ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIII, 2007, f.1

GENERAL NATURAL KÄHLER STRUCTURES OF CONSTANT HOLOMORPHIC SECTIONAL CURVATURE ON TANGENT BUNDLES

BY

S. DRUŢĂ* **and V. OPROIU†**

Dedicated to Prof. R. Miron on the occasion of his 80-th birthday
and to the memory of
Prof. Gh. Gheorghiev on the occasion of his 100-th birthday

Abstract. We study some properties of the curvature tensor field of the natural Kählerian structure (G, J) of general type on TM. Namely, we are interested in finding the conditions under which the Kählerian manifold (TM, G, J) has constant holomorphic sectional curvature. The main result is that a certain proportionality factor is expressed as a rational function of the two essential parameters, involved in the definition of the integrable almost complex structure J on TM, their derivatives and the values of the constant sectional curvature of the base manifold (M, g) and the constant holomorphic sectional curvature of (TM, G, J).

Mathematics Subject Classification 2000: primary 53C55, 53C15, 53C05.
Key words: tangent bundle, Riemannian metric, general natural lift.

1. Introduction. The tangent bundle TM of a Riemannian manifold (M, g) has many nice geometric properties, and furnishes important examples arising in various geometric classifications.

It is well knnown (see [17], [21], [22]) that the splitting of the tangent bundle to TM into the vertical and horizontal distributions, defined by the

*Partially supported by the Grant AT 191/2006, CNCSIS, Ministerul Educaţiei şi Cercetării, România

†Partially supported by the Grant 36/1463/2006, CNCSIS, Ministerul Educaţiei şi Cercetării, România

Levi Civita connection of g on M, and the corresponding Sasaki metric lead to an almost Kähler structure on TM. Moreover, the possibility to consider vertical, complete and horizontal lifts on TM leads to interesting geometric structures, studied in the last years (see [1], [2], [7], [8], [19], [20]), and to interesting relations with some problems in Lagrangian and Hamiltonian mechanics.

The second author has studied some properties of a natural lift G, of diagonal type, of the Riemannian metric g and a natural almost complex structure J of diagonal type on TM (see [12], [11], [10], and see also [14], [15]). The condition for (TM, G, J) to be a Kähler Einstein manifold leads to the conditions for (M, g) to have constant sectional curvature, and for (TM, G, J) to have constant sectional holomorphic curvature or to be a locally symmetric space.

In the paper [9], the second author has presented a general expression of the natural almost complex structures on TM. In the definition of the natural almost complex structure J of general type there are involved eight parameters (smooth functions of the density energy on TM). However, from the condition for J to define an almost complex structure, four of the above parameters can be expressed as (rational) functions of the other four parameters. A Riemannian metric G which is a natural lift of general type of the metric g depends on other six parameters. From the conditions for G to be Hermitian with respect to J, one gets that these six parameters can be expressed with the help of the first eight parameters involved in definition of J and two proportionality factors. From the integrability condition for J, we get (beside the condition for the base manifold to be of constant sectional curvature) that other two parameters involved in the definition of J can be expressed as functions of other two essential parameters and their first order derivatives. Thus a natural Hermitian structure (G, J) of general type depends on four essential parameters (two essential parameters involved in the definition of the integrable almost complex structure J and two proportionality factors). From the condition for (G, J) to be almost Kählerian, we get that the second proportionality factor is the derivative of the first one. The family of natural Kählerian structures (G, J) of general type on TM depends on three essential coefficients (two are involved in the expression of J, and the third one is the first proportionality coefficient).

In the present paper we study some properties of the curvature tensor field of the natural Kählerian structure (G, J) of general type on TM. Namely, we are interested in finding the conditions under which the Kählerian manifold (TM, G, J) has constant holomorphic sectional curvature. The main result is that the first proportionality factor is expressed as a rational function of the first two essential parameters, their derivatives and the values of the constant sectional curvature of the base manifold (M, g) and the constant holomorphic sectional curvature of (TM, G, J). Some quite long computations have been done by using the Mathematica package RICCI for doing tensor calculations.

The manifolds, tensor fields and other geometric objects we consider in this paper are assumed to be differentiable of class C^∞ (i.e. smooth). We use the computations in local coordinates in a fixed local chart but many results may be expressed in an invariant form by using the vertical and horizontal lifts. The well known summation convention is used throughout this paper, the range of the indices h, i, j, k, l, r being always $\{1, \dots, n\}$.

2. Preliminary results. Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \longrightarrow M$. Recall that TM has a structure of a $2n$-dimensional smooth manifold, induced from the smooth manifold structure of M. This structure is obtained by using local charts on TM induced from usual local charts on M. If $(U, \varphi) = (U, x^1, \dots, x^n)$ is a local chart on M, then the corresponding induced local chart on TM is $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$, where the local coordinates x^i, y^j, $i, j = 1, \dots, n$, are defined as follows. The first n local coordinates of a tangent vector $y \in \tau^{-1}(U)$ are the local coordinates in the local chart (U, φ) of its base point, i.e. $x^i = x^i \circ \tau$, by an abuse of notation. The last n local coordinates y^j, $j = 1, \dots, n$, of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis in $T_{\tau(y)}M$ defined by the local chart (U, φ). Due to this special structure of differentiable manifold for TM, it is possible to introduce the concept of M-tensor field on it. The M-tensor fields are defined by their components with respect to the induced local charts on TM (hence they are defined locally), but they can be interpreted as some (partial) usual tensor fields on TM. However, the essential quality of an M-tensor field on TM is that the local coordinate change rule of its components with respect to the change of induced local charts is the same as the local coordinate change rule of the components of a usual tensor field on M with respect to the change of local charts on M. More precisely, an M-tensor field of type (p, q) on TM is defined by sets of n^{p+q} components (functions depending on x^i and y^i), with p upper indices and q lower indices, assigned to induced local charts $(\tau^{-1}(U), \Phi)$ on TM, such that the local coordinate change rule of these components (with respect to induced local charts on TM) is that of the local coordinate components of a tensor field of type (p, q) on the base manifold M (with respect to usual local charts on M), when a change of local charts on M (and hence on TM) is performed (see [6] for further details); e.g., the components y^i, $i = 1, \dots, n$, corresponding to the last n local coordinates of a tangent vector y, assigned to the induced local chart $(\tau^{-1}(U), \Phi)$ define an M-tensor field of type $(1, 0)$ on TM. A usual tensor field of type (p, q) on M may be thought of as an M-tensor field of type (p, q) on TM. If the considered tensor field on M is covariant only, the corresponding M-tensor field on TM may be identified with the induced (pullback by τ) tensor field on TM. Some useful M-tensor fields on TM may be obtained as follows. Let $u : [0, \infty) \longrightarrow \mathbb{R}$ be a smooth function and let $\|y\|^2 = g_{\tau(y)}(y, y)$ be the square of the norm of the tangent vector $y \in \tau^{-1}(U)$. If δ^i_j are the Kronecker symbols (in fact, they are the local coordinate components of the identity tensor field I on M), then the components $u(\|y\|^2)\delta^i_j$ define an M-tensor field of type $(1, 1)$ on TM. Similarly, if $g_{ij}(x)$ are the local coordinate components of the metric tensor field g on M in the local chart (U, φ), then the components $u(\|y\|^2)g_{ij}$ define a symmetric M-tensor field of type $(0, 2)$ on TM. The components $g_{0i} = y^k g_{ki}$ define an M-tensor field of type $(0, 1)$ on TM.

