{-1})^{1/2}Z + 2^{-1}k_2^{-1}(I + 2\alpha A^{-1})^{1/2}H(X)\|^2 \\ &\quad + \langle (1-\beta)4^{-1}BX - 4^{-1}k_2^{-2}(I + 2\alpha A^{-1})H(X), H(X)\rangle \\ &\quad + \left\langle \left[\frac{\alpha}{2}I - k_2^2(I + 2\alpha A^{-1})\right] Z, Z\right\rangle \end{aligned}$$

By Lemmas 1,2 and 3, we obtain

$$\begin{aligned} U_2 \geq & \left\{\int_0^1 \sigma \int_0^1 \underline{X}^T \left[\frac{1-\beta}{4}\tilde{B}\right.\right. \\ & \left. -\frac{1}{4k_1^2}\left(\alpha\tilde{I} + \tilde{A}\right) JH(\sigma X)\right] JH(\tau\sigma X)\underline{X}d\tau d\sigma \\ & \left. + \underline{Y}^T\left[\alpha\tilde{B} - k_1^2(\alpha\tilde{I} + \tilde{A})\right]\underline{Y}\right\}, \end{aligned} \tag{4.9a}$$

and

$$\begin{aligned} U_3 \geq & \left\{\int_0^1 \sigma \int_0^1 \underline{X}^T \left[\frac{1-\beta}{4}\tilde{B}\right.\right. \\ & \left. -\frac{1}{4k_2^2}\left(\alpha\tilde{I} + 2\alpha\tilde{A}^{-1}\right) JH(\sigma X)\right] JH(\tau\sigma X)\underline{X}d\tau d\sigma \\ & \left. + \underline{Z}^T\left[\frac{\alpha}{2}\tilde{I} - k_2^2(\tilde{I} + 2\alpha\tilde{A}^{-1})\right]\underline{Z}\right\}. \end{aligned} \tag{4.9b}$$

Furthermore, by using Lemmas 2 and 3, we obtain

$$\begin{aligned} U_2 \geq & \left\{\delta_h\left[\frac{1-\beta}{4}\delta_b - \frac{1}{4k_1^2}(\alpha + \Delta_a)\Delta_h\right]\|X\|^2\right. \\ & \left. + \left[\alpha\delta_b - k_1^2(\alpha + \Delta_a)\right]\|Y\|^2\right\}, \end{aligned} \tag{4.9c}$$

and

$$\begin{aligned} U_3 \geq & \left\{\delta_h\left[\frac{1-\beta}{4}\delta_b - \frac{1}{4k_2^2}(1 + 2\alpha\delta_a^{-1})\Delta_h\right]\|X\|^2\right. \\ & \left. + \left[\frac{\alpha}{2} - k_2^2(1 + 2\alpha\delta_a^{-1})\right]\|Z\|^2\right\}. \end{aligned} \tag{4.9d}$$

Thus, we obtain, for all X, Y in $\mathcal{M}$,

$$U_2 \geq 0 \tag{4.10a}$$

if $k_1^2 \leq \frac{\alpha\delta_b}{\alpha+\Delta_a}$ with

$$\Delta_h \leq \frac{k_1^2(1-\beta)\delta_b}{(\alpha+\Delta_a)} \leq \frac{\alpha(1-\beta)\delta_b^2}{(\alpha+\Delta_a)^2} \tag{4.11a}$$

and for all X, Z in $\mathcal{M}$,

$$U_3 \geq 0 \tag{4.10b}$$

if $k_2^2 \leq \frac{\alpha\delta_a}{2(\delta+2\alpha)}$ with

$$\Delta_h \leq \frac{k_2^2(1-\beta)\delta_a\delta_b}{(2\alpha+\delta_a)} \leq \frac{\alpha(1-\beta)\delta_a^2\delta_b}{2(2\alpha+\delta_a)^2}. \tag{4.11b}$$

Combining all the inequalities in (4.10) and (4.11), we have for all X, Y, Z in $\mathcal{M}, U_2 \geq 0$ and $U_3 \geq 0$, if

$$\Delta_h \leq k\delta_a\delta_b$$

with

$$k = \min\left\{\frac{\alpha(1-\beta)\delta_b}{\delta_a(\alpha+\Delta_a)^2};\ \frac{\alpha(1-\beta)\delta_a}{2(\delta_a+2\alpha)^2}\right\} < 1. \tag{4.12}$$

This completes the proof of Lemma 4. $\square$

Finally, we are left with estimates for U_1 and U_4. From (4.7), we clearly have

$$\begin{aligned} U_1 &= \frac{1-\beta}{2}\int_0^1 \underline{X}^T\tilde{B}JH(\sigma X)\underline{X}\,d\sigma + \beta\underline{Y}^T\tilde{A}\tilde{B}\underline{Y} + \frac{\alpha}{2}\underline{Z}^T\underline{Z} \\ &\geq \frac{1-\beta}{2}\delta_b\delta_h\|X\|^2 + \beta\delta_a\delta_b\|Y\|^2 + \frac{\alpha}{2}\|Z\|^2 \\ &\geq \delta_4(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \end{aligned} \tag{4.13}$$

where $\delta_4 = \min\left\{\frac{\delta_b}{2}\delta_h(1-\beta); \beta\delta_a\delta_b; \frac{\alpha}{2}\right\}$. Since $P(t, X, Y, Z)$ satisfies (3.6), by Schwarz's inequality, we obtain

$$
\begin{aligned}
|U_4| &\leq \{(1-\beta)\Delta_b\|X\| + (\alpha + \Delta_a)\|Y\| \\
&+(1 + 2\alpha\delta_a^{-1})\|Z\|\}\|P(t, X, Y, Z)\| \\
&\leq \delta_5(\|X\| + \|Y\| + \|Z\|)[\delta_0 + \delta_1(\|X\| + \|Y\| + \|Z\|)] \\
&\leq 3\delta_1\delta_5(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \\
&+3^{1/2}\delta_0\delta_5(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{1/2},
\end{aligned}
\tag{4.14}
$$

where $\delta_5 = \max\{(1-\beta)\Delta_b; (\alpha + \Delta_a); (1 + 2\alpha\delta_a^{-1})\}$.

Combining inequalities (4.10), (4.13) and (4.14) in (4.7), we obtain

$$
\dot{V} \leq -2\delta_6(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + \delta_7(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{1/2}, \tag{4.15}
$$

where $\delta_6 = \frac{1}{2}(\delta_4 - 3\delta_1\delta_5)$, $\delta_1 < 3^{-1}\delta_5^{-1}\delta_4$, $\delta_7 = 3^{1/2}\delta_0\delta_5$.

If we choose $(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{1/2} \geq \delta_8 = \delta_7\delta_6^{-1}$, inequality (4.15) implies that

$$
\dot{V} \leq -\delta_6(\|X\|^2 + \|Y\|^2 + \|Z\|^2). \tag{4.16}
$$

Then there exists δ_9 such that

$$
\dot{V} \leq -1 \text{ if } \|X\|^2 + \|Y\|^2 + \|Z\|^2 \geq \delta_9^2.
$$

The remainder of the proof of Theorem 1 may now be obtained by use of the estimates (4.5) and (4.16) and an obvious adaptation of the Yoshizawa type reasoning employed in [5]. □

5. Proof of Theorem 2. The proof of this theorem follows as in the proof of Theorem 3([5]). □

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