Denote by $\dot{\nabla}$ the Levi Civita connection of the Riemannian metric g on M. Then we have the direct sum decomposition

$$TTM = VTM \oplus HTM \tag{1}$$

of the tangent bundle to TM into the vertical distribution $VTM = \mathrm{Ker}\ \tau_*$ and the horizontal distribution HTM defined by $\dot{\nabla}$. The set of vector fields $(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n})$ on $\tau^{-1}(U)$ defines a local frame field for VTM and for HTM we have the local frame field $(\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n})$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i}\frac{\partial}{\partial y^h}, \quad \Gamma^h_{0i} = y^k \Gamma^h_{ki},$$

and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of g.

The set $(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n})$ defines a local frame on TM, adapted to the direct sum decomposition (1). Remark that

$$\frac{\partial}{\partial y^i} = \left(\frac{\partial}{\partial x^i}\right)^V, \quad \frac{\delta}{\delta x^i} = \left(\frac{\partial}{\partial x^i}\right)^H,$$

where X^V and X^H denote the vertical and horizontal lift of the vector field X on M respectively. We can use the vertical and horizontal lifts in order to obtain invariant expressions for some results in this paper. However, we should prefer to work in local coordinates since the formulas are obtained easier and, in a certain sense, they are more natural.

We can easily obtain the following

Lemma 2.1. *If $n > 1$ and u, v are smooth functions on TM such that*

$$ug_{ij} + vg_{0i}g_{0j} = 0,$$

on the domain of any induced local chart on TM, then $u = 0,\ v = 0$.

Remark. In a similar way we obtain from the condition

$$u\delta^i_j + vg_{0j}y^i = 0$$

the relations $u = v = 0$.

Consider the energy density of the tangent vector y with respect to the Riemannian metric g

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y, y) = \frac{1}{2}g_{ik}(x)y^iy^k, \quad y \in \tau^{-1}(U). \tag{2}$$

Obviously, we have $t \in [0, \infty)$ for all $y \in TM$.

Denote by $C = y^i\frac{\partial}{\partial y^i}$ the Liouville vector field on TM and by $\widetilde{C} = y^i\frac{\delta}{\delta x^i}$ the similar horizontal vector field on TM. Consider the real valued smooth functions $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ defined on $[0, \infty) \subset \mathbf{R}$. A natural 1-st order almost complex structure on TM, defined by the Riemannian metric g, is obtained just like the natural 1-st order lifts of g to TM are obtained in [5], [4].

Theorem 2.2. *The natural tensor field J of type $(1,1)$ on TM, given by*

$$\begin{cases} J\frac{\delta}{\delta x^i} = a_1(t)\frac{\partial}{\partial y^i} + b_1(t)g_{0i}C + a_4(t)\frac{\delta}{\delta x^i} + b_4(t)g_{0i}\widetilde{C}, \\ \\ J\frac{\partial}{\partial y^i} = a_3(t)\frac{\partial}{\partial y^i} + b_3(t)g_{0i}C - a_2(t)\frac{\delta}{\delta x^i} - b_2(t)g_{0i}\widetilde{C}, \end{cases} \tag{3}$$

defines an almost complex structure on TM, if and only if $a_4 = -a_3, b_4 = -b_3$ and the coefficients a_1, a_2, a_3, b_1, b_2 and b_3 are related by

$$a_1a_2 = 1 + a_3^2\ , \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1 + (a_3 + 2tb_3)^2. \tag{4}$$

Remark. From the conditions (4) we have that the coefficients $a_1, a_2, a_1+2tb_1, a_2+2tb_2$ cannot vanish and have the same sign. We assume that $a_1 > 0,\ a_2 > 0,\ a_1+2tb_1 > 0,\ a_2+2tb_2 > 0$ for all $t \geq 0$.

Remark. The relations (4) allow us to express two of the coefficients $a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3$ as functions of the other four; e.g. we have:

$$a_2 = \frac{1+a_3^2}{a_1}, \quad b_2 = \frac{2a_3b_3 - a_2b_1 + 2tb_3^2}{a_1+2tb_1}. \tag{5}$$

The integrability condition for the above almost complex structure J on a manifold M is characterized by the vanishing of its Nijenhuis tensor field N_J, defined by

$$N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y],$$

for all vector fields X and Y on M.

The integrability conditions obtained in [9] are given in the theorem:

Theorem 2.3. *Let (M,g) be an $n(>2)$-dimensional connected Riemannian manifold. The almost complex structure J defined by (3) on TM is integrable if and only if (M,g) has constant sectional curvature c and the coefficients b_1, b_2, b_3 are given by:*

$$\begin{cases} b_1 = \frac{2c^2ta_2^2+2cta_1a_2'+a_1a_1'-c+3ca_3^2}{a_1-2ta_1'-2cta_2-4ct^2a_2'}, \\ b_2 = \frac{2ta_3'^2-2ta_1'a_2'+ca_2^2+2cta_2a_2'+a_1a_2'}{a_1-2ta_1'-2cta_2-4ct^2a_2'}, \\ b_3 = \frac{a_1a_3'+2ca_2a_3+4cta_2'a_3-2cta_2a_3'}{a_1-2ta_1'-2cta_2-4ct^2a_2'}. \end{cases} \tag{6}$$

Remark. The second relation in (5) (or in (4))is identically fulfilled by the expressions b_1, b_2, b_3 in (6).

Remark. In the case where $a_3 = 0$ it follows $b_3 = 0$ too, and we have:

$$a_2 = \frac{1}{a_1},\ b_1 = \frac{a_1a_1' - c}{a_1 - 2ta_1'},\ b_2 = \frac{c - a_1a_1'}{a_1(a_1^2 - 2ct)}$$

(compare with the corresponding expressions from [11] and [18]).

In the paper [9], the second author studied the conditions under which a Riemannian metric G of natural type on TM, defined by

$$
\begin{aligned}
G\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right) &= c_1 g_{ij} + d_1 g_{0i} g_{0j} = G^{(1)}_{ij},\\
G\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right) &= c_2 g_{ij} + d_2 g_{0i} g_{0j} = G^{(2)}_{ij},\\
G\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right) &= G\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\right) = c_3 g_{ij} + d_3 g_{0i} g_{0j} = G^{(3)}_{ij},
\end{aligned}
\tag{7}
$$

where $c_1, c_2, c_3, d_1, d_2, d_3$ are six smooth functions of the density energy on TM, is almost Hermitian with respect to the general almost complex structure J, i.e.

$$G(JX, JY) = G(X, Y),$$

for all vector fields X, Y on TM. He proved the following result

Theorem 2.4. *The family of natural, Riemannian metrics G on TM such that (TM, G, J) is an almost Hermitian manifold, is given by* (7), *provided that the coefficients $c_1, c_2, c_3, d_1, d_2,$ and d_3 are related to the coefficients $a_1, a_2, a_3, b_1, b_2,$ and b_3 by the following proportionality relations*

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} = \frac{c_3}{a_3} = \lambda$$

$$\frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \frac{c_3 + 2td_3}{a_3 + 2tb_3} = \lambda + 2t\mu,$$

where the proportionality coefficients $\lambda > 0$ and $\lambda + 2t\mu > 0$ are functions depending on t.

Remark. In the case where $a_3 = 0$, it follows that $c_3 = d_3 = 0$ and we obtain the almost Hermitian structure considered in [18]. Remark that the functions used in [18] are slightly different of the functions used in the present paper. Moreover, if $\lambda = 1$ and $\mu = 0$, we obtain the almost Kählerian structure considered in [12].

Considering the two-form Ω defined by the almost Hermitian structure (G, J) on TM

$$\Omega(X, Y) = G(X, JY),$$

for all vector fields X, Y on TM, the second author obtained the following result:

Proposition 2.5. *The expression of the* 2*-form* Ω *in the local adapted frame* $(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n})$ *on* TM*, is given by*

$$\Omega\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0,\ \Omega\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 0,\ \Omega\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = \lambda g_{ij} + \mu g_{0i} g_{0j}$$

or, equivalently

$$\Omega = (\lambda g_{ij} + \mu g_{0i} g_{0j}) \dot{\nabla} y^i \wedge dx^j, \tag{8}$$

where $\dot{\nabla} y^i = dy^i + \Gamma^i_{0h} dx^h$ *is the absolute differential of* y^i*.*

Next, by calculating the exterior differential of Ω, he obtained:

Theorem 2.6. *The almost Hermitian structure* (TM, G, J) *is almost Kählerian if and only if*

$$\mu = \lambda'.$$

Thus the family of general almost Kählerian structures on TM depends on five essential coefficients $a_1, a_3, b_1, b_3, \lambda$. Combining the results from the theorems (2.4), (2.3) and (2.6), we obtain that a general natural Kählerian structure (G, J) on TM is defined by three essential coefficients a_1, a_3, λ. However, these coefficients must satisfy the supplementary conditions $a_1 > 0, a_1 + 2tb_1 > 0, \lambda > 0, \lambda + 2t\mu > 0$. Examples of such structures can be found in [18] (see also [12]).

3. General natural Kähler structures of constant holomorphic sectional curvature on tangent bundles. The Levi-Civita connection ∇ of the Riemannian manifold (TM, G) is obtained from the formula

$$\begin{aligned} 2G(\nabla_X Y, Z) &= X(G(X,Z)) + Y(G(X,Z)) - Z(G(X,Y)) \\ &+ G([X,Y],Z) - G([X,Z],Y) - G([Y,Z],X); \\ &\forall X, Y, Z \in \chi(M) \end{aligned}$$

and is characterized by the conditions

$$\nabla G = 0,\ T = 0,$$

where T is the torsion tensor of ∇.

In the case of the tangent bundle TM we can obtain the explicit expression of ∇. The symmetric $2n \times 2n$ matrix

$$\begin{pmatrix} G^{(1)}_{ij} & G^{(3)}_{ij} \\ G^{(3)}_{ij} & G^{(2)}_{ij} \end{pmatrix}$$

associated to the metric G in the base $(\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n})$ has the inverse

$$\begin{pmatrix} H^{ij}_{(1)} & H^{ij}_{(3)} \\ H^{ij}_{(3)} & H^{ij}_{(2)} \end{pmatrix}$$

where the entries are the blocks

$$\begin{aligned} H^{kl}_{(1)} &= p_1 g^{kl} + q_1 y^k y^l \\ H^{kl}_{(2)} &= p_2 g^{kl} + q_2 y^k y^l \\ H^{kl}_{(3)} &= p_3 g^{kl} + q_3 y^k y^l. \end{aligned} \tag{9}$$

Here g^{kl} are the components of the inverse of the matrix (g_{ij}) and $p_1, q_1, p_2, q_2, p_3, q_3 : [0, \infty) \to \mathbb{R}$, some real smooth functions. Their expressions are obtained by solving the system:

$$\begin{cases} G^{(1)}_{ih} H^{hk}_{(1)} + G^{(3)}_{ih} H^{hk}_{(3)} = \delta^k_i \\ G^{(1)}_{ih} H^{hk}_{(3)} + G^{(3)}_{ih} H^{hk}_{(2)} = 0 \\ G^{(3)}_{ih} H^{hk}_{(1)} + G^{(2)}_{ih} H^{hk}_{(3)} = 0 \\ G^{(3)}_{ih} H^{hk}_{(3)} + G^{(2)}_{ih} H^{hk}_{(2)} = \delta^k_i, \end{cases} \tag{10}$$

in which we substitute the relations (7) and (9). By using Lemma 1, we get p_1, p_2, p_3 as functions of c_1, c_2, c_3

$$p_1 = \frac{c_2}{c_1 c_2 - c_3^2}, \quad p_2 = \frac{c_1}{c_1 c_2 - c_3^2}, \quad p_3 = -\frac{c_3}{c_1 c_2 - c_3^2} \tag{11}$$

and q_1, q_2, q_3 as functions of $c_1, c_2, c_3,\ d_1, d_2, d_3,\ p_1, p_2, p_3$

$$\begin{aligned} q_1 &= -\frac{c_2 d_1 p_1 - c_3 d_3 p_1 - c_3 d_2 p_3 + c_2 d_3 p_3 + 2 d_1 d_2 p_1 t - 2 d_3^2 p_1 t}{c_1 c_2 - c_3^2 + 2 c_2 d_1 t + 2 c_1 d_2 t - 4 c_3 d_3 t + 4 d_1 d_2 t^2 - 4 d_3^2 t^2}, \\ q_2 &= -\frac{d_2 p_2 + d_3 p_3}{c_2 + 2 d_2 t} \end{aligned} \tag{12}$$

$$+\frac{(c_3+2d_3t)[(d_3p_1+d_2p_3)(c_1+2d_1t)-(d_1p_1+d_3p_3)(c_3+2d_3t)]}{(c_2+2d_2t)[(c_1+2d_1t)(c_2+2d_2t)-(c_3+2d_3t)^2]},$$
$$q_3=-\frac{(d_3p_1+d_2p_3)(c_1+2d_1t)-(d_1p1+d_3p_3)(c_3+2d_3t)}{(c_1+2d_1t)(c_2+2d_2t)-(c_3+2d_3t)^2}$$

Next we can obtain the expression of the Levi Civita connection of the Riemannian metric G on TM.

Theorem 3.1. *The Levi-Civita connection ∇ of G has the following expression in the local adapted frame* $\left(\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^n},\frac{\delta}{\delta x^1},\dots,\frac{\delta}{\delta x^n}\right)$

$$\begin{cases}\nabla_{\frac{\partial}{\partial y^i}}\dfrac{\partial}{\partial y^j}=Q^h_{ij}\dfrac{\partial}{\partial y^h}+\widetilde{Q}^h_{ij}\dfrac{\delta}{\delta x^h}, & \nabla_{\frac{\delta}{\delta x^i}}\dfrac{\partial}{\partial y^j}=(\Gamma^h_{ij}+\widetilde{P}^h_{ji})\dfrac{\partial}{\partial y^h}+P^h_{ji}\dfrac{\delta}{\delta x^h}\\ \nabla_{\frac{\partial}{\partial y^i}}\dfrac{\delta}{\delta x^j}=P^h_{ij}\dfrac{\delta}{\delta x^h}+\widetilde{P}^h_{ij}\dfrac{\partial}{\partial y^h}, & \nabla_{\frac{\delta}{\delta x^i}}\dfrac{\delta}{\delta x^j}=(\Gamma^h_{ij}+\widetilde{S}^h_{ij})\dfrac{\delta}{\delta y^h}+S^h_{ij}\dfrac{\partial}{\partial y^h},\end{cases}$$

where Γ^h_{ij} are the Christoffel symbols of the connection $\dot\nabla$ and M-tensor fields appearing as coefficients in the above expressions are given as

(13)
$$\begin{cases}Q^h_{ij}=\frac12(\partial_iG^{(2)}_{jk}+\partial_jG^{(2)}_{ik}-\partial_kG^{(2)}_{ij})H^{kh}_{(2)}+\frac12(\partial_iG^{(3)}_{jk}+\partial_jG^{(3)}_{ik})H^{kh}_{(3)},\\ \widetilde{Q}^h_{ij}=\frac12(\partial_iG^{(2)}_{jk}+\partial_jG^{(2)}_{ik}-\partial_kG^{(2)}_{ij})H^{kh}_{(3)}+\frac12(\partial_iG^{(3)}_{jk}+\partial_jG^{(3)}_{ik})H^{kh}_{(1)},\\ \\ P^h_{ij}=\frac12(\partial_iG^{(3)}_{jk}-\partial_kG^{(3)}_{ij})H^{kh}_{(3)}+\frac12(\partial_iG^{(1)}_{jk}+R^l_{0jk}G^{(2)}_{li})H^{kh}_{(1)},\\ \widetilde{P}^h_{ij}=\frac12(\partial_iG^{(3)}_{jk}-\partial_kG^{(3)}_{ij})H^{kh}_{(2)}+\frac12(\partial_iG^{(1)}_{jk}+R^l_{0jk}G^{(2)}_{li})H^{kh}_{(3)},\\ \\ S^h_{ij}=-\frac12(\partial_kG^{(2)}_{ij}+R^l_{0ij}G^{(2)}_{lk})H^{kh}_{(2)}+c_3R_{i0jk}H^{kh}_{(3)},\\ \widetilde{S}^h_{ij}=-\frac12(\partial_kG^{(1)}_{ij}+R^l_{0ij}G^{(2)}_{lk})H^{kh}_{(3)}+c_3R_{i0jk}H^{kh}_{(1)},\end{cases}$$

where R^h_{kij} are the components of the curvature tensor field of the Levi Civita connection $\dot\nabla$ of the base manifold (M,g).

Taking into account the expressions (7), (9) and by using the formulas (11), (12) we can obtain the detailed expressions of $P^h_{ij},Q^h_{ij},S^h_{ij},\widetilde{P}^h_{ij},\widetilde{Q}^h_{ij},\widetilde{S}^h_{ij}$.

The curvature tensor field K of the connection ∇ is defined by the well known formula

$$K(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z,\quad X,Y,Z\in\Gamma(TM).$$

By using the local adapted frame $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right)$, $i,j=1,\dots,n$ we obtain, after a standard straightforward computation

$$\begin{aligned}
K\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} &= XXXX^h_{kij}\frac{\delta}{\delta x^h} + XXXY^h_{kij}\frac{\partial}{\partial y^h},\\
K\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} &= XXYX^h_{kij}\frac{\delta}{\delta x^h} + XXYY^h_{kij}\frac{\partial}{\partial y^h},\\
K\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\delta}{\delta x^k} &= YYXX^h_{kij}\frac{\delta}{\delta x^h} + YYXY^h_{kij}\frac{\partial}{\partial y^h},\\
K\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k} &= YYYX^h_{kij}\frac{\delta}{\delta x^h} + YYYY^h_{kij}\frac{\partial}{\partial y^h},\\
K\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} &= YXXX^h_{kij}\frac{\delta}{\delta x^h} + YXXY^h_{kij}\frac{\partial}{\partial y^h},\\
K\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} &= YXYX^h_{kij}\frac{\delta}{\delta x^h} + YXYY^h_{kij}\frac{\partial}{\partial y^h},
\end{aligned}$$

where the M-tensor fields appearing as coefficients are given by

$$\begin{aligned}
XXXX^h_{kij} &= \widetilde{S}^h_{il}\widetilde{S}^l_{jk} + P^h_{li}S^l_{jk} - \widetilde{S}^h_{jl}\widetilde{S}^l_{ik} - P^h_{lj}S^l_{ik} + R^h_{kij} + R^l_{0ij}P^h_{lk}\\
XXXY^h_{kij} &= \widetilde{S}^l_{jk}S^h_{il} + \widetilde{P}^h_{li}S^l_{jk} - \widetilde{S}^l_{ik}S^h_{jl} - \widetilde{P}^h_{lj}S^l_{ik} + \widetilde{P}^h_{lk}R^l_{0ij}\\
&\quad - \frac{1}{2}\dot{\nabla}_i R^r_{0jk}G^{(2)}_{rl}H^{(3)}_{hl} + c_3\dot{\nabla}_i R_{j0kh},\\
XXYX^h_{kij} &= \widetilde{P}^l_{kj}P^h_{li} + P^l_{kj}\widetilde{S}^h_{il} - \widetilde{P}^l_{ki}P^h_{lj} - P^l_{ki}\widetilde{S}^h_{jl} + R^l_{0ij}\widetilde{Q}^h_{lk},\\
XXYY^h_{kij} &= \widetilde{P}^l_{kj}\widetilde{P}^h_{li} + P^l_{kj}S^h_{il} - \widetilde{P}^l_{ki}\widetilde{P}^h_{lj} - P^l_{ki}S^h_{jl} + R^l_{0ij}Q^h_{lk} + R^h_{kij},\\
YYXX^h_{kij} &= \partial_i P^h_{jk} - \partial_j P^h_{ik} + \widetilde{P}^l_{jk}\widetilde{Q}^h_{il} + P^l_{jk}P^h_{il} - \widetilde{P}^l_{ik}\widetilde{Q}^h_{jl} - P^l_{ik}P^h_{jl},\\
YYXY^h_{kij} &= \partial_i \widetilde{P}^h_{jk} - \partial_j \widetilde{P}^h_{ik} + \widetilde{P}^l_{jk}Q^h_{il} + P^l_{jk}\widetilde{P}^h_{il} - \widetilde{P}^l_{ik}Q^h_{jl} - P^l_{ik}\widetilde{P}^h_{jl},\\
YYYX^h_{kij} &= \partial_i \widetilde{Q}^h_{jk} - \partial_j \widetilde{Q}^h_{ik} + Q^l_{jk}\widetilde{Q}^h_{il} + \widetilde{Q}^l_{jk}P^h_{il} - Q^l_{ik}\widetilde{Q}^h_{jl} - \widetilde{Q}^l_{ik}P^h_{jl},\\
YYYY^h_{kij} &= \partial_i Q^h_{jk} - \partial_j Q^h_{ik} + Q^l_{jk}Q^h_{il} + \widetilde{Q}^l_{jk}\widetilde{P}^h_{il} - Q^l_{ik}Q^h_{jl} - \widetilde{Q}^l_{ik}\widetilde{P}^h_{jl},\\
YXXX^h_{kij} &= \partial_i \widetilde{S}^h_{jk} + S^l_{jk}\widetilde{Q}^h_{il} + \widetilde{S}^l_{jk}P^h_{il} - \widetilde{P}^l_{ik}P^h_{lj} - P^l_{ik}\widetilde{S}^h_{jl}\\
&\quad - \dot{\nabla}_j R^r_{0ik}G^{(2)}_{rl}H^{(3)}_{hl},\\
YXXY^h_{kij} &= \partial_i S^h_{jk} + S^l_{jk}Q^h_{il} + \widetilde{S}^l_{jk}\widetilde{P}^h_{il} - \widetilde{P}^l_{ik}\widetilde{P}^h_{lj} - P^l_{ik}S^h_{jl}\\
&\quad - \dot{\nabla}_j R^r_{0ik}G^{(2)}_{rl}H^{(1)}_{hl},
\end{aligned}$$

$$YXYX^h_{kij} = \partial_i P^h_{kj} + \widetilde{P}^l_{kj}\widetilde{Q}^h_{il} + P^l_{kj}P^h_{il} - Q^l_{ik}P^h_{lj} - \widetilde{Q}^l_{ik}\widetilde{S}^h_{jl},$$
$$YXYY^h_{kij} = \partial_i \widetilde{P}^h_{kj} + \widetilde{P}^l_{kj}Q^h_{il} + P^l_{kj}\widetilde{P}^h_{il} - Q^l_{ik}\widetilde{P}^h_{lj} - \widetilde{Q}^l_{ik}S^h_{jl}.$$

Remark that, due to the condition for (M,g) to have constant sectional curvature, we have $\dot{\nabla}R^h_{kij}=0$, and the above formulas become simpler. In order to obtain the final expressions of the above M-tensor fields, we need the first and second order partial derivatives with respect to the tangential coordinates y^i of the usual tensor fields involved in the definition of the Riemannian metric G.

$$\begin{aligned}
\partial_i G^{(\alpha)}_{jk} &= c'_\alpha g_{0i}g_{jk} + d'_\alpha g_{0i}g_{0j}g_{0k} + d_\alpha g_{ij}g_{0k} + d_\alpha g_{0i}g_{jk}\\
\partial_i H^{jk}_{(\alpha)} &= p'_\alpha g^{jk}g_{0i} + q'_\alpha g_{0i}y^jy^k + q_\alpha \delta^j_i y^k + q_\alpha y^j\delta^k_i\\
\partial_i\partial_j G^{(\alpha)}_{kl} &= c''_\alpha g_{0i}g_{0j}g_{kl} + c'_\alpha g_{ij}g_{kl} + d''_\alpha g_{0j}g_{0k}g_{0l} + d'_\alpha g_{ij}g_{0k}g_{0l}\\
&\quad + d'_\alpha g_{0j}g_{ik}g_{0l} + d'_\alpha g_{0j}g_{0k}g_{il} + d'_\alpha g_{0i}g_{jk}g_{0l}\\
&\quad + d'_\alpha g_{0i}g_{0k}g_{jl} + d_\alpha g_{jk}g_{il} + d_\alpha g_{ik}g_{jl}, \qquad \alpha = 1,2,3
\end{aligned}$$

Next we get the first order partial derivatives with respect to the tangential coordinates y^i of the M-tensor fields $P^h_{ij}, Q^h_{ij}, S^h_{ij}, \widetilde{P}^h_{ij}, \widetilde{Q}^h_{ij}, \widetilde{S}^h_{ij}$

$$\begin{aligned}
\partial_i Q^h_{jk} &= \frac{1}{2}\partial_i H^{hl}_{(2)}(\partial_j G^{(2)}_{kl} + \partial_k G^{(2)}_{jl} - \partial_l G^{(2)}_{jk})\\
&\quad + \frac{1}{2}H^{hl}_{(2)}(\partial_i\partial_j G^{(2)}_{kl} + \partial_i\partial_k G^{(2)}_{jl} - \partial_i\partial_l G^{(2)}_{jk})\\
&\quad + \frac{1}{2}\partial_i H^{hl}_{(3)}(\partial_j G^{(3)}_{kl} + \partial_k G^{(3)}_{jl}) + \frac{1}{2}H^{hl}_{(3)}(\partial_i\partial_j G^{(3)}_{kl} + \partial_i\partial_k G^{(3)}_{jl}),\\
\partial_i \widetilde{Q}^h_{jk} &= \frac{1}{2}\partial_i H^{hl}_{(3)}(\partial_j G^{(2)}_{kl} + \partial_k G^{(2)}_{jl} - \partial_l G^{(2)}_{jk})\\
&\quad + \frac{1}{2}H^{hl}_{(3)}(\partial_i\partial_j G^{(2)}_{kl} + \partial_i\partial_k G^{(2)}_{jl} - \partial_i\partial_l G^{(2)}_{jk})\\
&\quad + \frac{1}{2}\partial_i H^{hl}_{(1)}(\partial_j G^{(3)}_{kl} + \partial_k G^{(3)}_{jl}) + \frac{1}{2}H^{hl}_{(1)}(\partial_i\partial_j G^{(3)}_{kl} + \partial_i\partial_k G^{(3)}_{jl}),\\
\partial_i \widetilde{P}^h_{jk} &= \frac{1}{2}\partial_i H^{hl}_{(2)}(\partial_j G^{(3)}_{kl} - \partial_l G^{(3)}_{jk}) + \frac{1}{2}H^{hl}_{(2)}(\partial_i\partial_j G^{(3)}_{kl} - \partial_i\partial_l G^{(3)}_{jk})\\
&\quad + \frac{1}{2}\partial_i H^{hl}_{(3)}(\partial_j G^{(1)}_{kl} + R^r_{0kl}G^{(2)}_{rj})\\
&\quad + \frac{1}{2}H^{hl}_{(3)}(\partial_i\partial_j G^{(1)}_{kl} + R^r_{ikl}G^{(2)}_{rj} + R^r_{0kl}\partial_i G^{(2)}_{rj}),
\end{aligned}$$

$$\partial_i P^h_{jk} = \frac{1}{2}\partial_i H^{hl}_{(3)}(\partial_j G^{(3)}_{kl} - \partial_l G^{(3)}_{jk}) + \frac{1}{2}H^{hl}_{(3)}(\partial_i\partial_j G^{(3)}_{kl} - \partial_i\partial_l G^{(3)}_{jk})$$
$$+ \frac{1}{2}\partial_i H^{hl}_{(1)}(\partial_j G^{(1)}_{kl} + R^r_{0kl}G^{(2)}_{rj})$$
$$+ \frac{1}{2}H^{hl}_{(1)}(\partial_i\partial_j G^{(1)}_{kl} + R^r_{ikl}G^{(2)}_{rj} + R^r_{0kl}\partial_i G^{(2)}_{rj}),$$
$$\partial_i S^h_{jk} = -\frac{1}{2}\{(\partial_i\partial_r G^{(1)}_{jk} + R^l_{ijk}G^{(2)}_{lr})H^{rh}_{(2)} + (\partial_r G^{(1)}_{jk} + R^l_{0jk}G^{(2)}_{lr})\partial_i H^{rh}_{(2)}\}$$
$$+ c'_3 g_{0i}R_{j0kr}H^{rh}_{(3)} + c_3(R_{jikr}H^{rh}_{(3)} + R_{j0kr}\partial_i H^{rh}_{(3)}),$$
$$\partial_i \widetilde{S}^h_{jk} = -\frac{1}{2}\{(\partial_i\partial_r G^{(1)}_{jk} + R^l_{ijk}G^{(2)}_{lr})H^{rh}_{(3)} + (\partial_r G^{(1)}_{jk} + R^l_{0jk}G^{(2)}_{lr})\partial_i H^{rh}_{(3)}\}$$
$$+ c'_3 g_{0i}R_{j0kr}H^{rh}_{(1)} + c_3(R_{jikr}H^{rh}_{(1)} + R_{j0kr}\partial_i H^{rh}_{(1)}).$$

Now we have to replace these derivatives, next the explicit expressions of the M-tensor fields $P^h_{ij}, Q^h_{ij}, S^h_{ij}, \widetilde{P}^h_{ij}, \widetilde{Q}^h_{ij}, \widetilde{S}^h_{ij}$, the expressions (11), (12) of the functions p_1, p_2, p_3, q_1, q_2, q_3 and of their derivatives in order to obtain the components of the curvature tensor as functions of a_1, a_2, a_3 and their derivatives of first, second and third order only. The expressions are obtained by using the Mathematica package RICCI. It was not convenient to think $a_1, a_2, a_3, b_1, b_2, b_3$, as well as $c_1, c_2, c_3, d_1, d_2, d_3$ and $p_1, p_2, p_3, q_1, q_2, q_3$ as functions of t since RICCI did not make some useful factorizations after the command TensorSimplify. We decided to consider these functions as well as their derivatives of first, second and third order, as constants, the tangent vector y as a first order tensor, the components $G^{(1)}_{ij}, G^{(2)}_{ij}, G^{(3)}_{ij}$, $H^{ij}_{(1)}, H^{ij}_{(2)}, H^{ij}_{(3)}$ as second order tensors and so on, on the Riemannian manifold M, the associated indices being h, i, j, k, l, r, s.

The holomorphic sectional curvature of a Kälerian manifold (TM, G, J) is given by the formula:

$$K_0(X,Y)Z = \frac{k}{4}[G(Y,Z)X - G(X,Z)Y$$
$$+ G(JY,Z)JX - G(JX,Z)JY + 2G(X,JY)JZ]$$

In the case of the Kählerian manifold (TM, G, J), we obtain, after a standard straightforward computation

$$K_0\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} = XXXX0^h_{kij}\frac{\delta}{\delta x^h} + XXXY0^h_{kij}\frac{\partial}{\partial y^h},$$

$$K_0\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k}=XXYX0^h_{kij}\frac{\delta}{\delta x^h}+XXYY0^h_{kij}\frac{\partial}{\partial y^h},$$
$$K_0\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\delta}{\delta x^k}=YYXX0^h_{kij}\frac{\delta}{\delta x^h}+YYXY0^h_{kij}\frac{\partial}{\partial y^h},$$
$$K_0\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k}=YYYX0^h_{kij}\frac{\delta}{\delta x^h}+YYYY0^h_{kij}\frac{\partial}{\partial y^h},$$
$$K_0\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k}=YXXX0^h_{kij}\frac{\delta}{\delta x^h}+YXXY0^h_{kij}\frac{\partial}{\partial y^h},$$
$$K_0\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k}=YXYX0^h_{kij}\frac{\delta}{\delta x^h}+YXYY0^h_{kij}\frac{\partial}{\partial y^h},$$

where

$$\begin{aligned}
XXXX0^h_{kij}=\frac{k}{4}\Big[&G^{(1)}_{jk}\delta^h_i-G^{(1)}_{ik}\delta^h_j-J^{(3)h}_i(J^{(1)l}_jG^{(3)}_{lk}-J^{(3)l}_jG^{(1)}_{lk})\\
&+J^{(3)h}_j(J^{(1)l}_iG^{(3)}_{lk}-J^{(3)l}_iG^{(1)}_{lk})-2J^{(3)h}_k(J^{(1)l}_jG^{(3)}_{il}-J^{(3)l}_jG^{(1)}_{il})\Big],\\
XXXY0^h_{kij}=\frac{k}{4}\Big[&J^{(1)h}_i(J^{(1)l}_jG^{(3)}_{lk}-J^{(3)l}_jG^{(1)}_{lk})\\
&-J^{(1)h}_j(J^{(1)l}_iG^{(3)}_{lk}-J^{(3)l}_iG^{(1)}_{lk})+2J^{(1)h}_k(J^{(1)l}_jG^{(3)}_{il}-J^{(3)l}_jG^{(1)}_{il})\Big],\\
XXYX0^h_{kij}=\frac{k}{4}\Big[&G^{(3)}_{jk}\delta^h_i-G^{(3)}_{ik}\delta^h_j-J^{(3)h}_i(J^{(1)l}_jG^{(2)}_{lk}-J^{(3)l}_jG^{(3)}_{lk})\\
&+J^{(3)h}_j(J^{(1)l}_iG^{(2)}_{lk}-J^{(3)l}_iG^{(3)}_{lk})-2J^{(2)h}_k(J^{(1)l}_jG^{(3)}_{il}-J^{(3)l}_jG^{(1)}_{il})\Big],\\
XXYY0^h_{kij}=\frac{k}{4}\Big[&J^{(1)h}_i(J^{(1)l}_jG^{(2)}_{lk}-J^{(3)l}_jG^{(3)}_{lk})\\
&-J^{(1)h}_j(J^{(1)l}_iG^{(2)}_{lk}-J^{(3)l}_iG^{(3)}_{lk})+2J^{(3)h}_k(J^{(1)l}_jG^{(3)}_{il}-J^{(3)l}_jG^{(1)}_{il})\Big],\\
YYXX0^h_{kij}=\frac{k}{4}\Big[&J^{(2)h}_i(J^{(2)l}_jG^{(1)}_{lk}-J^{(3)l}_jG^{(3)}_{lk})\\
&-J^{(2)h}_j(J^{(2)l}_iG^{(1)}_{lk}-J^{(3)l}_iG^{(3)}_{lk})-2J^{(3)h}_i(J^{(3)l}_jG^{(2)}_{il}-J^{(2)l}_jG^{(3)}_{il})\Big],\\
YYXY0^h_{kij}=\frac{k}{4}\Big[&G^{(3)}_{jk}\delta^h_i-G^{(3)}_{ik}\delta^h_j-J^{(3)h}_i(J^{(2)l}_jG^{(1)}_{lk}-J^{(3)l}_jG^{(3)}_{lk})\\
&+J^{(3)h}_j(J^{(2)l}_iG^{(1)}_{lk}-J^{(3)l}_iG^{(3)}_{lk})+2J^{(1)h}_i(J^{(3)l}_jG^{(2)}_{il}-J^{(2)l}_jG^{(3)}_{il})\Big],\\
YYYX0^h_{kij}=\frac{k}{4}\Big[&J^{(2)h}_i(J^{(2)l}_jG^{(1)}_{lk}-J^{(3)l}_jG^{(3)}_{lk})
\end{aligned}$$

$$
\begin{aligned}
&\quad - J_j^{(2)h}(J_i^{(2)l}G_{lk}^{(1)} - J_i^{(3)l}G_{lk}^{(3)}) - 2J_i^{(3)h}(J_j^{(3)l}G_{il}^{(2)} - J_j^{(2)l}G_{il}^{(3)})\Big],\\
YYYY0_{kij}^h &= \frac{k}{4}\Big[G_{jk}^{(2)}\delta_i^h - G_{ik}^{(2)}\delta_j^h - J_i^{(3)h}(J_j^{(2)l}G_{lk}^{(3)} - J_j^{(3)l}G_{lk}^{(2)})\\
&\quad + J_j^{(3)h}(J_i^{(2)l}G_{lk}^{(3)} - J_i^{(3)l}G_{lk}^{(2)}) - 2J_k^{(3)h}(J_j^{(2)l}G_{il}^{(3)} - J_j^{(3)l}G_{il}^{(2)})\Big],\\
YXXX0_{kij}^h &= \frac{k}{4}\Big[- G_{jk}^{(3)}\delta_i^h - J_i^{(3)h}(J_j^{(2)l}G_{lk}^{(1)} - J_j^{(3)l}G_{lk}^{(3)})\\
&\quad - J_j^{(2)h}(J_i^{(1)l}G_{lk}^{(3)} - J_i^{(3)l}G_{lk}^{(1)}) - 2J_k^{(3)h}(J_j^{(2)l}G_{il}^{(1)} - J_j^{(3)l}G_{il}^{(3)})\Big],\\
YXXY0_{kij}^h &= \frac{k}{4}\Big[G_{ik}^{(1)}\delta_j^h + J_i^{(1)h}(J_j^{(2)l}G_{lk}^{(1)} - J_j^{(3)l}G_{lk}^{(3)})\\
&\quad + J_j^{(3)h}(J_i^{(1)l}G_{lk}^{(3)} - J_i^{(3)l}G_{lk}^{(1)}) + 2J_k^{(1)h}(J_j^{(2)l}G_{il}^{(1)} - J_j^{(3)l}G_{il}^{(3)})\Big],\\
YXYX0_{kij}^h &= \frac{k}{4}\Big[J_i^{(3)h}(J_j^{(1)l}G_{lk}^{(2)} - J_j^{(3)l}G_{lk}^{(3)})\\
&\quad - J_j^{(3)h}(J_i^{(2)l}G_{lk}^{(3)} - J_i^{(3)l}G_{lk}^{(2)}) + 2J_k^{(2)h}(J_j^{(1)l}G_{il}^{(2)} - J_j^{(3)l}G_{il}^{(3)})\Big],\\
YXYY0_{kij}^h &= \frac{k}{4}\Big[G_{jk}^{(3)}\delta_i^h - G_{ik}^{(2)}\delta_j^h + J_i^{(3)h}(J_j^{(1)l}G_{lk}^{(2)} - J_j^{(3)l}G_{lk}^{(3)})\\
&\quad + J_j^{(1)h}(J_i^{(2)l}G_{lk}^{(3)} - J_i^{(3)l}G_{lk}^{(2)}) + 2J_k^{(3)h}(J_j^{(1)l}G_{lk}^{(3)} - J_i^{(3)l}G_{lk}^{(2)})\Big].
\end{aligned}
$$

In order to get the conditions under which (TM, G, J) is a Kählerian manifold of constant holomorphic sectional curvature, we study the vanishing of the components of the difference $K - K_0$. In this study it is useful the following generic result similar to the lemma 2.1

Lemma 3.2. *If $\alpha_1, \ldots, \alpha_{10}$ are smooth functions on TM such that*

$$
\begin{aligned}
&\alpha_1\delta_i^h g_{jk} + \alpha_2\delta_j^h g_{ik} + \alpha_3\delta_k^h g_{ij} + \alpha_4\delta_k^h g_{0i}g_{0j} + \alpha_5\delta_j^h g_{0i}g_{0k} + \alpha_6\delta_i^h g_{0j}g_{0k}\\
&\quad + \alpha_7 g_{jk}g_{0i}y^h + \alpha_8 g_{ik}g_{0j}y^h + \alpha_9 g_{ij}g_{0k}y^h + \alpha_{10}g_{0i}g_{0j}g_{0k}y^h = 0,
\end{aligned}
$$

then $\alpha_1 = \cdots = \alpha_{10} = 0$.

After a detailed analysis of several terms in the vanishing problem of the components of the above difference we have chosen the difference $YXYY_{kij}^h - YXYY0_{kij}^h$ since in this expression there appear two terms with shorter expressions. Namely, from the second term (which contain $g_{ik}\delta_j^h$), by imposing the condition to be zero, we get

$$
\lambda' = \frac{4a_1'c - \lambda k(2a_1a_1' + 2c + 4a_3a_3'ct + 2a_3^2c)}{k(a_1^2 + 2ct + 2a_3^2ct)}, \tag{14}
$$

Replacing this value in the third one (which contains $g_{ij}\delta_k^h$) we obtain the value of λ :

$$\lambda = \frac{4a_1c}{k(a_1^2 + 2ct + 2a_3^2ct)} \tag{15}$$

Next, by using this expression of λ as well as the values of λ',λ'' and λ''', obtained by straightforward computation, we get with RICCI that all the components of the difference $K - K_0$ are zero. We should mention the serious technical difficulties in the computation of the differences $YXYY^h_{kij} - YXYY0h_{kij}, YXYX^h_{kij} - YXYX0^h_{kij}, YXXY^h_{kij} - YXXY0^h_{kij}$, and $YXXX^h_{kij} - YXXX0^h_{kij}$.

Hence we may state

Theorem 3.3. *The Kählerian manifold (TM, G, J) with G and J obtained as natural lifts of general type of the Riemannian metric g on the Riemannian manifold (M, g), has constant holomorphic sectional curvature k if and only if the parameter λ is expressed by the expression* (15).

Remark. If $a_3 = 0$ we obtain the well known condition for (TM, G, J) to have constant holomorphic sectional curvature in the case where G, J are natural lifts of diagonal type (see [13], [3], and see also [16]).

REFERENCES

1. ABBASSI, M.T.K.; SARIH, M. – *On some hereditary properties of Riemannian g-natural metrics on tangent bundles of Riemannian manifolds,* Diff. Geom. And its Appl. 22 (2005), 19-47.
2. ABBASSI, M.T.K.; SARIH, M. – *On Riemannian g-natural metrics of the form $a.g^s + b.g^h + c.g^v$ on the tangent bundle of a Riemannian manifold (M,g).* Mediterranean J. Math. 2 (2005), 19-43.
3. BEJAN, C.L.; OPROIU, V. – *Tangent bundles of quasi-constant holomorphic sectional curvatures*, Balkan J. Geom. Applic., 11 (2006), 11-22.
4. KOLÁŘ, I.; MICHOR, P.; SLOVAK, J. – *Natural Operations in Differential Geometry*, Springer Verlag, Berlin, 1993, vi, 434 pp.
5. KOWALSKI, O.; SEKIZAWA, M. – *Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles-a classification,* Bull. Tokyo Gakugei Univ. (4), 40 (1988), 1-29.

6. Mok, K.P.; Patterson, E.M.; Wong, Y.C. – *Structure of symmetric tensors of type (0,2) and tensors of type (1,1) on the tangent bundle*, Trans. Amer. Math. Soc., 234 (1977), 253-278.
7. Munteanu, M. – *Cheeger Gromoll type metrics on the tangent bundle,* Proceedings of Fifth International Symposium BioMathsPhys, Iasi , June 16-17, 2006, 9pp.
8. Munteanu, M. – *Old and New Structures on the Tangent Bundle,* Proceedings of the Eighth International Conference on Geometry, Integrability and Quantization, June 9-14, 2006, Varna, Bulgaria, Ed. I. M. Mladenov and M. de Leon, Sofia 2007 264-278.
9. Oproiu, V. – *A generalization of natural almost Hermitian structures on the tangent bundles*, Math. J. Toyama Univ., 22 (1999), 1-14.
10. Oproiu, V. – *A locally symmetric Kaehler Einstein structure on the tangent bundle of a space form*, Beiträge Algebra Geom/Contributions to Algebra and Geometry, 40 (1999), 363-372.
11. Oproiu, V. – *A Kaehler Einstein structure on the tangent bundle of a space form*, Int. J. Math. Math. Sci. 25 (3) (2001), 183-195.
12. Oproiu, V. – *Some new geometric structures on the tangent bundles*, Publ. Math. Debrecen, 55/3-4 (1999), 261-281.
13. Oproiu, V. – *Bochner flat tangent bundles*, An.St. Univ. "Al.I. Cuza" Iasi, Matematica, 52 (2006), 25-36.
14. Oproiu, V.; Papaghiuc, N. – *A Kaehler structure on the nonzero tangent bundle of a space form,* Diff. Geom. Appl. 11 (1999), 1-14.
15. Oproiu, V.; Papaghiuc, N. – *Some classes of almost anti-Hermitian structures on the tangent bundle*, Mediterranean Journal of Mathematics 1 (3) (2004), 269-282.
16. Oproiu, V.; Poroşniuc, D. – *A class of Kaehler Einstein structures on the cotangent bundle of a space form*, Publ. Math. Debrecen 66 (2005), 457-478.
17. Sasaki, S. – *On the differential geometry of the tangent bundle of Riemannian manifolds*, Tôhoku Math. J., 10 (1958), 238-354.
18. Tahara, M.; Watanabe,Y. – *Natural almost Hermitian, Hermitian and Kähler metrics on the tangent bundles*, Math. J. Toyama Univ., 20 (1997), 149-160.
19. Tahara, M., Marchiafava, S., Watanabe, Y. – *Quaternion Kähler structures on the tangent bundle of a complex space form*, Rend. Ist. Mat. Univ. Trieste, Suppl. vol 30 (1999), 163-175.
20. Tahara, M., Vanhecke, L., Watanabe, Y. – *New structures on tangent bundles*, Note di Matematica (Lecce), 18 (1998), 131-141.
21. Yano, K., Ishihara, S. – *Tangent and Cotangent Bundles*, M. Dekker Inc., New York, 1973.

22. Yano, K.; Kobayashi, S. – *Prolongations of tensor fields and connections to tangent bundles, I. General Theory*, Jour. Math. Soc. Japan, 18 (1966), 194-210.

Received: 18.I.2007

Faculty of Mathematics,
University "Al.I. Cuza",
Iaşi, 700 506
ROMÂNIA
simonadruta@yahoo.com

Faculty of Mathematics,
University "Al.I. Cuza",
Iaşi, 700 506,
ROMÂNIA
voproiu@uaic.ro